# Extensions of Elementary Cause-Effect Structures
# Extended abstract


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Before formulation of some extensions of elementary cause-effect (c-e) structures (see References), let us outline their concept by examples. A c-e structure (both elementary - a counterpart of 1-safe Petri nets - and extended) is a directed graph with predecessors and successors of every vertex (node) grouped into families of sets, as shows left graph in Fig.1: predecessors of $e$: $\{\{a,b\},\{b,c\},\{d\}\}$, successors: $\{\{f,g\},\{h\}\}$. In the right graph, the node symbols are subscripted and superscripted with expressions called formal polynomials, that determine the grouping, so that the "operator of multiplication •" collects the arguments into a group, whereas "operator of addition +" separates the groups. Symbol $\theta$ denotes the empty family. Thus, this graph is the set $\{a_e^\theta, b_e^\theta, c_e^\theta, d_e^\theta, e^{a\bullet b+b\bullet c+d}_{f\bullet g+h}, f_\theta^e, g_\theta^e, h_\theta^e\}$. Each c-e structure can be represented by a set of such annotated nodes. The arrows, though helpful to understand its dynamics, are a superfluous information. Informally, the dynamics is a "token game": node $e$ can receive signals (represented by tokens) simultaneously from $a$ and $b$ or simultaneously from $b$ and $c$ or only from $d$, and send signals simultaneously to $f$ and g or only to $h$. Thus, the operator "•" means simultaneity, while "+" - exclusive choice. As a realistic example, consider the c-e structure $ROAD$ in Fig.2, describing a traffic through the bridge $B$ on the two-lane road, each lane for vehicles heading in the opposite directions. The bridge can hold only one vehicle at a time.

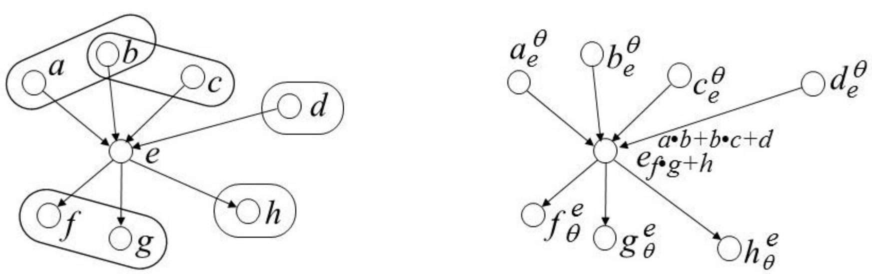


**Fig. 1.** left: graph with encircled families of predecessors and successors of $e$; right: graph with subscripted and superscripted symbols of nodes.

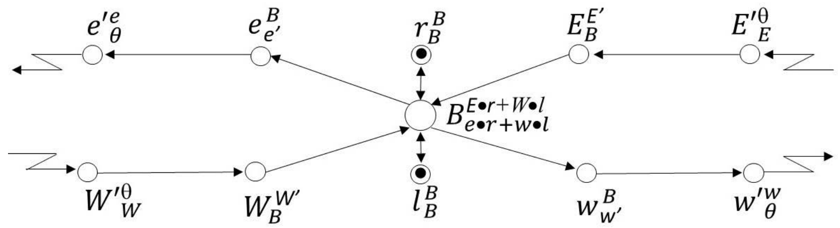


**Fig. 2.** c-e structure $ROAD$. Traffic from the East: $E' \to E \to B \to e \to e'$ and from the West: $W' \to W \to B \to w \to w'$. Nodes $r$ and $l$ prevent the U-turn on the bridge: a token in $r$ makes impossible move $E \to B \to w$, while in $l$ - impossible move $W \to B \to e$.

Thus, in the set-like notation,
$ROAD = \{E'^{\theta}_{E},\ E^{E'}_{B},\ B^{E\bullet r+W\bullet l}_{e\bullet r+w\bullet l},\ r^{B}_{B},\ e^{B}_{e'},\ e'^{e}_{\theta},\ W'^{\theta}_{W},\ W^{W'}_{B},\ l^{B}_{B},\ w^{B}_{w'},\ w'^{w}_{\theta}\}$.
Anticipating the formal definitions, notice that this is a combination $ROAD = EW \bullet R + WE \bullet L$, where $EW = \{E'^{\theta}_{E}, E^{E'}_{B}, B^{E}_{e}, e^{B}_{e'}, e'^{e}_{\theta}\}$,
$WE = \{W'^{\theta}_{W}, W^{W'}_{B}, B^{W}_{w}, w^{B}_{w'}, w'^{w}_{\theta}\}$, $r = \{r^{B}_{B}, B^{r}_{r}\}$, $l = \{l^{B}_{B}, B^{l}_{l}\}$, or pictorially, a combination of the c-e structures in Fig.3. So, the "multiplication" and "ad-

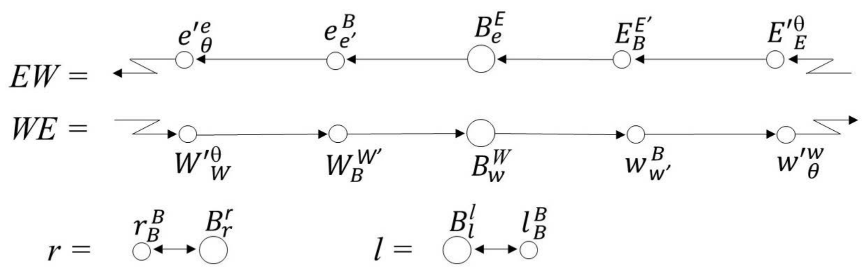


**Fig. 3.** traffic East → West and West → East, and no-U-turn control.

dition" are now extended from the formal polynomials onto c-e structures, so that "$\bullet$" and "$+$" mean making union of sets being their arguments, with formal product and sum of subscripts/superscripts of nodes identically named in both sets. Now, the formal definitions.

**Definition 1** (set $\boldsymbol{F}[\mathbb{X}]$, quasi semiring of formal polynomials)

Let $\mathbb{X}$ be a non-empty enumerable set. Their elements, called *nodes*, are counterparts of places in Petri nets [Rei 85]. Let $\theta \notin \mathbb{X}$ be a symbol called *neutral*. It will play a role of neutral element for operations on expressions, called *formal polynomials over* $\mathbb{X}$. The names of nodes, symbol $\theta$, operators $+$, $\bullet$ and parentheses are symbols out of which formal polynomials are formed in the usual (infix) way. Their set is denoted by $\boldsymbol{F}[\mathbb{X}]$. Stronger binding of $\bullet$ than $+$, allows for dropping some parentheses. Addition and multiplication of $K, L \in \boldsymbol{F}[\mathbb{X}]$ is defined as follows: $K \oplus L = (K + L)$, $K \odot L = (K \bullet L)$. Let us use $+$ and $\bullet$ instead of $\oplus$ and $\odot$. It is required that the system $\langle \boldsymbol{F}[\mathbb{X}], +, \bullet, \theta \rangle$ obeys the following equality axioms for all $K, L, M \in \boldsymbol{F}[\mathbb{X}]$, $x \in \mathbb{X}$:

| | | | |
|---|---|---|---|
| $(+)$ | $\theta + K = K + \theta = K$ | $(\bullet)$ | $\theta \bullet K = K \bullet \theta = K$ |
| $(++)$ | $K + K = K$ | $(\bullet\bullet)$ | $x \bullet x = x$ |
| $(+++)$ | $K + L = L + K$ | $(\bullet\bullet\bullet)$ | $K \bullet L = L \bullet K$ |
| $(++++)$ | $K + (L + M) = (K + L) + M$ | $(\bullet\bullet\bullet\bullet)$ | $K \bullet (L \bullet M) = (K \bullet L) \bullet M$ |

$(+\bullet)$ If $L \neq \theta \Leftrightarrow M \neq \theta$ then $K \bullet (L + M) = K \bullet L + K \bullet M$

Algebraic system which obeys these axioms will be referred to as a *quasi-semiring of formal polynomials*.[3] □

The system $\langle \boldsymbol{F}[\mathbb{X}], +, \bullet, \theta \rangle$ has a "family of sets" model shown above, thus is consistent. Its peculiarity, in contrast to the ordinary semiring, is axiom $(+\bullet)$ - the conditional distributivity of multiplication over addition, and the neutral $\theta$ for both operations. These assumptions make c-e structures behaviourally equivalent to Petri nets.

**Definition 2** (cause-effect structure, carrier, set $\boldsymbol{CE}$)

A cause-effect structure (c-e structure) over $\mathbb{X}$ is a pair $U = (C, E)$ of total and injective functions:

$C$: $\mathbb{X} \to \boldsymbol{F}[\mathbb{X}]$ (*cause function*; nodes occuring in $C(x)$ are *causes* of $x$)
$E$: $\mathbb{X} \to \boldsymbol{F}[\mathbb{X}]$ (*effect function*; nodes occuring in $E(x)$ are *effects* of $x$)

such that $x$ occurs in the formal polynomial $C(y)$ iff $y$ occurs in $E(x)$. *Carrier* of $U$ is the set $car(U) = \{x \in \mathbb{X} : C(x) \neq \theta \vee E(x) \neq \theta\}$. $U$ is of *finite carrier* iff $|car(U)| < \infty$ ($|\ ...\ |$ denotes cardinality). The set of all c-e structures over $\mathbb{X}$ is denoted by $\boldsymbol{CE}[\mathbb{X}]$. Since $\mathbb{X}$ is fixed, we write $\boldsymbol{CE}$ - wherever this makes no confusion. □

Since functions $C$ and $E$ are total, each c-e structure comprises all nodes from $\mathbb{X}$, also the isolated ones - those from outside of its carrier. Presenting c-e structures graphically, only their carriers are pictured.

**Definition 3** (addition and multiplication, monomial c-e structure)

For c-e structures $U = (C_U, E_U)$, $V = (C_V, E_V)$ define:
$U + V = (C_{U+V}, E_{U+V}) = (C_U + C_V, E_U + E_V)$ where
$(C_U + C_V)(x) = C_U(x) + C_V(x)$
$U \bullet V = (C_{U \bullet V}, E_{U \bullet V}) = (C_U \bullet C_V, E_U \bullet E_V)$ where
$(C_U \bullet C_V)(x) = C_U(x) \bullet C_V(x)$

[3] In the early papers on cause-effect structures, the term "near-semiring" has been used. But in the meantime some authors used it in different meaning, so, we use term "quasi-semiring" for this axiomatic system.

(The same symbol is used for multiplication of c-e structures, functions and polynomials)

$U$ is a *monomial* c-e structure iff each polynomial $C_U(x)$ and $E_U(x)$ is a monomial, i.e. does not comprise non-reducible "+". $\square$

Evidently $U + V \in \boldsymbol{CE}$ and $U \bullet V \in \boldsymbol{CE}$ that is, in the resulting structures, $x$ occurs in $C_{U+V}(y)$ iff $y$ occurs in $E_{U+V}(x)$ and the same for $U \bullet V$. Thus, addition and multiplication of c-e structures yield correct c-e structures.

The set $\boldsymbol{CE}$ with addition, multiplication and a distinguished element denoted also by $\theta$ and understood as the empty c-e structure $(\theta, \theta)$, where $\theta$ is a constant function $\theta(x) = \theta$ for all $x \in \mathbb{X}$, makes an algebraic system similar to that in Definition 1.

**Proposition 1** (quasi semiring of c-e structures)

The system $\langle \boldsymbol{CE}[\mathbb{X}], +, \bullet, \theta \rangle$ obeys the following equations for all $U, V, W \in \boldsymbol{CE}[\mathbb{X}]$, $x, y \in \mathbb{X}$:

(+) $\theta + U = U + \theta = U$ (•) $\theta \bullet U = U \bullet \theta = U$

(++) $U + U = U$ (••) $(x \to y) \bullet (x \to y) = x \to y$

(+++) $U + V = U + V$ (•••) $U \bullet V = V \bullet U$

(++++) $U + (V + W) = (U + V) + W$ (••••) $U \bullet (V \bullet W) = (U \bullet V) \bullet W$

(+•) If $C_V(x) \neq \theta \Leftrightarrow C_W(x) \neq \theta$ and $E_V(x) \neq \theta \Leftrightarrow E_W(x) \neq \theta$ then $U \bullet (V + W) = U \bullet V + U \bullet W$ $\square$

This follows directly from definition of c-e structures and definitions of adding and multiplying c-e structures. The operations on c-e structures make possible to combine small c-e structures into large ones.

**Definition 4** (partial order $\leq$; substructure, set $\boldsymbol{SUB}[V]$)

For $U, V \in \boldsymbol{CE}$ let $U \leq V \Leftrightarrow V = U + V$; obviously, $\leq$ is a partial order in $\boldsymbol{CE}$. If $U \leq V$ then $U$ is a *substructure* of $V$; $\boldsymbol{SUB}[V] = \{U : U \leq V\}$ is the set of all substructures of $V$. For $A \subseteq \boldsymbol{CE}$: $V \in A$ is *minimal* (w.r.t. $\leq$) in $A$ iff $\forall W \in A$: $(W \leq V \Rightarrow W = V)$ $\square$

The crucial notion for behaviour of c-e structures is firing component, a counterpart of transition in Petri nets, i.e. a state transformer. It is, however, not a primitive notion but derived from the definition of c-e structures, and is introduced regardless of any particular c-e structure:

**Definition 5** (firing component, set $\boldsymbol{FC}$, pre-set and post-set)

A minimal in $\boldsymbol{CE} \backslash \{\theta\}$ c-e structure $Q = (C_Q, E_Q)$ is a *firing component* iff $Q$ is a monomial c-e structure and $C_Q(x) = \theta \Leftrightarrow E_Q(x) \neq \theta$ for any $x \in car(Q)$. The set of all firing components is denoted by $\boldsymbol{FC}$, thus the set of all firing components of $U \in \boldsymbol{CE}$ is $\boldsymbol{FC}[U] = \boldsymbol{SUB}[U] \cap \boldsymbol{FC}$. Following the standard Petri net notation, let for $Q \in \boldsymbol{FC}$ and $G \subseteq \boldsymbol{FC}$:

$^\bullet Q = \{x \in car(Q) : C_Q(x) = \theta\}$ (*pre-set* of $Q$)
$Q^\bullet = \{x \in car(Q) : E_Q(x) = \theta\}$ (*post-set* of $Q$)
$^\bullet Q^\bullet = {}^\bullet Q \cup Q^\bullet$ □

So, the firing component is a connected graph, due to the required minimality. Elements of the pre-set are its *causes* and elements of the post-set are its *effects*. Of many conclusions from above definitions, some are worth to point out:

**Proposition 2**

(a) $U_1 \leq V_1 \wedge U_2 \leq V_2 \Rightarrow U_1 + U_2 \leq V_1 + V_2$ (monotonicity of +)
(b) $U \bullet (V + W) \leq U \bullet V + U \bullet W$ but equality not always holds
(c) $U \leq V \Rightarrow \boldsymbol{FC}[U] \subseteq \boldsymbol{FC}[V]$ but converse implication not always holds
(d) $\boldsymbol{FC}[U] \cup \boldsymbol{FC}[V] \subseteq \boldsymbol{FC}[U + V]$ but converse inclusion not always holds

Point (d) states that new firing components may appear when summing up c-e structures. For instance, let $U = \{a_{x+y}, b_{x\bullet y}, x^{a\bullet b}, y^{a\bullet b}\}$, $V = \{a_{x\bullet y}, x^a, y^a\}$, thus $\boldsymbol{F}\mathbf{C}[U] = \emptyset$, $\boldsymbol{F}\mathbf{C}[V] = \{V\}$,
$\boldsymbol{F}\mathbf{C}[U + V] = \{\{a_x, x^a\}, \{a_y, y^a\}, V, \{a_{x\bullet y}, b_{x\bullet y}, x^{a\bullet b}, y^{a\bullet b}\}\}$, thus
$\boldsymbol{F}\mathbf{C}[U] \cup \boldsymbol{FC}[V] \neq \boldsymbol{F}\mathbf{C}[U + V]$. The phenomenon of creation new firing components when assembling c-e structures from smaller parts, reflects a general observation: compound systems may sometimes reveal behaviours absent in their parts.

**Definition 6** (state of c-e structure)

A state of c-e structure $U$ is a total injective function $s : car(U) \rightarrow \mathbb{N}_\omega$, thus a multiset over $car(U)$ ($\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$, where $\omega$ symbolises infinity, that is $\omega > n$ for each $n \in \mathbb{N}$; $\mathbb{N}$ is the set of natural numbers, with 0). The set of all states of $U$ is denoted by $\mathbb{S}$. □

**Definition 7** (weights of monomials and capacity of nodes)

Given a c-e structure $U = (C, E)$ and its firing component $Q \in \boldsymbol{FC}[U]$, let along with the pre-set $^\bullet Q$ and post-set $Q^\bullet$ of $Q$, some multisets $\overline{^\bullet Q}$: $^\bullet Q \rightarrow \mathbb{N}_\omega \backslash \{0\}$ and $\overline{Q^\bullet}$: $Q^\bullet \rightarrow \mathbb{N}_\omega \backslash \{0\}$ be given as additional information. The value $\overline{^\bullet Q}(x)$ is called a *weight* (or *multiplicity*) of monomial $E_Q(x)$ and the value $\overline{Q^\bullet}(x)$ - a *weight* (or *multiplicity*) of monomial $C_Q(x)$. Let *cap* be a total injective function $cap : car(U) \rightarrow \mathbb{N}_\omega$, assigning a *capacity* to each node in the set $car(U)$. A c-e structure with such enhanced firing components is called a *c-e structure-with-weights of monomials and capacity of nodes.* □

**Definition 8** (firing components enabled and with inhibitors)

For a firing component $Q \in \boldsymbol{FC}[U]$, the set $inh[Q] = \{x \in {}^\bullet Q : \overline{^\bullet Q}(x) = \omega\}$ is the collection of nodes in the pre-set of $Q$, whose effect monomials $E_Q(x)$ are of weight $\omega$. The nodes in $inh[Q]$ will play role of *inhibiting nodes* of firing component $Q$, as follows. For $Q$ and state $s$ let us define the formula: $enabled[Q](s) \overset{\text{def}}{\Leftrightarrow}$

$\forall x \in inh[Q] : s(x) = 0 \wedge$
$\forall x \in {}^{\bullet}Q \backslash inh[Q] : \overline{{}^{\bullet}Q}(x) \leq s(x) \leq cap(x) \wedge$
$\forall x \in Q^{\bullet} : \overline{Q^{\bullet}}(x) + s(x) \leq cap(x)$ □

So, $Q$ is enabled at the state $s$ iff none of inhibiting nodes $x \in {}^{\bullet}Q$ contains a token and each remaining node in ${}^{\bullet}Q$ does, with no fewer tokens than is the weight of its effect monomial $E_Q(x)$ and no more than capacity of each $x \in {}^{\bullet}Q$. Moreover, none of $x \in Q^{\bullet}$ holds more tokens than their number, when increased by the weight of its cause monomial $C_Q(x)$, exceedes capacity of $x$. The inhibiting nodes of a firing components will be called its *inhibitors*.

Fig.4(a) shows a firing component $Q$ with weighted (multiplied) effect monomials $E_Q(a) = 5 \otimes x$, $E_Q(b) = \omega \otimes (x \bullet y)$, $E_Q(c) = 3 \otimes y$ and weighted cause monomials $C_Q(x) = 2 \otimes (a \bullet b)$, $C_Q(y) = 4 \otimes (b \bullet c)$. The inhibitor of $Q$ is node $b$. Fig.4(b) shows the behaviourally equivalent single transition in Petri net with weights and inhibitor arrow.

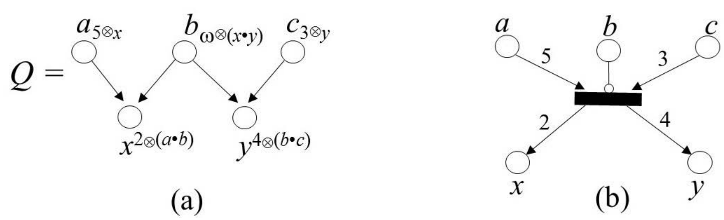


**Fig. 4.** (a) Firing component $Q$ with weights; $2 \otimes (a \bullet b)$, $\omega \otimes (x \bullet y)$, etc. denote multiplicity of the product $a \bullet b$ by the factor $2 = \overline{Q^{\bullet}}(x)$ and product $x \bullet y$ by factor $\omega = \overline{{}^{\bullet}Q}(b)$. (b) Behaviourally equivalent Petri net transition.

**Definition 9** (semantics [[ ]] of c-e structures with inhibitors)

For $Q \in \boldsymbol{FC}[U]$, let $[[Q]] \subseteq \mathbb{S} \times \mathbb{S}$ be a binary relation defined as: $(s,t) \in [[Q]]$ iff $enabled[Q](s) \wedge t = (s - \overline{{}^{\bullet}Q}) + \overline{Q^{\bullet}} \leq cap$ ($Q$ transforms state $s$ into $t$). *Semantics* $[[U]]$ of $U \in \boldsymbol{CE}$ is $[[U]] = \bigcup_{Q \in \boldsymbol{FC}[U]} [[Q]]$. Closure, reachability and computation: $(s,t) \in [[U]]^*$ iff $s = t$ or there is a sequence of states $s_0, s_1, ..., s_n$ with $s = s_0$, $t = s_n$ and $(s_j, s_{j+1}) \in [[U]]$ for $j = 0, 1, ..., n-1$. State $t$ is *reachable* from $s$ in semantics [[ ]] and the sequenece $s_0, s_1, ..., s_n$ is a *computation* in $U$. □

In the c-e structure which presents a ride throught the bridge $B$, the priority ride from the East can be enforced using inhibitor, i.e. node $E$ in the pre-set of firing component $\{W_B^{\theta}, E_{\omega \otimes B}^{\theta}, l_B^{\theta}, B_{\theta}^{W \bullet E \bullet l}\}$, as shown in Fig.5.

Firing components $\{E_B^{\theta}, r_B^{\theta}, B_{\theta}^{E \bullet r}\}$ and $\{W_B^{\theta}, E_{\omega \otimes B}^{\theta}, l_B^{\theta}, B_{\theta}^{W \bullet E \bullet l}\}$ of the c-e structure in Fig.5 have Petri nets (with inhibitor arcs) counterparts as two transitions shown in Fig.6.

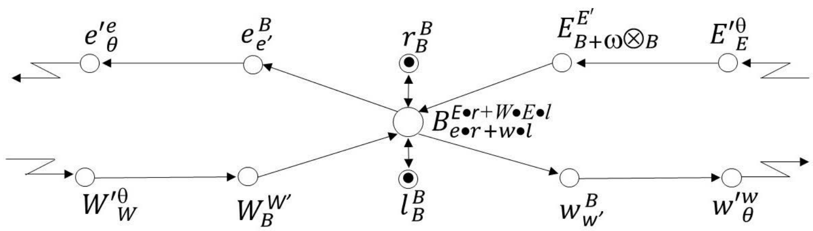


**Fig. 5.** If at $E$ and $W$ are vehicles (tokens) and none at $B$, then only the one in $E$ will get entry permit at $B$, since only firing component $\{E^{\theta}_{B}, r^{\theta}_{B}, B^{E\bullet r}_{\theta}\}$ can fire in such state, not this one: $\{W^{\theta}_{B}, E^{\theta}_{\omega\otimes B}, l^{\theta}_{B}, B^{W\bullet E\bullet l}_{\theta}\}$ - due to its inhibiting node $E$ if it contains a token.

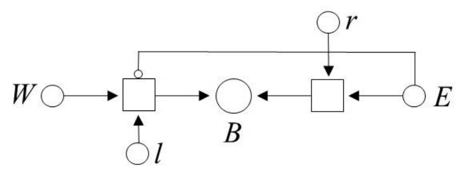


**Fig. 6.** Petri net counterpart of two firing components with place $B$ of c-e structure shown in Fig.3.2. Inhibitor arc leads from place $E$ to the left transition.

**Example (the Readers/Writers problem)**

A set of $n$ sequential agents run concurrently under constraint: writing to a common file by the $j$th ($j = 1, 2, ..., n$) agent prevents all remaining from reading and writing, but not from their private (internal) activity. Reading may proceed in parallel. Fig.7 shows three agents with the following meaning of nodes: $Aj$ - agent of number $j = 1, 2, 3$ is active (holds a token) if it is neither reading nor writing; $Rj$ - is active if the $j$th agent is reading; $Wj$ - is active if the $j$th agent is writing. $Wj$ and $Rj$ play both roles: of the ordinary nodes or of the inhibitors, dependently which firing component they belong to.

A few semantic properties of c-e structures are in:

**Proposition 3**

For any c-e structures $U, V \in \boldsymbol{CE}$:

(a) $U \leq V \Rightarrow \boldsymbol{FC}[U] \subseteq \boldsymbol{FC}[V] \Rightarrow [[U]] \subseteq [[V]] \Rightarrow [[U]]^* \subseteq [[V]]^*$

(b) $[[U]] \cup [[V]] \subseteq [[U+V]]$, but the reverse inclusion not always holds

(c) $\boldsymbol{FC}[U] \cup \boldsymbol{FC}[V] = \boldsymbol{FC}[U+V] \Rightarrow [[U]] \cup [[V]] = [[U+V]]$ but not conversely.

(d) $\boldsymbol{FC}[U] \cup \boldsymbol{FC}[V] = \boldsymbol{FC}[U+V]$ and $[[U]]^* \cup [[V]]^* = [[U+V]]^*$ are unrelated by implication. □

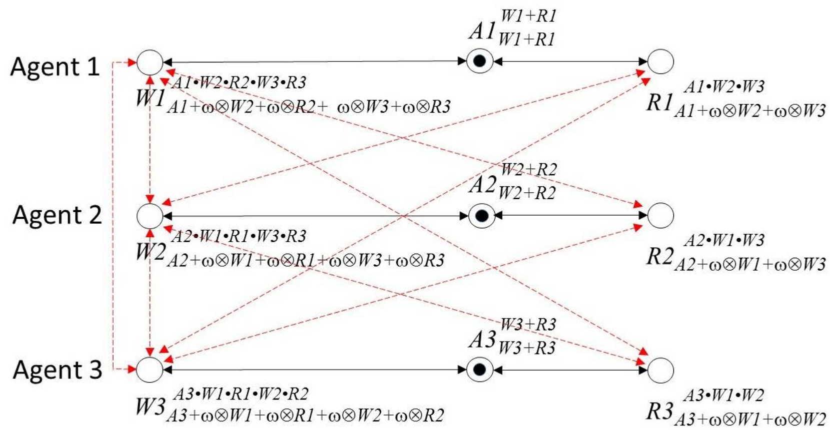


**Fig. 7.** Three agents' Readers/Writers system as a c-e structure $RW$ with inhibitors; the dashed arrows denote usage of inhibitors. Initially, the agents are neither reading nor writing (tokens in $A1, A2, A3$).

## Another extension: c-e structures with time.

Time models are different from those in Petri nets with time, where time is usually treated as necessary or admissible period of activity of a node (site or action). Here, the minimal time model is considered, where capacity of nodes equal 1, and with each node a time period of ***mandatory*** stay of a token is associated. This is the shortest time during which the node ***must*** hold the token. On expiry of this period, the token ***can*** leave the node (if other necessary conditions for this "move" are met). Lapse of time may be related either to individual firing components or to the whole c-e structure. The period of a token stay at a node is set up on entering this token into it and decreases by one time unit ("tick") of the timer referred to by the node, until permission to leave this node. On expiry of the mandatory residing time at this node, the token can leave it if all other conditions for this action are met. Any c-e structure with the minimal time model can be simulated ("implemented") by a c-e structure without time but with some additional nodes associated with every original node. A number of these supplementing and linearly ordered nodes represent duration of mandatory stay of a token in the original node.

**Definition 10** (min-time c-e structure, set $T_{min}\boldsymbol{CE}$)
$U = \langle C, E, T_{min} \rangle$ is a *minimal-time* c-e structure iff $\langle C, E \rangle$ c-e structure with capacity of nodes equal 1, and $T_{min}$: $car(U) \to \mathbb{N} \backslash \{0\}$ is a *minimal time function* of the meaning: $T_{min}(x)$ is the least number of time units indicated by a timer referred to by the node $x$, during which a token ***must*** stay at $x$ since its appearance. The timer is associated to a particular node. The set of all the min-time c-e structures over is denoted by $T_{min}\boldsymbol{CE}$ □

**Definition 11** (state of min-time c-e structure)

State is a function $s : \mathbb{X} \to \mathbb{N}$ with the informal meaning: $s(x) = 0$ if there is no token at the node $x$ and $s(x) > 1$ is a remaining time (a number of ticks of the timer referred to by $x$) during which the token ***must*** remain at $x$; $s(x) = 1$ indicates that the time of ***compulsory*** residence of a token at the node $x$, prescribed by $T_{min}(x)$ has elapsed, thus, the token ***can*** be moved further - if other conditions for this are satisfied. The set of all states is $\mathbb{S} = \mathbb{N}^{\mathbb{X}}$ (state-space). □

So, now, the $s(x)$ is not a number of tokens residing at the node, but a current time lapse.

**Definition 12** (min-time semantics: a firing rule)

For $Q \in \boldsymbol{FC}[U]$, let $[[Q]] \subseteq \mathbb{S} \times \mathbb{S}$ be a binary relation defined as: $(s, t) \in [[Q]]$ if and only if:

$\forall x \in {}^{\bullet}Q : [s(x) = 1 \wedge t(x) = 0 \wedge \forall y \in Q^{\bullet} : [s(y) = 0 \wedge t(y) = T_{min}(y)] \vee$
$\exists x \in {}^{\bullet}Q^{\bullet} : [s(x) > 1 \wedge t(x) = s(x) - 1]$

Semantics $[[U]]$ of $U \in T_{min}CE$ is the union of relations $[[U]] = \bigcup_{Q \in FC[U]} [[Q]]$.

$[[U]]^*$ is the reflexive and transitive closure of $[[U]]$ □

The formula $\exists x \in {}^{\bullet}Q^{\bullet} : [s(x) > 1 \wedge t(x) = s(x) - 1]$ expresses decrease by one time unit of token's stay at a certain node $x$ of $Q$, if the minimal time of this token has not expired in the state $s$. The minimal time can be simulated by c-e structures without time constraints. An exemplary simulation of the c-e structure in Fig.8 (firing component) depicts Fig.9.

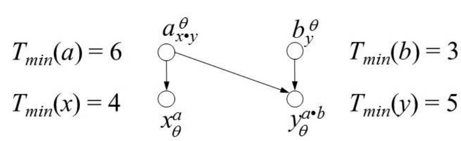


**Fig. 8.** c-e structure (a firing component) with min-time assigned to nodes.

If the time is taken from a timer common to all nodes (violation of distributed systems' principles!), the semantics is re-interpreted: the decreasing elapse of time now concerns all nodes in $car(U)$, not only a given firing component. Thus, the formula $\exists x \in {}^{\bullet}Q^{\bullet} : [s(x) > 1 \wedge t(x) = s(x) - 1]$ would be replaced with $\exists x \in car(Q) : [s(x) > 1 \wedge t(x) = s(x) - 1]$. An example of this case, taken from music, is in Fig.10.

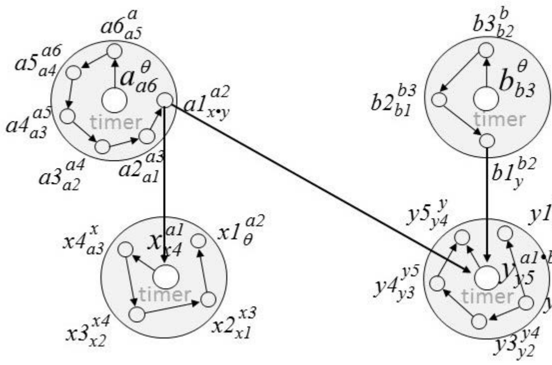


**Fig. 9.** A possible simulation of the c-e structure in Fig.8 with the minimal time of nodes, by means of no-time c-e structure. The separate timer (each with perhaps diverse progress rate of time) is associated with every node. The counterclockwise direction of a token's motion inside the timers, simulates elapse of time controlled by the timers associated with nodes $a, b, x, y$.

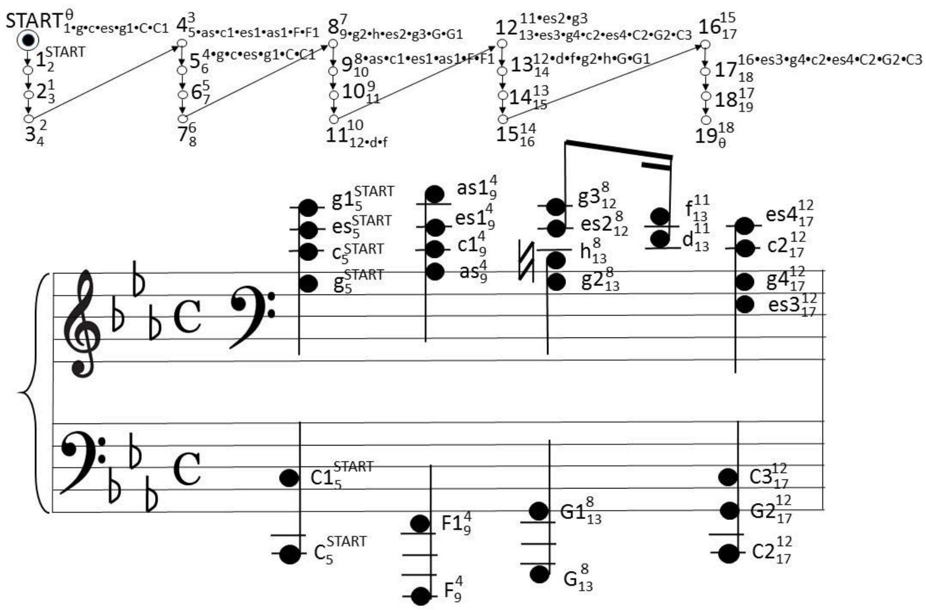


**Fig. 10.** First bar of the score of Prelude c-minor by Chopin, in the form of the min-time c-e-structure. The notes are represented as nodes with assigned duration periods, implemented by the control mechanism above the music text. The chords are accomplished by synchronization vertical notes, using multiplication "•"

The graphic examples have been tested by a computer program comprising editor and simulator of the cause-effect structures [Chm 2003].

A number of problems and properties of extended c-e structures, not presented in this short note, can be transferred from elementary c-e structures (see References). For instance such issues as:

- Decomposition of c-e structures
- Relation to Petri nets and to other models of concurrency
- C-e structures as lattices - their lattice properties
- Processes generated by c-e structures, monoid of processes
- Formal languages of c-e structure processes: the analysis and synthesis problems

**Summarizing**: the main motivation to develop the algebra (the quasi semiring) of c-e structures, was to combine structuring mechanism and transformation rules it provides, with appeal of simple pictorial and animated presentation of modelled real life systems. This algebra is a formal background for combining small c-e structures of easy to understand behaviour, into large system models, whose behavioral properties might be inferred from behaviour of their small parts. Such feature is called a *compositionality* (here conditional) - a counterpart of the *extensionality* in formal logic. Also, absence of explicit appearance of transitions and adjacent arrows - as is the case of Petri nets - provides more monitor space for graphic presentation.

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