The Influence of the Test Operator on the Expressive Powers of PDL-Like Logics

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Abstract. Berman and Paterson proved that Test-Free PDL is weaker than PDL. As the description logics $\mathcal{ALC}_{trans}$ and $\mathcal{ALC}_{reg}$ are, respectively, variants of Test-Free PDL and PDL, there is a concept of $\mathcal{ALC}_{reg}$ that is not equivalent to any concept of $\mathcal{ALC}_{trans}$. Generalizing this, we show that there is a concept of $\mathcal{ALC}_{reg}$ that is not equivalent to any concept of the logic that extends $\mathcal{ALC}_{trans}$ with inverse roles, nominals, qualified number restrictions, the universal role and local reflexivity of roles. We also provide some results for the case with RBoxes and TBoxes. One of them states that tests can be eliminated from TBoxes of the deterministic Horn fragment of $\mathcal{ALC}_{reg}$.

1 Introduction

Propositional Dynamic Logic (PDL) is a well-known modal logic for reasoning about computer programs [5,7]. Its variant $\mathcal{ALC}_{reg}$ is a description logic (DL) for reasoning about terminological knowledge [16]. Berman and Paterson [2] proved that Test-Free PDL is weaker than PDL. In particular, they gave a formula of PDL that is not equivalent to any formula of Test-Free PDL. This means that there is a concept of $\mathcal{ALC}_{reg}$ that is not equivalent to any concept of $\mathcal{ALC}_{trans}$ (a variant of Test-Free PDL). While bisimulations are usually used for separating the expressive powers of modal and description logics (see, e.g., [3,4,10]), the proof given by Berman and Paterson [2] exploits the fact that "over a single symbol alphabet, the regular sets are precisely those which are ultimately periodic" (see [6, Theorem 3.1.2]) and is somehow similar to the proof of that connectivity is inexpressible in first-order logic.

Generalizing the result and method of Berman and Paterson, in Section 3 we prove that there is a concept of $\mathcal{ALC}_{reg}$ that is not equivalent to any concept of the DL $\mathcal{ALCIOQU}\mathit{Self}_{trans}$, which extends $\mathcal{ALC}_{trans}$ with inverse roles ($\mathcal{I}$), nominals ($\mathcal{O}$), qualified number restrictions ($\mathcal{Q}$), the universal role ($\mathcal{U}$) and local reflexivity of roles ($\mathcal{S}\mathit{elf}$) as of the DL $\mathcal{SROIQ}$ [8]. That is, extending $\mathcal{ALC}_{trans}$ with the features $\mathcal{I}$, $\mathcal{O}$, $\mathcal{Q}$, $\mathcal{U}$ and $\mathcal{S}\mathit{elf}$ does not help in expressing the test operator. Modifying the proof of Berman and Paterson [2] for dealing with the features $\mathcal{O}$, $\mathcal{Q}$, $\mathcal{U}$ and $\mathcal{S}\mathit{elf}$ can be done in a rather straightforward way (see our Lemmas 1, 3, 4 and their proofs). However, dealing with inverse roles ($\mathcal{I}$)

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requires an advanced refinement, as regular sets over an alphabet consisting of an atomic role and its inverse need not be ultimately periodic. The proof of our Lemma 2 is more sophisticated than the proof of [6, Theorem 3.1.2].

In Section 4, we provide a result stating that using regular RBoxes and acyclic TBoxes for $\mathcal{ALCIOQUSelf}_{trans}$ does not help in expressing tests, but using simple stratified TBoxes under the stratified semantics on the background allows us to express every concept by another without tests. A further result states that tests can be eliminated from TBoxes of the deterministic Horn fragment of $\mathcal{ALC}_{reg}$. This suggests that tests can be eliminated from tractable[1] Horn fragments of PDL-like logics.

2 Preliminaries

This section provides notions and definitions related with syntax and semantics of DLs [1]. We denote the sets of concept names, role names and individual names by $\mathbf{C}$, $\mathbf{R}_+$ and $\mathbf{I}$, respectively. A concept name is an *atomic concept*, a role name is an *atomic role*. Let $\mathbf{R} = \mathbf{R}_+ \cup \mathbf{R}_-$, where $\mathbf{R}_- = \{\overline{r} \mid r \in \mathbf{R}_+\}$ and $\overline{r}$ is called the *inverse* of r. We call elements of $\mathbf{R}$ *basic roles*. We distinguish a subset of $\mathbf{R}_+$ whose elements are called *simple roles*. If $r \in \mathbf{R}_+$ is a simple role, then $\overline{r}$ is also a simple role. The set $\Sigma = \mathbf{C} \cup \mathbf{R}_+ \cup \mathbf{I}$ is called the *signature*.

Let $\Phi \subseteq \{\mathcal{I}, \mathcal{O}, \mathcal{Q}, \mathcal{U}, \mathcal{S}elf\}$, where the symbols mean inverse roles, nominals, qualified number restrictions, the universal role and local reflexivity of roles, respectively. Roles and concepts of the DLs $\mathcal{ALC}$, $\mathcal{ALC}{+}\Phi$, $(\mathcal{ALC}{+}\Phi)_{trans}$ and $(\mathcal{ALC}{+}\Phi)_{reg}$ are defined as follows.

If $\mathcal{L} = \mathcal{ALC}$, then:

- if $r \in \mathbf{R}_+$, then r is a role of $\mathcal{L}$,
- if $A \in \mathbf{C}$, then A is a concept of $\mathcal{L}$,
- $\top$ and $\bot$ are concepts of $\mathcal{L}$,
- if C and D are concepts of $\mathcal{L}$ and R is a role of $\mathcal{L}$, then $\neg C$, $C \sqcup D$, $C \sqcap D$, $\exists R.C$ and $\forall R.C$ are concepts of $\mathcal{L}$.

If $\mathcal{L} = \mathcal{ALC}{+}\Phi$, then additionally:

- if $\mathcal{I} \in \Phi$ and R is a role of $\mathcal{L}$, then $\overline{R}$ is a role of $\mathcal{L}$,
- if $\mathcal{O} \in \Phi$ and $a \in \mathbf{I}$, then $\{a\}$ is a concept of $\mathcal{L}$,
- if $\mathcal{Q} \in \Phi$, $n \in \mathbb{N}$, C is a concept of $\mathcal{L}$, R is a simple role of $\mathcal{L}$ (i.e., a simple role that is a role of $\mathcal{L}$), then $\geq n\, R.C$ and $\leq n\, R.C$ are concepts of $\mathcal{L}$,
- if $U \in \Phi$, then U is a role of $\mathcal{L}$,
- if $\mathcal{S}elf \in \Phi$ and $r \in \mathbf{R}_+$, then $\exists r.\mathcal{S}elf$ is a concept of $\mathcal{L}$.

If $\mathcal{L} = (\mathcal{ALC}{+}\Phi)_{trans}$, then additionally:

- ε is a role of $\mathcal{L}$,
- if R and S are roles of $\mathcal{L}$ and are different from U, then $R \sqcup S$, $R \circ S$ and R^* are roles of $\mathcal{L}$.

[1] I.e., with a PTime or lower data complexity.

$\bot^{\mathcal{I}} = \emptyset \quad \top^{\mathcal{I}} = \Delta^{\mathcal{I}} \quad \{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\} \quad (\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$	$\overline{R}^{\mathcal{I}} = (R^{\mathcal{I}})^{-1}$
$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}} \quad (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$	$\varepsilon^{\mathcal{I}} = \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$
$(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y\,(\langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}})\}$	$(R \circ S)^{\mathcal{I}} = R^{\mathcal{I}} \circ S^{\mathcal{I}}$
$(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y\,(\langle x, y\rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}})\}$	$(R \sqcup S)^{\mathcal{I}} = R^{\mathcal{I}} \cup S^{\mathcal{I}}$
$(\exists R.\mathcal{S}elf)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \langle x, x\rangle \in R^{\mathcal{I}}\}$	$(R^*)^{\mathcal{I}} = (R^{\mathcal{I}})^*$
$(\geq n\, R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \geq n\}$	$(C?)^{\mathcal{I}} = \{\langle x, x\rangle \mid x \in C^{\mathcal{I}}\}$
$(\leq n\, R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \leq n\}$	$U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

Fig. 1. Semantics of complex concepts and complex roles.

If $\mathcal{L} = (\mathcal{ALC}+\Phi)_{reg}$, then additionally:

- if C is a concept of $\mathcal{L}$, then $C?$ is a role of $\mathcal{L}$.
  This constructor is called the *test operator*.

When $\Phi = \emptyset$, we shorten the names $(\mathcal{ALC}+\Phi)_{trans}$ and $(\mathcal{ALC}+\Phi)_{reg}$ to $\mathcal{ALC}_{trans}$ and $\mathcal{ALC}_{reg}$, respectively. Similarly, we write $\mathcal{ALCIOQU}\mathcal{S}elf_{trans}$ to denote $(\mathcal{ALC}+\Phi)_{trans}$ with $\Phi = \{\mathcal{I}, \mathcal{O}, \mathcal{Q}, \mathcal{U}, \mathcal{S}elf\}$, and so on.

We denote atomic concepts by letters like A or B, atomic roles by letters like r or s, and individual names by letters like a or b. We use letters C and D to denote (arbitrary) concepts, R and S to denote (arbitrary) roles.

An *interpretation* is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set, called the *domain*, and $\cdot^{\mathcal{I}}$ is the *interpretation function* of $\mathcal{I}$ that maps each $a \in \mathbf{I}$ to $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, each $A \in \mathbf{C}$ to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, and each $r \in \mathbf{R}_+$ to a relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The function $\cdot^{\mathcal{I}}$ is extended to interpret complex roles and concepts as specified in Figure 1.

Concepts C and D are *equivalent*, denoted by $C \equiv D$, if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretations $\mathcal{I}$. Similarly, roles R and S are *equivalent*, denoted by $R \equiv S$, if $R^{\mathcal{I}} = S^{\mathcal{I}}$ for all interpretations $\mathcal{I}$.

If $\mathcal{L}$ is a sublogic of $\mathcal{L}'$ (like $(\mathcal{ALC}+\Phi)_{trans}$ is a sublogic of $(\mathcal{ALC}+\Phi)_{reg}$), then we say that $\mathcal{L}$ is *weaker* (or *less expressive*) than $\mathcal{L}'$ (in expressing concepts) if there exists a concept C of $\mathcal{L}'$ that is not equivalent to any concept of $\mathcal{L}$.

2.1 RBoxes

A finite set $\mathcal{S}$ of context-free production rules over $\mathbf{R}$ is called a *context-free semi-Thue system* over $\mathbf{R}$. It is *symmetric* if $\overline{R} \to \overline{S}_k \ldots \overline{S}_1$ belongs to $\mathcal{S}$ for every production rule $R \to S_1 \ldots S_k$ of $\mathcal{S}$.[2] It is *regular* if the language consisting of words derivable from any $R \in \mathbf{R}$ is regular. Assume that R is derivable from itself.

[2] If $k = 0$, then the RHS (right hand side) of each of the rules represents the empty word ε.

A *role inclusion axiom* (RIA) has the form $S_1 \circ \cdots \circ S_k \sqsubseteq R$, where $k \geq 0$ and $S_1, \ldots, S_k, R$ are basic roles. If $k = 0$, then the LHS (left hand side) of the inclusion stands for ε.

A *(regular) RBox* is a finite set $\mathcal{R}$ of RIAs such that $\mathcal{S} = \{R \to S_1 \ldots S_k \mid (S_1 \circ \cdots \circ S_k \sqsubseteq R) \in \mathcal{R}\}$ is a regular and symmetric semi-Thue system with the property that only ε and words with length 1 can be derived from any simple role $R \in \mathbf{R}$. An RBox is allowed for a DL $\mathcal{L}$ if it uses inverse roles only when they are allowed for $\mathcal{L}$. Since it is undecidable whether a context-free semi-Thue system is regular, we assume that each RBox $\mathcal{R}$ is accompanied by a mapping $\pi_{\mathcal{R}}$ that associates each $R \in \mathbf{R}$ with a regular expression $\pi_{\mathcal{R}}(R)$ that generates the set of words derivable from R using the rules of the corresponding semi-Thue system.

If $S_1 \circ \cdots \circ S_k \sqsubseteq R$ is a RIA of $\mathcal{R}$, then we call R an *intensional predicate* specified by $\mathcal{R}$. An interpretation $\mathcal{I}$ *validates* a RIA $S_1 \circ \cdots \circ S_k \sqsubseteq R$ if $(S_1 \circ \cdots \circ S_k)^{\mathcal{I}} \subseteq R^{\mathcal{I}}$. It is a *model* of an RBox $\mathcal{R}$ if it validates all RIAs of $\mathcal{R}$.

2.2 TBoxes

A *TBox axiom* (or *terminological axiom*) is either a general concept inclusion (GCI) $C \sqsubseteq D$ or a concept equivalence $C \doteq D$. A concept equivalence $A \doteq D$ (where $A \in \mathbf{C}$) is called a *concept definition.* A *TBox* is a finite set of TBox axioms. It is allowed for a DL $\mathcal{L}$ if it uses only concepts of $\mathcal{L}$. An interpretation $\mathcal{I}$ *validates* $C \sqsubseteq D$ (resp. $C \doteq D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ (resp. $C^{\mathcal{I}} = D^{\mathcal{I}}$). It is a *model* of a TBox $\mathcal{T}$ if it validates all axioms of $\mathcal{T}$.

A TBox $\mathcal{T}$ is *acyclic* if there exist concept names $A_1, \ldots, A_n$ such that $\mathcal{T}$ consists of n axioms and the i-th axiom of $\mathcal{T}$ is of the form $A_i \doteq C$, $C \sqsubseteq A_i$ or $A_i \sqsubseteq C$, where C does not use the concept names $A_i, \ldots, A_n$. The concept names $A_1, \ldots, A_n$ are called *intensional predicates* specified by $\mathcal{T}$.

A TBox $\mathcal{T}$ is called a *simple stratified TBox* if there exists a partition $(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ of $\mathcal{T}$, called a *stratification* of $\mathcal{T}$, such that, for each $1 \leq i \leq n$, $\mathcal{T}_i = \{C_{i,j} \sqsubseteq A_{i,j} \mid 1 \leq j \leq n_i\}$, where each $A_{i,j}$ is a concept name that does not occur in $\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}$ and may occur at the LHS of $\sqsubseteq$ in the axioms of $\mathcal{T}_i$ only under the scope of $\sqcap$, $\sqcup$ and $\exists$. The concept names $A_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq n_i$, are called *intensional predicates* specified by $\mathcal{T}$.

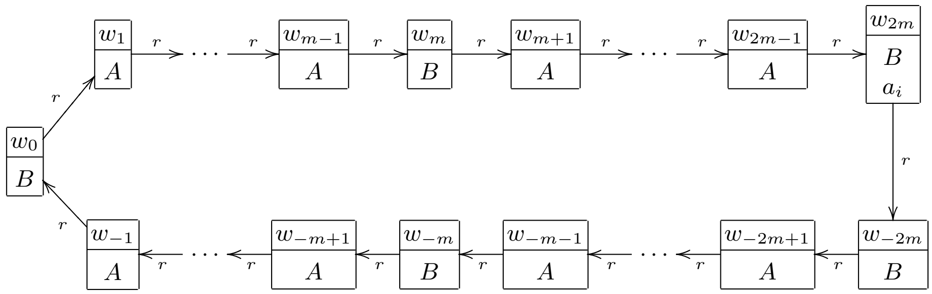
Note that negation ($\neg$) is allowed at the LHS of $\sqsubseteq$ in GCIs of a simple stratified TBox, but it can be applied only to concepts that do not use the predicates defined in the current or later strata.

3 The First Result

In this section, we prove the following theorem:

Theorem 1. *There is no concept of* $\mathcal{ALCIOQUS}elf_{trans}$ *equivalent to the concept* $C = \exists((r \circ A?)^* \circ r \circ B? \circ r \circ A?).\top$ *or* $C = \exists((r \circ A?)^* \circ r \circ (\neg A)? \circ r \circ A?).\top$ *of* $\mathcal{ALC}_{reg}$.

To prove this theorem we will use a family of interpretations $\mathcal{I}_m = \langle \Delta^{\mathcal{I}_m}, \cdot^{\mathcal{I}_m} \rangle$, $m > 1$, illustrated and specified as follows:



- $\Delta^{\mathcal{I}_m} = \{w_{-2m}, w_{-2m+1}, \ldots, w_{2m}\}$,
- $r^{\mathcal{I}_m} = \{\langle w_i, w_{i+1}\rangle, \langle w_{2m}, w_{-2m}\rangle \mid -2m \leq i < 2m\}$,
- $s^{\mathcal{I}_m} = \emptyset$ for $s \in \mathbf{R}_+ - \{r\}$,
- $B^{\mathcal{I}_m} = \{w_{-2m}, w_{-m}, w_0, w_m, w_{2m}\}$,
- $A^{\mathcal{I}_m} = \Delta^{\mathcal{I}_m} - B^{\mathcal{I}_m}$,
- $C^{\mathcal{I}_m} = \emptyset$ for $C \in \mathbf{C} - \{A, B\}$,
- $a^{\mathcal{I}_m} = w_{2m}$ for $a \in \mathbf{I}$.

Note that $|\Delta^{\mathcal{I}_m}| = 4m + 1$. Comparing $\mathcal{I}_m$ with the structure $\mathcal{A}_m$ used in [2], note that the domain of $\mathcal{A}_m$ has the size $2m+1$, $\mathcal{A}_m$ does not deal with nominals, and only one propositional variable is interpreted in $\mathcal{A}_m$ as a non-empty subset of the domain.

Observe that, for C being one of the two concepts mentioned in Theorem 1, $w_0 \in C^{\mathcal{I}_m}$ but $w_m \notin C^{\mathcal{I}_m}$. The structure of the proof of Theorem 1 is as follows. Given any concept D of $\mathcal{ALCIOQUS}elf_{trans}$, we first transform it to a concept D_2 of $\mathcal{ALCIO}_{trans}$ over the signature $\{r, A, B, a\}$ such that $D_2^{\mathcal{I}_m} = D^{\mathcal{I}_m}$ for all $m > 1$ (see Lemma 1). We then transform D_2 to a concept D_3 such that $D_3^{\mathcal{I}_m} = D_2^{\mathcal{I}_m}$ for all $m > 1$, the $*$ operator is used only for r^n and $\overline{r}^n$ for some n (see Lemma 2), and for every subconcept $\exists R.D_3'$ or $\forall R.D_3'$ of D_3, R is of the form r, $\overline{r}$, $(r^n)^*$ or $(\overline{r}^n)^*$ for some $n \geq 1$ (see Lemma 3). Next, we show that there exists $m > 1$ such that $w_0 \in D_3^{\mathcal{I}_m} \Leftrightarrow w_m \in D_3^{\mathcal{I}_m}$ (see Lemma 4). Thus, for that m, $C^{\mathcal{I}_m} \neq D_3^{\mathcal{I}_m}$, and therefore, C is not equivalent to D (since $D_3^{\mathcal{I}_m} = D_2^{\mathcal{I}_m} = D^{\mathcal{I}_m}$).

Lemma 1. *For any concept C of $\mathcal{ALCIOQUS}elf_{trans}$, there exists a concept D of $\mathcal{ALCIO}_{trans}$ over the signature $\{r, A, B, a\}$ such that $D^{\mathcal{I}_m} = C^{\mathcal{I}_m}$ for all $m > 1$.*

Proof. Let D be the concept obtained from C by:

- replacing every subconcept

- $\geq n\, R.E$, where $n \geq 2$, by $\bot$,
- $\geq 1\, R.E$ by $\exists R.E$,
- $\geq 0\, R.E$ by $\top$,
- $\leq n\, R.E$, where $n \geq 1$, by $\top$,
- $\leq 0\, R.E$ by $\forall R.\neg E$,
- $\exists U.E$ by $\exists r^*.E$,
- $\forall U.E$ by $\forall r^*.E$,
- $\exists R.\mathit{Self}$ by $\bot$,

– replacing every concept name different from A and B by $\bot$,
– replacing every nominal $\{b\}$, where $b \neq a$, by $\{a\}$,
– replacing every role name s different from r by $\emptyset$,
– repeatedly replacing every role $\emptyset \sqcup R$ or $R \sqcup \emptyset$ by R, every role $\emptyset^*$ by ε, and every role $\overline{\emptyset}$, $\emptyset \circ R$ or $R \circ \emptyset$ by $\emptyset$,
– replacing every subconcept $\exists\emptyset.E$ by $\bot$, and every $\forall\emptyset.E$ by $\top$.

It is easy to see that D satisfies the properties mentioned in the lemma. $\square$

We treat a word $R_1 \ldots R_k$ over the alphabet $\{r, \overline{r}\}$ as the role $R_1 \circ \cdots \circ R_k$, and by R^n we denote the composition of n copies of R. Thus, $R^0 = \varepsilon$. Conversely, a role R without tests that uses only basic roles r and $\overline{r}$ is treated as a regular expression over the alphabet $\{r, \overline{r}\}$ (where $\sqcup$ stands for $\cup$, and $\circ$ for ;). For such a role R, by $\mathcal{L}(R)$ we denote the regular language generated by R. For a word R over the alphabet $\{r, \overline{r}\}$, by $|R|$ we denote the *length* of R (defined in the usual way), and by $||R||$ we denote the *norm* of R, which is defined as follows: $||\varepsilon|| = 0$, $||r|| = 1$, $||\overline{r}|| = -1$, $||RS|| = ||R|| + ||S||$. Observe that, for words R and S over the alphabet $\{r, \overline{r}\}$, if $||R|| = ||S||$, then $R^{\mathcal{I}_m} = S^{\mathcal{I}_m}$ for all $m > 1$.

Lemma 2. *Let R be a role without tests that uses only basic roles r and $\overline{r}$. Then, there exists a role S such that $S^{\mathcal{I}_m} = R^{\mathcal{I}_m}$ for all $m > 1$ and the $*$ operator can be used in S only for r^n and $\overline{r}^n$ for some n.*

Proof. Since $\mathcal{L}(R)$ is a regular language, by the pumping lemma, there exists an integer $p > 0$ such that every word from $\mathcal{L}(R)$ of length at least p can be represented as xyz such that $|y| > 0$, $|xy| \leq p$ and $xy^iz \in \mathcal{L}(R)$ for all $i \geq 0$.

Let $n = p(p-1)\cdots 2 \cdot 1$ and let $\mathcal{L}'$ be the language obtained from $\mathcal{L}(R)$ by deleting all words y such that there exists $x \in \mathcal{L}(R)$ with $|x| < |y|$ and $||x|| = ||y||$. By *pumping*(x, y, z) we denote the formula

$$xyz \in \mathcal{L}' \wedge |y| > 0 \wedge |xy| \leq p \wedge \forall i \geq 0\ xy^iz \in \mathcal{L}(R).$$

Observe that, if $w' = xyz \in \mathcal{L}'$ and *pumping*(x, y, z) holds, then:

– $||y|| \neq 0$ because otherwise we would have $xz \in \mathcal{L}(R)$, $|xz| < |w'|$ and $||xz|| = ||w'||$, which contradict the definition of $\mathcal{L}'$;
– if $||y|| > 0$ then, for all $i \geq 0$, there exists $u \in \mathcal{L}(R)$ with $||u|| = ||w'(r^n)^i||$;
– if $||y|| < 0$ then, for all $i \geq 0$, there exists $u \in \mathcal{L}(R)$ with $||u|| = ||w'(\overline{r}^n)^i||$.

Denote this observation by $(\star)$. For each integer j, $0 \leq j < n$, let

$$K_j^+ = \{||xyz|| : \mathit{pumping}(x, y, z),\ n \mid (||xyz|| - j) \text{ and } ||y|| > 0\}$$
$$K_j^- = \{||xyz|| : \mathit{pumping}(x, y, z),\ n \mid (||xyz|| - j) \text{ and } ||y|| < 0\}.$$

For intuition, informally, we intend to define S to be the role

$$\bigsqcup \mathcal{S}_1 \sqcup ((\bigsqcup \mathcal{S}_2) \circ (r^n)^*) \sqcup ((\bigsqcup \mathcal{S}_3) \circ (\overline{r}^n)^*), \tag{1}$$

where $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are the finite sets of words over the alphabet $\{r, \overline{r}\}$ constructed as follows:

- $\mathcal{S}_1 := \{x \in \mathcal{L}' : |x| < p\}$, $\mathcal{S}_2 := \emptyset$, $\mathcal{S}_3 := \emptyset$;
- for each j from 0 to $n-1$ do
  - if $K_j^+ \neq \emptyset$ then
    - if K_j^+ does not have a minimum then add r^j to both $\mathcal{S}_2$ and $\mathcal{S}_3$;
    - else: let $k = \min K_j^+$, if $k \geq 0$ then $\mathcal{S}_2 := \mathcal{S}_2 \cup \{r^k\}$ else $\mathcal{S}_2 := \mathcal{S}_2 \cup \{(\overline{r})^{-k}\}$ (for this second case, notice that $-k$ is a positive integer);
  - if $K_j^- \neq \emptyset$ then
    - if K_j^- does not have a maximum then add r^j to both $\mathcal{S}_2$ and $\mathcal{S}_3$;
    - else: let $k = \max K_j^-$, if $k \geq 0$ then $\mathcal{S}_3 := \mathcal{S}_3 \cup \{r^k\}$ else $\mathcal{S}_3 := \mathcal{S}_3 \cup \{(\overline{r})^{-k}\}$.

Formally, we define S to be the role obtained from (1) by deleting any i-th main disjunct such that $\mathcal{S}_i$ is empty, for $i \in \{1,2,3\}$. To prove that $S^{\mathcal{I}_m} = R^{\mathcal{I}_m}$ for all $m > 1$ it is sufficient to show that:

1. if $w \in \mathcal{L}(S)$, then there exists $u \in \mathcal{L}(R)$ such that $||u|| = ||w||$,
2. if $w \in \mathcal{L}(R)$, then there exists $u \in \mathcal{L}(S)$ such that $||u|| = ||w||$.

Consider the assertion (1) and let $w \in \mathcal{L}(S)$. There are the following cases:

- Case $w \in \mathcal{S}_1$: We have that $w \in \mathcal{L}' \subseteq \mathcal{L}(R)$. Just take $u = w$.
- Case $w = r^j(r^n)^h$, $K_j^+ \neq \emptyset$ and K_j^+ does not have a minimum: Thus, there exists $w' = xyz \in \mathcal{L}'$ such that *pumping*(x,y,z) holds, $||y|| > 0$ and $||w'|| = j + n \cdot h'$ for some $h' < h$. By $(\star)$, there exists $u \in \mathcal{L}(R)$ such that $||u|| = ||w||$.
- Case $w = r^k(r^n)^h$, $K_j^+ \neq \emptyset$, $k = \min K_j^+$ and $k \geq 0$: Thus, there exists $w' = xyz \in \mathcal{L}'$ such that *pumping*(x,y,z) holds, $||y|| > 0$ and $||w'|| = k$. By $(\star)$, there exists $u \in \mathcal{L}(R)$ such that $||u|| = ||w||$.
- Case $w = (\overline{r})^{-k}(r^n)^h$, $K_j^+ \neq \emptyset$, $k = \min K_j^+$ and $k < 0$: Thus, there exists $xyz \in \mathcal{L}'$ such that *pumping*(x,y,z) holds, $||y|| > 0$ and $||w'|| = k$. Notice that $||(\overline{r})^{-k}|| = k$. By $(\star)$, there exists $u \in \mathcal{L}(R)$ such that $||u|| = ||w||$.
- Case $w = r^j(r^n)^h$, $K_j^- \neq \emptyset$ and K_j^- does not have a maximum: Thus, there exists $w' = xyz \in \mathcal{L}'$ such that *pumping*(x,y,z) holds, $||y|| < 0$ and $||w'|| = j + n \cdot h'$ for some $h' > h$. By $(\star)$, there exists $u \in \mathcal{L}(R)$ such that $||u|| = ||w||$.
- The four previous cases are related to $\mathcal{S}_2$. The four remaining cases, which are related to $\mathcal{S}_3$, can be dealt with in a similar way.

Consider the assertion (2) and let $w \in \mathcal{L}(R)$. There exists $w' \in \mathcal{L}'$ such that $||w'|| = ||w||$. If $|w'| < p$, then $w' \in \mathcal{S}_1$ and we can just take $u = w'$. Suppose $|w'| \geq p$. Thus, w' can be represented as xyz such that *pumping*(x,y,z) holds. There are the following cases:

– Case $||y|| > 0$: There exists $0 \leq j < n$ such that $||w'|| \in K_j^+$ and $||w'|| = j + n \cdot i$ for some integer i. Consider the following subcases.
  - Case K_j^+ does not have a minimum: Thus, $r^j \in \mathcal{S}_2$. Taking $u = r^j(r^n)^i$, we have that $u \in \mathcal{L}(S)$ and $||u|| = ||w'|| = ||w||$.
  - Case $k = \min K_j^+$ and $k \geq 0$: Thus, $r^k \in \mathcal{S}_2$. Observe that $||w'|| \geq k$ and $n \,|\, (||w'|| - k)$. Taking $u = r^{||w'||}$, we have that $u \in \mathcal{L}(S)$ and $||u|| = ||w'|| = ||w||$.
  - Case $k = \min K_j^+$ and $k < 0$: Thus, $(\overline{r})^{-k} \in \mathcal{S}_2$. Observe that $||w'|| \geq k$ and $n \,|\, (||w'|| - k)$. Taking $u = (\overline{r})^{-k}(r^{||w'||-k})$, we have that $u \in \mathcal{L}(S)$ and $||u|| = ||w'|| = ||w||$.

– The case when $||y|| < 0$ is dual to the above case and can be dealt with analogously. □

Let $\mathfrak{C}$ denote the set of concepts C of $\mathcal{ALCIO}_{trans}$ over the signature $\{r, A, B, a\}$ such that, for every subconcept $\exists R.D$ or $\forall R.D$ of C, R is of the form r, $\overline{r}$, $(r^n)^*$ or $(\overline{r}^n)^*$ for some $n \geq 1$.

Lemma 3. *For any concept C of $\mathcal{ALCIO}_{trans}$ over the signature $\{r, A, B, a\}$, there exists a concept $D \in \mathfrak{C}$ such that $D^{\mathcal{I}_m} = C^{\mathcal{I}_m}$ for all $m > 1$.*

Proof. Let E be the concept obtained from C by replacing every role R by a role S that satisfies the conditions mentioned in Lemma 2. We have $E^{\mathcal{I}_m} = C^{\mathcal{I}_m}$ for all $m > 1$. Then, let D be obtained from E by repeatedly applying the following transformations:

$$\begin{array}{rlrl} \exists(R \sqcup S).F \equiv \exists R.F \sqcup \exists R.F & & \forall(R \sqcup S).F \equiv \forall R.F \sqcap \forall R.F \\ \exists(R \circ S).F \equiv \exists R.\exists S.F & & \forall(R \circ S).F \equiv \forall R.\forall S.F \\ \exists\varepsilon.F \equiv F & & \forall\varepsilon.F \equiv F. \end{array}$$

It is clear that $D \in \mathfrak{C}$ and $D^{\mathcal{I}_m} = E^{\mathcal{I}_m} = C^{\mathcal{I}_m}$ for all $m > 1$. □

For a concept $C \in \mathfrak{C}$, by $n_r(C)$ we denote the number of occurrences of $\exists r$, $\exists\overline{r}$, $\forall r$ and $\forall\overline{r}$ in C.

Lemma 4. *For any concept $C \in \mathfrak{C}$ and integers m and k such that $m > 1$, $4m + 1$ is prime and $n_r(C) < m - |k|$, we have $w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}$.*

Proof. This proof is similar to the one of [2, Lemma 3]. The intuition is as follows:

– a concept C' can distinguish w_k and w_{k+m} only if $n_r(C')$ is large enough so that the checking can recognize that the neighborhood of w_k differs from the corresponding neighborhood of w_{k+m}, in particular, to recognize that the first one contains w_{m+1} (resp. w_{-m-1}) and the second one contains w_{-2m} (resp. w_{2m}); the reason is that, since $4m + 1$ is prime, either $((r^n)^*)^{\mathcal{I}_m} = \Delta^{\mathcal{I}_m} \times \Delta^{\mathcal{I}_m}$ or $\langle w_i, w_j \rangle \in ((r^n)^*)^{\mathcal{I}_m}$ iff $j = i$;

– since $n_r(C) < m - |k|$, C cannot distinguish w_k and w_{k+m}.

Observe that $-m < k < m$. We prove this lemma by induction on the structure of C. The cases when C is A, B, $\top$ or $\bot$ are trivial. The cases when C is of the form $D \sqcap E$ or $\forall R.D$ are reduced to the cases of $\neg(\neg D \sqcup \neg E)$ and $\neg\exists R.\neg D$, respectively.

- Case $C = \{a\}$: Since $a^{\mathcal{I}_m} = w_{2m}$, $w_k \notin C^{\mathcal{I}_m}$ and $w_{k+m} \notin C^{\mathcal{I}_m}$.
- Case $C = \neg D$: We have $n_r(D) = n_r(C)$. By induction, $w_k \in D^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in D^{\mathcal{I}_m}$, and hence, $w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}$.
- Case $C = D \sqcup E$: We have $n_r(D) \leq n_r(C)$ and $n_r(E) \leq n_r(C)$. By induction, $w_k \in D^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in D^{\mathcal{I}_m}$ and $w_k \in E^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in E^{\mathcal{I}_m}$, which imply that $w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}$.
- Case $C = \exists r.D$: We have $n_r(D) = n_r(C) - 1 < m - |k| - 1 \leq m - |k+1|$. By induction, $w_{k+1} \in D^{\mathcal{I}_m} \Leftrightarrow w_{k+1+m} \in D^{\mathcal{I}_m}$. Hence, $w_k \in (\exists r.D)^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in (\exists r.D)^{\mathcal{I}_m}$, which means $w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}$.
- Case $C = \exists \overline{r}.D$: We have $n_r(D) = n_r(C) - 1 < m - |k| - 1 \leq m - |k-1|$. By induction, $w_{k-1} \in D^{\mathcal{I}_m} \Leftrightarrow w_{k-1+m} \in D^{\mathcal{I}_m}$. Similarly to the previous case, this implies that $w_k \in (\exists \overline{r}.D)^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in (\exists \overline{r}.D)^{\mathcal{I}_m}$, which means $w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}$.
- Case $C = \exists (r^n)^*.D$ and $(4m+1)|n$: We have $\langle w_i, w_j \rangle \in ((r^n)^*)^{\mathcal{I}_m}$ iff $j = i$. Hence,
$$w_k \in (\exists (r^n)^*.D)^{\mathcal{I}_m} \Leftrightarrow w_k \in D^{\mathcal{I}_m}$$
$$w_{k+m} \in (\exists (r^n)^*.D)^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in D^{\mathcal{I}_m}.$$
We have $n_r(D) = n_r(C)$. By induction, $w_k \in D^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in D^{\mathcal{I}_m}$. Therefore, $w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}$.
- Case $C = \exists (r^n)^*.D$ and $(4m+1) \not| \, n$: Since $4m+1$ is prime, $0, n, 2n, 3n, \ldots, (4m)n$ have all $4m+1$ different residues modulo $4m+1$. Hence, $\langle w_i, w_j \rangle \in ((r^n)^*)^{\mathcal{I}_m}$ for all $w_i, w_j \in \Delta^{\mathcal{I}_m}$, and
$$w_k \in (\exists (r^n)^*.D)^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in (\exists (r^n)^*.D)^{\mathcal{I}_m},$$
because they are both equivalent to that there exists $w_j \in D^{\mathcal{I}_m}$. Therefore,
$$w_k \in C^{\mathcal{I}_m} \Leftrightarrow w_{k+m} \in C^{\mathcal{I}_m}.$$
- The case $C = \exists (\overline{r}^n)^*.D$ is similar to the two previous cases. □

We now recall and prove Theorem 1.

Theorem 1 *There is no concept of $\mathcal{ALCIOQUS}elf_{trans}$ equivalent to the concept $C = \exists((r \circ A?)^* \circ r \circ B? \circ r \circ A?).\top$ or $C = \exists((r \circ A?)^* \circ r \circ (\neg A)? \circ r \circ A?).\top$ of $\mathcal{ALC}_{reg}$.*

Proof. For a contradiction, suppose D is a concept of $\mathcal{ALCIOQUS}elf_{trans}$ equivalent to C. By Lemma 1, there exists a concept D_2 of $\mathcal{ALCIO}_{trans}$ over the signature $\{r, A, B, a\}$ such that $D_2^{\mathcal{I}_m} = D^{\mathcal{I}_m}$ for all $m > 1$. By Lemma 3, there exists a concept $D_3 \in \mathfrak{C}$ such that $D_3^{\mathcal{I}_m} = D_2^{\mathcal{I}_m}$ for all $m > 1$. Let m be an integer such that $m > n_r(D_3)$ and $4m+1$ is prime. By Lemma 4, $w_0 \in D_3^{\mathcal{I}_m} \Leftrightarrow w_m \in D_3^{\mathcal{I}_m}$. This contradicts the facts that $D_3^{\mathcal{I}_m} = D_2^{\mathcal{I}_m} = D^{\mathcal{I}_m} = C^{\mathcal{I}_m}$, $w_0 \in C^{\mathcal{I}_m}$ and $w_m \notin C^{\mathcal{I}_m}$. □

Corollary 1. *For any* $\Phi \subseteq \{\mathcal{I}, \mathcal{O}, \mathcal{Q}, \mathcal{U}, \mathcal{S}elf\}$*,* $(\mathcal{ALC}+\Phi)_{trans}$ *is weaker than* $(\mathcal{ALC}+\Phi)_{reg}$ *in expressing concepts.*

4 Dealing with RBoxes and TBoxes

The result of the previous section roughly states that, without using RBoxes and TBoxes, it is hard to eliminate tests, at least it is impossible to eliminate tests from $\mathcal{ALCIOQUS}elf_{reg}$ without decreasing the expressive power. As expected, using acyclic TBoxes that consist only of concept definitions do not help in expressing tests. The first result of this section states that using RBoxes and acyclic TBoxes that are defined more liberally as in Section 2 does not help either. The second result states that, however, using simple stratified TBoxes under the stratified semantics on the background, it is possible to express every concept by another without tests. The third result states that tests can be eliminated from the deterministic Horn fragment of $\mathcal{ALC}_{reg}$. Due to the lack of space, proofs of these results are provided only in the long version [11] of the current paper.

4.1 The Case with RBoxes and Acyclic TBoxes

We say that a concept C is *inexpressible in a DL* $\mathcal{L}$ *even when using RBoxes and acyclic TBoxes* if, for every concept D, every RBox $\mathcal{R}$ and every acyclic TBox $\mathcal{T}$ of $\mathcal{L}$ such that the intensional predicates specified by $\mathcal{R}$ and $\mathcal{T}$ do not occur in C, there exists a model $\mathcal{I}$ of $\mathcal{R}$ and $\mathcal{T}$ such that $C^{\mathcal{I}} \neq D^{\mathcal{I}}$.

Proposition 1. *The concept* $C = \exists((r \circ A?)^* \circ r \circ B? \circ r \circ A?).\top$ *or* $C = \exists((r \circ A?)^* \circ r \circ (\neg A)? \circ r \circ A?).\top$ *of* $\mathcal{ALC}_{reg}$ *is inexpressible in* $\mathcal{ALCIOQUS}elf_{trans}$ *even when using RBoxes and acyclic TBoxes.*

4.2 Eliminating Tests from Concepts by Simple Stratified TBoxes

Let $\mathcal{T}$ be a simple stratified TBox. An interpretation $\mathcal{I}$ is called a *standard model* of $\mathcal{T}$ (under the stratified semantics) if there exist a partition $(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ of $\mathcal{T}$ and interpretations $\mathcal{J}_0, \ldots, \mathcal{J}_n$ such that:

- $\mathcal{T}_i = \{C_{i,j} \sqsubseteq A_{i,j} \mid 1 \leq j \leq n_i\}$ for $1 \leq i \leq n$,
- $\mathcal{J}_n = \mathcal{I}$ and $\Delta^{\mathcal{J}_i} = \Delta^{\mathcal{I}}$ for all $0 \leq i < n$,
- $x^{\mathcal{J}_0} = x^{\mathcal{I}}$ for all $x \in \Sigma - \{A_{i,j} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i\}$,
- for each $1 \leq i \leq n$, $x^{\mathcal{J}_i} = x^{\mathcal{J}_{i-1}}$ for all $x \in \Sigma - \{A_{i',j} \mid i \leq i' \leq n, 1 \leq j \leq n_{i'}\}$ and $A_{i,j}^{\mathcal{J}_i}$, for $1 \leq j \leq n_i$, are the smallest subsets of $\Delta^{\mathcal{J}_i}$ such that $A_{i,j}^{\mathcal{J}_i} = C_{i,j}^{\mathcal{J}_i}$.

It can be shown that, for every interpretation $\mathcal{J}_0$, there exists a unique standard model $\mathcal{I}$ of $\mathcal{T}$ such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}_0}$ and $x^{\mathcal{I}} = x^{\mathcal{J}_0}$ for all $x \in \Sigma - \{A_{i,j} \mid 1 \leq i \leq n$ and $1 \leq j \leq n_i\}$. We call it the *standard model of* $\mathcal{T}$ *based on* $\mathcal{J}_0$.

In what follows, let $\Phi \subseteq \{\mathcal{I}, \mathcal{O}, \mathcal{Q}, \mathcal{U}, \mathcal{S}elf\}$ (in general, extending Φ with other features does not affect Proposition 2 given below). Let C be a concept of $(\mathcal{ALC}+\Phi)_{reg}$, D a concept and $\mathcal{T}$ a simple stratified TBox of $(\mathcal{ALC}+\Phi)_{trans}$ such that the intensional predicates specified by $\mathcal{T}$ do not occur in C. We say that C is *expressed by D and $\mathcal{T}$ under the stratified semantics* if, for every standard model $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} = D^{\mathcal{I}}$.

Proposition 2. *Every concept of $(\mathcal{ALC}+\Phi)_{reg}$ can be expressed by a concept and a simple stratified TBox of $(\mathcal{ALC}+\Phi)_{trans}$ under the stratified semantics.*

4.3 Eliminating Tests from Horn TBoxes

The previous subsection deals with eliminating tests from a standing alone concept by using a simple stratified TBox under the stratified semantics. Roughly speaking, it suggests that tests in PDL-like roles can be eliminated by using fixpoints outside roles. The result of this subsection states that tests can be eliminated from TBoxes of the deterministic Horn fragment of $\mathcal{ALC}_{reg}$. This is possible because the traditional semantics of such TBoxes has a fixpoint characterization.

A role can be treated as a regular expression over the alphabet $\mathbf{R}_{+} \cup \{C? \mid C$ is a concept$\}$, where $\sqcup$ and $\circ$ stand for $\cup$ and semicolon, respectively. Conversely, a word over this alphabet can be treated as a role. Given a role R, let $\mathcal{L}(R)$ denote the regular language generated by R and let $\forall\exists R.C$ be a new concept constructor whose semantics in an interpretation $\mathcal{I}$ is specified as follows:

$$(\forall\exists R.C)^{\mathcal{I}} = \bigcap\{(\forall S.\exists S'.C)^{\mathcal{I}} \mid SS' \in \mathcal{L}(R)\}.$$

Observe that, if $R \in \mathbf{R}_{+}$, then $\forall\exists R.C \equiv \forall R.C \sqcap \exists R.C$.

The deterministic Horn fragment of $\mathcal{ALC}_{reg}$, denoted by D-Horn-$\mathcal{ALC}_{reg}$, is designed with the intention to be (probably) the most expressive fragment of $\mathcal{ALC}_{reg}$ that has a PTime data complexity (under the traditional semantics).

A *D-Horn-$\mathcal{ALC}_{reg}$ TBox axiom* is an expression of the form $C_l \sqsubseteq C_r$, where C_l and C_r are concepts defined by the following BNF grammar, with $A \in \mathbf{C}$ and $s \in \mathbf{R}_{+}$:

$$C_l ::= \top \mid A \mid C_l \sqcap C_l \mid C_l \sqcup C_l \mid \exists R_l.C_l \mid \forall\exists R_l.C_l \tag{2}$$
$$R_l ::= s \mid R_l \circ R_l \mid R_l \sqcup R_l \mid R_l^* \mid C_l? \tag{3}$$
$$C_r ::= \top \mid \bot \mid A \mid \neg C_l \mid C_r \sqcap C_r \mid \neg C_l \sqcup C_r \mid \exists R_r.C_r \mid \forall R_l.C_r \tag{4}$$
$$R_r ::= s \mid R_r \circ R_r \mid C_r? \tag{5}$$

A *D-Horn-$\mathcal{ALC}_{reg}$ TBox* is a finite set of D-Horn-$\mathcal{ALC}_{reg}$ TBox axioms.

Remark 1. A (reduced) *ABox* is a finite set of *assertions* of the form $A(a)$, $\neg A(a)$ or $r(a, b)$ (where $A \in \mathbf{C}$ and $r \in \mathbf{R}_{+}$). A *knowledge base* in D-Horn-$\mathcal{ALC}_{reg}$ is a pair $\langle\mathcal{T}, \mathcal{A}\rangle$ consisting of a D-Horn-$\mathcal{ALC}_{reg}$ TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. The notion of whether an interpretation is a *model* of an ABox or a knowledge base is defined in the usual way. A knowledge base is *satisfiable* if it has a model. It can be proved that checking whether a given knowledge base $\langle\mathcal{T}, \mathcal{A}\rangle$ in D-Horn-$\mathcal{ALC}_{reg}$

is satisfiable is solvable in polynomial time in the size of the ABox $\mathcal{A}$.[3] That is, D-Horn-$\mathcal{ALC}_{reg}$ has a PTime data complexity. If $\forall\exists$ in (2) is replaced by $\forall$, then instead of D-Horn-$\mathcal{ALC}_{reg}$ we obtain the general Horn fragment of $\mathcal{ALC}_{reg}$ with a NP-hard data complexity for the satisfiability problem.[4] □

The following theorem states that tests can be eliminated from D-Horn-$\mathcal{ALC}_{reg}$.

Theorem 2. *For every D-Horn-$\mathcal{ALC}_{reg}$ TBox $\mathcal{T}$ over a signature Σ, there exists a D-Horn-$\mathcal{ALC}_{reg}$ TBox $\mathcal{T}'$ without tests over a signature $\Sigma' \supseteq \Sigma$ such that:*

1. *for every model $\mathcal{I}$ of $\mathcal{T}$, there exists a model $\mathcal{I}'$ of $\mathcal{T}'$ such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$ and $x^{\mathcal{I}} = x^{\mathcal{I}'}$ for all $x \in \Sigma$,*
2. *for every model $\mathcal{I}'$ of $\mathcal{T}'$, the interpretation $\mathcal{I}$ over Σ specified by $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$ and $x^{\mathcal{I}} = x^{\mathcal{I}'}$ for all $x \in \Sigma$ is a model of $\mathcal{T}$.*

5 Conclusions

Generalizing the result and method of Berman and Paterson [2], we have proved that there is a concept of $\mathcal{ALC}_{reg}$ that is not equivalent to any concept of the DL that extends $\mathcal{ALC}_{trans}$ with inverse roles, nominals, qualified number restrictions, the universal role and local reflexivity of roles. This implies, among others, that CPDL (Converse-PDL) is more expressive than Test-Free CPDL, and GCPDL (Graded Converse-PDL) is more expressive than Test-Free GCPDL. Extending our result by applying the technique of [2], it can also be proved that CPDL_{n+1} (CPDL with at most $n+1$ levels of nesting of tests) is more expressive than CPDL_n, and similarly for GCPDL.

The other results of this paper state that, on the other hand, using simple stratified TBoxes under the stratified semantics on the background, it is possible to express every concept by another without tests. Furthermore, tests can be eliminated from the deterministic Horn fragment D-Horn-$\mathcal{ALC}_{reg}$ of $\mathcal{ALC}_{reg}$. If one extends D-Horn-$\mathcal{ALC}_{reg}$ with other features (e.g., $\mathcal{I}$, $\mathcal{O}$, $\mathcal{Q}$, $\mathcal{U}$ and $\mathcal{S}elf$) appropriately so that the resulting language still has a PTime data complexity (cf. Horn-$\mathcal{SHIQ}$ [9], Horn-$\mathcal{SROIQ}$ [15] and Horn-DL [14]), then our elimination technique (presented in the long version [11] of the current paper) can still be applied. Besides, it is hard to define a fragment of $\mathcal{ALC}_{reg}$ that is more expressive than D-Horn-$\mathcal{ALC}_{reg}$ and still has a PTime data complexity under the traditional semantics. So, we have a tendency to claim that tests can be eliminated from tractable Horn fragments of PDL-like logics.

[3] A more general result was proved in [13] for D-Horn-CPDL_{reg}, which extends D-Horn-$\mathcal{ALC}_{reg}$ with inverse roles and regular RBoxes.

[4] The hardness was shown for the general Horn fragment of $\mathcal{ALC}$ [12].

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