

# Data Retrieval and Noise Reduction by Fuzzy Associative Memories

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**Abstract.** A novel theoretical background of fuzzy associative memories (FAM) is proposed. A framework of formal concept analysis is used for a new working theory of FAM. Two principal activities of FAM are formalized : data retrieval and noise reduction. It is shown that the problem of data retrieval is connected with solvability and eigen sets of a certain system of fuzzy relation equations. The differentiation of FAM models according to their ability to reduce noise is defined. It is shown how the choice of formal context determines a type of noise that can be reduced by the corresponding retrieval mechanism. Finally, we propose a fast algorithm of data retrieval.

## 1 Introduction

One of the first publications devoted to *fuzzy associative memories* (FAM) has been made by Kosko - [6]. The FAM has been characterized as a single-layer feedforward neural net performing a nonlinear matrix-vector multiplication. This approach was later extended with the purpose to increase the storage capacity (e.g. [4]). Significant progress was achieved by introduction of learning implication rules [3, 5], that afterwards led to *implicative fuzzy associative memory* (IFAM) with *implicative fuzzy learning*. A justification of validity of a certain IFAM model was discussed in [14] where the characterization of one type of suppressed noise - eroded - was proposed.

In the current contribution, we use the framework of formal concept analysis [17] and propose a working theory of FAM. Let us remark that the language and technique of formal concept analysis is used in other theories, e.g., fuzzy property-oriented concept lattices, as well as in applications such as modeling and processing of incomplete knowledge in information systems [1, 2].

We formalize two principal activities of FAM: data retrieval and noise reduction. We use the proposed formalism and show that the problem of data retrieval is connected with solvability and eigen sets of a certain system of fuzzy relation equations. We differentiate FAM models according to their ability to

reduce noise and show how the choice of formal context determines a type of noise that can be reduced by the corresponding retrieval mechanism.

From the technical point of view, we extend the theory of FAM by using a general algebraic structure instead of a specific one (Łukasiewicz algebra in [14]) and by enlarging the set of autoregressive fuzzy associative memory models (AFAM). We propose a formal characterization of AFAM models and show the way of various modifications. We analyze the retrieval mechanism of AFAM and its ability to remove noise. We show that the larger is the amount of noise, the greater should be the fuzzy relation that models retrieval with noise reduction. Further, we show how the type of removable noise depends on which type of AFAM models is applied. Finally, we construct a fast algorithm of data retrieval and give illustration of the noise reduction.

## 2 Preliminaries

### 2.1 Implicative fuzzy associative memory

In this Section, we discuss underlying assumptions related to autoregressive fuzzy associative memories (AFAM) using denotation in [14]. We formalize the retrieval mechanism and the problem of noise reduction.

Let us propose the following formalization of AFAM. A database

$$D = \{\mathbf{x}^1, \dots, \mathbf{x}^p\},$$

of objects (images, patterns, signals, texts, etc.) is represented by normal fuzzy sets such that every  $\mathbf{x}^k$ ,  $k = 1, \dots, p$ , is a map  $\mathbf{x}^k : X \rightarrow [0, 1]$ , where  $X = \{u_1, \dots, u_n\}$  is a universe. The problem is to find a model of  $D$  together with a retrieval mechanism such that every element  $\mathbf{x}^k \in D$  can be successfully retrieved even from its noisy version.

According to [14], a *model of AFAM for database  $D$*  can be identified with the 3-tuple  $(W, \theta, \square)$ , consisting of a fuzzy relation  $W : X \times X \rightarrow [0, 1]$ , a bias vector  $\theta \in [0, 1]^n$ , and a set-relation composition  $\square : [0, 1]^{n \times n} \times [0, 1]^n \times [0, 1]^n \rightarrow [0, 1]^n$ , such that for all  $\mathbf{x}^k \in D$ ,

$$\mathbf{x}^k = W \square \mathbf{x}^k \vee \theta, k = 1, \dots, p. \quad (1)$$

We say that (1) represents a *retrieval mechanism in AFAM* and that this mechanism *reduces noise*, if

$$\mathbf{x}^k = W \square \tilde{\mathbf{x}}^k \vee \theta, k = 1, \dots, p, \quad (2)$$

where  $\tilde{\mathbf{x}}^k$  is a noisy version of  $\mathbf{x}^k$ .

In [14], an implicative model of AFAM has been proposed. The model uses Łukasiewicz algebra of operations on  $[0, 1]$ , so that fuzzy relation  $W$  is expressed in the implicative form

$$W(u_i, u_j) = \bigwedge_{k=1}^p (\mathbf{x}^k(u_i) \rightarrow \mathbf{x}^k(u_j)), \quad (3)$$

and other constituents are as follows:

$$\theta_i = \bigwedge_{k=1}^p \mathbf{x}^k(u_i), i = 1, \dots, n,$$

$\square$  is the sup  $-\otimes$ -composition.

For a given input  $\mathbf{x}$ , AFAM returns the output in accordance with (1) and the assignment given above, so that

$$W \square \mathbf{x} \vee \theta = \bigvee_{i=1}^n (W(u_i, u_j) \otimes \mathbf{x}(u_i)) \vee \theta.$$

The proposed in [14] AFAM is able to reduce an eroded noise, i.e. if  $\mathbf{x}$  is less than some database element, say  $\mathbf{x}^k$ , then the retrieved output is close to  $\mathbf{x}^k$  as well.

## 2.2 Algebraic background

In this Section, we step aside from the terminology of associative memories and introduce the algebraic background of what will be proposed as a new model of AFAM.

Let  $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1 \rangle$  be a fixed, complete, integral, residuated, commutative l-monoid (*a complete residuated lattice*). We remind the main characteristics of this structure:  $\langle L, \vee, \wedge, 0, 1 \rangle$  is a complete bounded lattice,  $\langle L, *, \rightarrow, 1 \rangle$  is a residuated, commutative monoid.

Let  $X$  be a non-empty set,  $L^X$  a class of ( $L$ -valued) *fuzzy sets* on  $X$  and  $L^{X \times X}$  a class of ( $L$ -valued) *fuzzy relations* on  $X$ . Fuzzy sets and fuzzy relations are identified with their membership functions, i.e. elements from  $L^X$  and  $L^{X \times X}$ , respectively. A fuzzy set  $A$  is *normal* if there exists  $x_A \in X$  such that  $A(x_A) = 1$ . The (ordinary) set  $Core(A) = \{x \in X \mid A(x) = 1\}$  is the *core* of the normal fuzzy set  $A$ . Fuzzy sets  $A \in L^X$  and  $B \in L^X$  are *equal* ( $A = B$ ), if for all  $x \in X$ ,  $A(x) = B(x)$ . A fuzzy set  $A \in L^X$  is *less than or equal* to a fuzzy set  $B \in L^X$  ( $A \leq B$ ), if for all  $x \in X$ ,  $A(x) \leq B(x)$ .

The lattice operations  $\vee$  and  $\wedge$  induce the union and intersection of fuzzy sets, respectively. The binary operations  $*$  and  $\rightarrow$  of  $\mathcal{L}$  are used for set-relation compositions of the types sup  $-*$  or inf  $-\rightarrow$  that are usually denoted by  $\circ$  and  $\triangleright$  where

$$(R \circ A)(y) = \bigvee_{x \in X} (R(x, y) * A(x)),$$

and

$$(R \triangleright A)(y) = \bigwedge_{x \in X} (R(x, y) \rightarrow A(x)).$$

We say that compositions  $\circ$  and  $\triangleright$  are *skew adjoint*, which means that for every  $A, B \in L^X$ ,  $R \in L^{X \times X}$ , the following holds:

$$R \circ A \leq B \leftrightarrow A \leq R^{op} \triangleright B,$$

where  $R^{op}(x, y) = R(y, x)$ .

Let us remind that the  $\circ$  composition was introduced by L. Zadeh [18] in the form  $\max - \min$ .

### 3 Fuzzy Preorders and Their Eigen Sets

In this Section, we recall basic facts about fuzzy preorder relations. Then we characterize eigen sets of fuzzy preorder relations.

Our interest to fuzzy preorder relations is connected with the analysis of the expression in (3) – a representation of the AFAM model. This is the representation of the so called Valverde (fuzzy) preorder [16].

#### 3.1 Fuzzy preorders

A binary fuzzy relation on  $X$  is a *\*-fuzzy preorder* of  $X$ , if it is reflexive and \*-transitive. The fuzzy preorder  $Q^* \in L^{X \times X}$ , where

$$Q^*(x, y) = \bigwedge_{i \in I} (A_i(x) \rightarrow A_i(y)), \quad (4)$$

is *generated* by an arbitrary family of fuzzy sets  $(A_i)_{i \in I}$  of  $X$ .

*Remark 1.* The fuzzy preorder  $Q^*$  is often called Valverde order determined by the family of fuzzy sets  $(A_i)_{i \in I}$  of  $X$  (see [16] for details).

If  $Q$  is a fuzzy preorder on  $X$ , then  $Q^{op}$  is a fuzzy preorder on  $X$  as well.

#### 3.2 Eigen sets of fuzzy preorders

In this Section, we show that fuzzy preorder  $Q^*$  (given by (4)) generated by fuzzy sets  $(A_i)_{i \in I} \subseteq L^X$ , is the greatest solution to the system of fuzzy relation equations

$$W \circ A_i = A_i, \quad i \in I, \quad (5)$$

where  $W$  denotes an unknown fuzzy relation. At the same time,  $(Q^*)^{op}$  is the greatest solution to the system of fuzzy relation equations

$$W \triangleright A_i = A_i, \quad i \in I. \quad (6)$$

Moreover, we show that there exists a binary preorder that gives a solution to (5) and (after the “transposition”) to (6).

**Proposition 1.** *Let  $(A_i)_{i \in I} \subseteq L^X$ , be a family of fuzzy sets of  $X$  and a fuzzy preorder  $Q^*$  be generated by this family in the sense of (4). Then  $Q^*$  is the greatest solution to the system of fuzzy relation equations (5) and  $(Q^*)^{op}$  is the greatest solution to the system of fuzzy relation equations (6).*

*Proof.* It is obvious that both systems are solvable - this is because the identity (fuzzy) relation is a solution to (5) and (6). This fact implies that fuzzy relation  $Q^*$  is the greatest solution to the system (5). Let us prove the second claim.

The following chain of equivalences can be easily obtained from the first claim:

$$\begin{aligned} (\forall y) \left( \bigvee_{x \in X} (Q^*(x, y) * A_i(x)) \leq A_i(y) \right) &\Leftrightarrow (\forall y)(\forall x)(Q^*(x, y) * A_i(x) \leq A_i(y)) \Leftrightarrow \\ (\forall x)(\forall y)(A_i(x) \leq Q^*(x, y) \rightarrow A_i(y)) &\Leftrightarrow (\forall x) \left( A_i(x) \leq \bigwedge_{y \in X} (Q^*(x, y) \rightarrow A_i(y)) \right). \end{aligned}$$

By reflexivity of  $Q^*$ ,

$$\bigwedge_{y \in X} (Q^*(x, y) \rightarrow A_i(y)) \leq A_i(x).$$

Therefore,

$$(Q^*)^{op} \triangleright A_i = A_i.$$

**Corollary 1.** *Let the assumptions of Proposition 1 be fulfilled. Then fuzzy sets  $A_i$ ,  $i \in I$ , are eigen fuzzy sets of the relation  $Q^*$  ( $(Q^*)^{op}$ ) with respect to composition  $\circ$  ( $\triangleright$ ).*

By Proposition 1, fuzzy relation  $Q^*$  ( $(Q^*)^{op}$ ) is the greatest solution of the system (5) (similarly, fuzzy relation  $(Q^*)^{op}$  is the greatest solution of the system (6)). Let us show that there are smaller fuzzy relations that solve the system (5) (resp. the system (6)). Moreover, these smaller relations are ordinary (binary) preorders on  $X$ .

**Proposition 2.** *Let  $(A_i)_{i \in I} \subseteq L^X$  be a family of fuzzy sets of  $X$  and fuzzy preorder  $Q^*$  be generated by this family in the sense of (4). Let  $\Delta : L \rightarrow L$  be the following unary operation on  $L$ :*

$$\Delta(a) = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

*Then  $\Delta(Q^*)$  is a solution to the system (5) and  $(\Delta(Q^*))^{op}$  is a solution to the system (6). Moreover,  $\Delta(Q^*)$  is an ordinary (binary) preorder on  $X$ .*

*Proof.* At first, we prove that  $\Delta(Q^*)$  is a solution to (5). This fact easily follows from the following two inequalities:

$$\begin{aligned} \Delta(Q^*) \circ A_i &\leq Q^* \circ A_i = A_i, \quad i \in I, \\ \Delta(Q^*) \circ A_i &\geq A_i. \end{aligned}$$

The first inequality is due to  $\Delta(Q^*) \leq Q^*$ . The second inequality follows from reflexivity of  $\Delta(Q^*)$ .

At second, we show that  $\Delta(Q^*)$  is a preorder relation on  $X$ . The property of reflexivity is inherited from  $Q^*$ . To prove transitivity we choose  $t, u, v \in X$  and consider the case  $\Delta(Q^*)(t, u) = 1$  and  $\Delta(Q^*)(u, v) = 1$ . It is easy to see that

$$\Delta(Q^*)(t, u) * \Delta(Q^*)(u, v) = Q^*(t, u) * Q^*(u, v) \leq Q^*(t, v).$$

We refer to  $\Delta(Q^*)$  as to a binary “skeleton” of  $Q^*$ .

## 4 New AFAM Models

In this Section, we put a bridge between the theory, presented in Section 3, and the theory of autoregressive fuzzy associative memories (AFAM), presented in Section 2. We propose a new concept of adjoint AFAM models that share a common fuzzy preorder relation. We characterize types of noise that can be reduced by retrieval in corresponding AFAM models.

### 4.1 Adjoined AFAM Models

Let us choose and fix complete residuated lattice with the support  $L = [0, 1]$  and database  $D = \{\mathbf{x}^1, \dots, \mathbf{x}^p\}$  of 2D  $[0, 1]$ -valued (gray scaled) images. We assume that the images in  $D$  are represented by  $n$ -dimensional vectors, so that each vector is a sequence of image rows. We identify every image with a fuzzy set on  $\bar{n} = \{1, 2, \dots, n\}$ , so that  $\mathbf{x}^k \in [0, 1]^n$ ,  $k = 1, \dots, p$ . We additionally assume that all fuzzy sets in  $D$  are normal.

**Definition 1.** We say that a pair  $(W, \square)$ , where  $W \in [0, 1]^{n \times n}$  is a fuzzy relation and  $\square : [0, 1]^{n \times n} \times [0, 1]^n \rightarrow [0, 1]^n$  is a set-relation composition, is an AFAM model of database  $D$ , if for all  $\mathbf{x}^k \in D$ ,

$$\mathbf{x}^k = W \square \mathbf{x}^k, k = 1, \dots, p. \quad (7)$$

We say that two AFAM models  $(W, \square)$  and  $(W, \tilde{\square})$  of  $D$  are adjoint, if there exists a complete residuated lattice  $\mathcal{L}$  such that  $\square$  is of the sup  $-*$  type,  $\tilde{\square}$  is of the inf  $- \rightarrow$  type and  $\square$  and  $\tilde{\square}$  are skew-adjoint.

By (7) and the terminology of AFAM, any element from  $D$  is successfully retrieved from its sample in a corresponding AFAM model. On the other hand, according to the terminology of set-relation compositions, the same equation (7) characterizes any element from  $D$  as an eigen set of the fuzzy relation with respect to a certain composition - both are constituents of the corresponding AFAM model.

*Example 1.* Let us choose a complete residuated lattice  $\mathcal{L}$  on  $[0, 1]$  and give two examples of adjoint AFAM models. Following (4), we construct the fuzzy preorder relation  $Q^*$ , such that

$$Q^*(i, j) = \bigwedge_{k=1}^p (\mathbf{x}^k(i) \rightarrow \mathbf{x}^k(j)), \quad (8)$$

and take its binary skeleton  $\Delta(Q^*)$ . By Proposition 1, the two pairs  $(Q^*, \circ)$  and  $((Q^*)^{op}, \triangleright)$  are examples of adjoint AFAM models. By Proposition 2, the two pairs  $(\Delta(Q^*), \circ)$  and  $(\Delta(Q^*)^{op}, \triangleright)$  are examples of adjoint AFAM models too.

## 4.2 AFAM and Noise Reduction

Let us characterize types of noise that can be removed/reduced by the retrieval mechanisms of adjoint AFAM models.

**Definition 2.** Let  $(W, \square)$  be a model of AFAM with respect to database  $D$ ,  $\tilde{\mathbf{x}} \in [0, 1]^n$ , and  $\tilde{\mathbf{x}} \notin D$ . We say that  $\tilde{\mathbf{x}}$  is a noisy version of some element  $\mathbf{x} \in D$ , that can be removed by the retrieval in  $(W, \square)$ , if

$$\mathbf{x} = W \square \tilde{\mathbf{x}}. \quad (9)$$

**Proposition 3.** Let  $D$  be a database,  $\mathcal{L}$  a complete residuated lattice on  $[0, 1]$ , and  $(W, \circ), (W, \triangleright)$  adjoint AFAM models. Let moreover, fuzzy relation  $W$  be reflexive. Then

- (i) if  $(W, \circ)$  removes a noisy version  $\tilde{\mathbf{x}}$  of the element  $\mathbf{x} \in D$ , then  $\tilde{\mathbf{x}}$  is an eroded version of  $\mathbf{x}$ , i.e.  $\tilde{\mathbf{x}} \leq \mathbf{x}$ ;
- (ii) if  $(W, \triangleright)$  removes a noisy version  $\tilde{\mathbf{x}}$  of the element  $\mathbf{x} \in D$ , then  $\tilde{\mathbf{x}}$  is a dilated version of  $\mathbf{x}$ , i.e.  $\tilde{\mathbf{x}} \geq \mathbf{x}$ .

*Proof.* We give the proof of the case (i). Assume that  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$  and  $W \circ \tilde{\mathbf{x}} = \mathbf{x}$ . Then for any  $j = 1, \dots, n$ ,

$$x_j = \bigvee_{i=1}^n (W(i, j) * \tilde{x}_i) \geq W(j, j) * \tilde{x}_j = \tilde{x}_j.$$

Therefore,  $\tilde{\mathbf{x}} \leq \mathbf{x}$ .

## 5 AFAM and Formal Concept Analysis

In this section, we explain the relationship between the proposed theory of AFAM and the theory of formal concept analysis (FCA) [17]. For this purpose, we adapt the terminology of FCA to the proposed above analysis of fuzzy associative memories.

We choose and fix a finite set  $X$  and a complete residuated lattice  $\mathbf{L}$ . The following *formal context*  $\mathcal{K} = (\mathcal{A}, \mathcal{R}, \mathcal{I}_{\square})$  where  $\mathcal{A} \subseteq L^X$  is a set of objects,  $\mathcal{R} \subseteq L^{X \times X}$  is a set of attributes, and  $\mathcal{I}_{\square}$  is an incidence relation on  $\mathcal{A} \times \mathcal{R}$ , is proposed. We say that an object (dataset)  $A$  possesses an attribute  $R$ , if the latter is an AFAM model of  $A$  via the composition  $\square$ . Equivalently and in accordance with Definition 1,  $\mathcal{I}_{\square}(A, R) = 1$  if and only if  $A \in \mathcal{A}$  is an eigen fuzzy set of  $R \in \mathcal{R}$  with respect to the composition  $\square$ .

Let us choose and fix a formal context  $\mathcal{K}$  with the given above specification. A couple  $(\mathbf{A}, \mathbf{W})$  is a *formal concept* of  $\mathcal{K}$ , if  $\mathbf{A} \subseteq \mathcal{A}$  is a dataset,  $\mathbf{W} \subseteq \mathcal{R}$  is a

set of fuzzy relations such that  $A \in \mathbf{A}$  if and only if  $A$  is an eigen fuzzy set of any  $W \in \mathbf{W}$  with respect to the composition  $\square$ , and vice versa,  $W \in \mathbf{W}$  if and only if  $(W, \square)$  is an AFAM model of  $\mathbf{A}$ .

Using both languages, we say that every element in dataset  $\mathbf{A}$  has every attribute in  $\mathbf{W}$ , i.e. can be successfully retrieved from its sample using any AFAM model  $(W, \square)$  where  $W \in \mathbf{W}$  and composition  $\square$  is clear from the context  $\mathcal{K}$ .

Below, we analyze, how the problems of data retrieval and noise reduction can be formalized in terms of context analysis. We formulate two assertions and characterize formal concepts where datasets as formal concept objects are connected with retrieval models as formal concept attributes. The proofs of both below given statements can be obtained from Propositions 1,2.

**Proposition 4.** *Let  $X$  be a set,  $\mathcal{L}$  a complete residuated lattice,  $\mathcal{K} = (L^X, L^{X \times X}, \mathcal{I}_\circ)$  a formal context and  $\mathbf{A} \subseteq L^X$  a dataset of objects. Then the smallest concept with objects from  $\mathbf{A}$  includes as attributes all fuzzy relations  $R \in L^{X \times X}$  such that  $\Delta(Q^*) \leq R \leq Q^*$  where  $Q^*$  and  $\Delta(Q^*)$  are specified in Propositions 1,2.*

**Proposition 5.** *Let  $X$  be a set,  $\mathcal{L}$  a complete residuated lattice,  $\mathcal{K} = (L^X, L^{X \times X}, \mathcal{I}_\triangleright)$  be a formal context and  $\mathbf{A} \subseteq L^X$  a dataset of elements. Then the smallest concept with elements from  $\mathbf{A}$  includes as attributes all fuzzy relations  $R \in L^{X \times X}$  such that  $\Delta(Q^*)^{op} \leq R \leq (Q^*)^{op}$  where  $(Q^*)^{op}$  and  $\Delta(Q^*)^{op}$  are specified in Propositions 1,2.*

The difference between Propositions 4 and 5 is in the choice of formal context. The latter determines a type of composition that connects an element from a dataset with the corresponding model.

By Definition 2, a noisy and ideal elements from  $\mathbf{A}$  are connected by equation (9). The following relationship is an easy consequence of (9): the larger is an amount of noise, the greater should be a fuzzy relation that models retrieval with noise reduction and vice versa. This fact is illustrated in Fig. 2 where we demonstrate two results of noise reduction: the one is based on the model  $(Q^*, \circ)$  and the other one - on the model  $(\Delta(Q^*), \circ)$ . It easily observed that model  $(Q^*, \circ)$  reduces a larger amount of noise than the other model  $(\Delta(Q^*), \circ)$ . This is because fuzzy relation  $Q^*$  is greater than fuzzy relation  $\Delta(Q^*)$ .

Let us construct a formal concept of  $\mathcal{K}$ , suitable for characterization of an AFAM model with the ability of noise reduction. For this purpose we differentiate AFAM models according to “degrees of fuzziness” of their fuzzy relations. The latter will be defined as

$$\delta(W) = \sum_{x,y \in X} W(x,y),$$

where  $W \in L^{X \times X}$ . It is easy to see that for two fuzzy relations  $Q^*$  (given by (4)) and its binary skeleton  $\Delta(Q^*)$ , the following inequality holds:  $\delta(\Delta(Q^*)) \leq \delta(Q^*)$ . A formal concept  $(\mathbf{A}, \mathbf{W}_D)$  of  $\mathcal{K}$  with the ability of noise reduction is a couple  $(\mathbf{A}, \mathbf{W}_D)$ , where  $A \in \mathbf{A}$ , if and only if  $A$  is an eigen fuzzy set of any  $W \in \mathbf{W}_D$  such that  $\delta(W) \geq D$ , and vice versa,  $W \in \mathbf{W}_D$ , if and only if  $\delta(W) \geq D$  and  $(W, \square)$  is an AFAM model of  $\mathbf{A}$ . The value of  $D$  regulates the amount of noise and should be less or equal than  $\delta(Q^*)$ .



## 6 Illustration

The aim of this Section is to give illustrations to the theoretical results of this paper. We use gray scaled images with the range  $[0, 1]$ , where 0 (1) represents the black (white) color. We choose two different datasets of images, both were artificially created from open access databases. These sets contain 2D images of  $40 \times 30$  and  $120 \times 90$  pixels, respectively. All images are represented by corresponding fuzzy sets with values in  $[0, 1]$ .

### 6.1 Algorithms of data retrieval

We tested the two AFAM models  $(Q^*, \circ)$  and  $(\Delta(Q^*), \circ)$ , given in Example 1. The first experiment is to verify data retrieval. Both models successfully passed the verification. For the model that is based on the relation  $\Delta(Q^*)$ , we elaborated a fast algorithm of data retrieval that uses the fact that  $\Delta(Q^*)$  is actually a binary relation.

We compare average execution time of the two corresponding algorithms and conclude that the algorithm based on  $\Delta(Q^*)$  is up to thirty time faster than that based on  $Q^*$ , see Table 1.

Image size	Run-time 1	Run-time 2
40 x 30	5.40	0.195
120 x 90	435.80	13.79

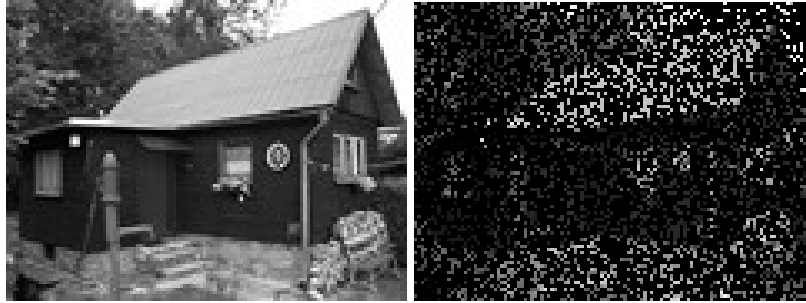
**Table 1.** Run-time (in seconds) of the two algorithms of data retrieval that are based on models  $(Q^*, \circ)$  (left column) and  $(\Delta(Q^*), \circ)$  (right column).

### 6.2 Reduction of Noise

The second experiment is to analyze the influence of different retrieval AFAM models  $(Q^*, \circ)$  and  $(\Delta(Q^*), \circ)$  on eroded noise. For this purpose, we added 70% pepper noise to the original image in Fig. 1. The algorithm of data retrieval, based on the model  $(Q^*, \circ)$ , was more efficient than that, based on the model  $(\Delta(Q^*), \circ)$ , see Fig. 2.

## 7 Conclusion

A new theory of autoregressive fuzzy associative memories (AFAM) has been proposed. It extends [14] by using general algebraic structures and new types of autoregressive fuzzy associative memory models. We showed how the proposed theory is connected with systems of fuzzy relation equations and eigen sets of their solutions. We proposed a new concept of adjoint AFAM models that share



**Fig. 1.** Original image without noise (left) and with 70 % pepper noise (right).



**Fig. 2.** Noise reduction based on the model  $(Q^*, \circ)$  (left) and on the model  $(\Delta(Q^*), \circ)$  (right).

a common fuzzy preorder relation. We characterized two types of noise that can be reduced by retrieval in corresponding AFAM models. Two problems have been discussed: data retrieval and noise reduction.

The relationship between the AFAM data retrieval and the formal concept analysis has been analyzed. We proposed a new working theory of FAM and formalized it in the language of formal concept analysis. We characterized two principal activities of FAM: data retrieval and noise reduction. We used the proposed formalism and showed that the problem of data retrieval is connected with solvability and eigen sets of a certain system of fuzzy relation equations. We differentiated FAM models according to their ability to reduce noise and showed how the choice of formal context determines a type of noise that can be reduced by the corresponding retrieval mechanism.

Finally, we proposed a fast algorithm of data retrieval that is based on an AFAM model with a binary fuzzy preorder.

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