The Curvilinear Search Algorithm for Solving Three-Person Game

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Abstract. We formulate the problem of finding a Nash equilibrium point for the non-zero sum three-person game as a nonconvex optimization problem by generalizing Mills's theorem [10]. For solving the problem, we propose the curvilinear algorithm which allows us to find global solutions. The proposed algorithm was tested numerically on some examples as well as on 3 competitive companies which share the bread market of the city Ulaanbataar.

Keywords: Game theory, optimality conditions, global optimization, the curvilinear search algorithm

Introduction

It is well known that game theory plays an important role in economics, optimization and operations research. Game theory is found to be a powerful mathematical tool for modeling of firm competitions at oligopoly markets where each firm maximizes its own profit using the same price which depends on the sum of quantities produced by the firms. Existence of Nash equilibrium points have been proved in [11, 12]. In recent years, computational methods of game theory or equivalently, finding Nash equilibrium points in various games have been intensively studying in the literature [2, 3, 6–10, 13–17]. As usual, finding Nash equilibrium points in zero-sum games leads to linear programming while in non-zero sum game it requires solving nonconvex optimization problems. Global search for Nash equilibrium points have been mainly studied for polymatrix [5] and hexamatrix games by global optimization techniques [13–15].

But it seems to us that a little attention has been paid to computational aspects of non-zero sum three-person game. Aim of this paper is to fulfill this gap. The paper is organized as follows. Section 1 is devoted to formulation of non-zero sum three-person game in mixed strategies and its reduction to a nonconvex optimization. The Curvilinear Search Algorithm has been considered in Section 2. Computational experiments has been examined in Section 3.

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1 Non-Zero Sum Three-person Game

Consider the three-person game in mixed strategies with payoff matrices (A, B, C) for players 1,2 and 3.

$$A = (a_{ijk}), \quad B = (b_{ijk}), \quad C = (c_{ijk}),$$

$$i = 1, \ldots, m, \quad j = 1, \ldots, n, \quad k = 1, \ldots, s.$$

Denote by D_q the set

$$D_q = \left\{ u \in \mathbb{R}^q \;\middle|\; \sum_{i=1}^{q} u_i = 1,\ u_i \geq 0,\ i = 1, \ldots, q \right\}.$$

A mixed strategy for player 1 is a vector $x = (x_1, x_2, \ldots, x_m) \in D_m$ representing the probability that player 1 uses a strategy i. Similarly, the mixed strategies for players 2 and 3 are $y = (y_1, y_2, \ldots, y_n) \in D_n$ and $z = (z_1, z_2, \ldots, z_s) \in D_s$. Their expected payoffs are given by:

$$f_1(x, y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} x_i y_j z_k,$$

$$f_2(x, y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} x_i y_j z_k,$$

$$f_3(x, y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} x_i y_j z_k.$$

Definition 1. *A triple of mixed strategies $x^* \in D_m,\ y^* \in D_n,\ z^* \in D_s$, is a Nash equilibrium if*

$$\begin{cases} f_1(x^*, y^*, z^*) \;\geq\; f_1(x, y^*, z^*), & \forall x \in D_m, \\ f_2(x^*, y^*, z^*) \;\geq\; f_2(x^*, y, z^*), & \forall y \in D_n, \\ f_3(x^*, y^*, z^*) \;\geq\; f_3(x^*, y^*, z), & \forall z \in D_s. \end{cases}$$

It is clear that

$$f_1(x^*, y^*, z^*) = \max_{x \in D_m} f_1(x, y^*, z^*),$$

$$f_2(x^*, y^*, z^*) = \max_{y \in D_n} f_2(x^*, y, z^*),$$

$$f_3(x^*, y^*, z^*) = \max_{z \in D_s} f_3(x^*, y^*, z).$$

For further purpose, it is useful to formulate the following statement.

Theorem 1. *A triple strategy (x^*, y^*, z^*) is a Nash equilibrium if and only if*

$$\begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} x_i^* y_j^* z_k^* \;\geq\; \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} y_j^* z_k^*, & i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} x_i^* y_j^* z_k^* \;\geq\; \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} x_i^* z_k^*, & j = 1, 2, \ldots, n, \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} x_i^* y_j^* z_k^* \;\geq\; \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} x_i^* y_j^*, & k = 1, 2, \ldots, s. \end{cases} \tag{1}$$

Proof. Necessity: Assume that (x^*, y^*, z^*) is a Nash equilibrium. Then by definition 1, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i^* y_j^* z_k^* \ \geq \ \sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i y_j^* z_k^*, \quad \forall x \in D_m, \tag{2}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i^* y_j^* z_k^* \ \geq \ \sum_{i=1}^{m}\sum_{k=1}^{s} a_{ijk}x_i^* y_j z_k^*, \quad \forall y \in D_n, \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i^* y_j^* z_k^* \ \geq \ \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ijk}x_i^* y_j^* z_k, \quad \forall z \in D_s. \tag{4}$$

In the first inequality (2), successively choose $x = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the m spots, in (3) choose $y = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the n spots, and in (4) choose $z = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the s spots. We can easily see that

$$f_1(x^*, y^*, z^*) \ \geq \ \sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk} y_j^* z_k^*, \quad i = 1, \ldots, m,$$

$$f_2(x^*, y^*, z^*) \ \geq \ \sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk} x_i^* z_k^*, \quad j = 1, \ldots, n,$$

$$f_3(x^*, y^*, z^*) \ \geq \ \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk} x_i^* y_j^*, \quad k = 1, \ldots, s.$$

Sufficiency: Suppose that for a triple $(x^*, y^*, z^*) \in D_m \times D_n \times D_s$, conditions (1) are satisfied. We choose $x \in D_m, \ y \in D_n$ and $z \in D_s$ and multiply (1) by x_i, y_j and z_k respectively. We obtain

$$\sum_{e=1}^{m} x_e \left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i^* y_j^* z_k^*\right] \ \geq \ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i y_j^* z_k^*,$$

$$\sum_{e=1}^{n} y_e \left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}x_i^* y_j^* z_k^*\right] \ \geq \ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}x_i^* y_j z_k^*,$$

$$\sum_{e=1}^{m} z_e \left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}x_i^* y_j^* z_k^*\right] \ \geq \ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}x_i^* y_j^* z_k.$$

Taking into account that $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = \sum_{k=1}^{s} z_k = 1$ we have

$$f_1(x^*, y^*, z^*) \ \geq \ f_1(x, y^*, z^*), \quad \forall x \in D_m,$$

$$f_2(x^*, y^*, z^*) \ \geq \ f_2(x^*, y, z^*), \quad \forall y \in D_n,$$

$$f_3(x^*, y^*, z^*) \ \geq \ f_3(x^*, y^*, z), \quad \forall z \in D_s,$$

which shows that (x^*, y^*, z^*) is a Nash equilibrium. The proof is complete.

Now we are ready to generalize Mills's theorem [10] formulated originally for the bimatrix game of two players for three-person matrix game as follows.

Theorem 2. *A triple strategy (x^*, y^*, z^*) is a Nash equilibrium for the non-zero sum three-person game if and only if there exist scalars (p^*, q^*, t^*) such that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a solution to the following nonconvex optimization problem:*

$$\max_{(x,y,z,p,q,t)} F(x, y, z, p, q, t) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(a_{ijk} + b_{ijk} + c_{ijk})x_i y_j z_k - p - q - t \tag{5}$$

subject to:

$$\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk} y_j z_k \ \le\ p, \quad i = 1, \dots, m, \tag{6}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk} x_i z_k \ \le\ q, \quad j = 1, \dots, n, \tag{7}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk} x_i y_j \ \le\ t, \quad k = 1, \dots, s, \tag{8}$$

$$\begin{aligned} &\sum_{i=1}^{m} x_i = 1, \quad x_i \ge 0, \quad i = 1, \dots, m,\\ &\sum_{j=1}^{n} y_j = 1, \quad y_j \ge 0, \quad j = 1, \dots, n,\\ &\sum_{k=1}^{s} z_k = 1, \quad z_k \ge 0, \quad k = 1, \dots, s. \end{aligned} \tag{9}$$

Proof. Necessity: Now suppose that (x^*, y^*, z^*) is a Nash equilibrium point. Choose scalars p^*, q^* and t^* as: $p^* = f_1(x^*, y^*, z^*)$, $q^* = f_2(x^*, y^*, z^*)$, and $t^* = f_3(x^*, y^*, z^*)$. We show that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a solution to problem (5)-(9). First, we show that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a feasible point for problem (5). By Theorem 1, the equivalent characterization of a Nash equilibrium point, we have

$$\begin{cases} \sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk} y_j^* z_k^* \ \ge\ f_1(x^*, y^*, z^*),\\ \sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk} x_i^* z_k^* \ \ge\ f_2(x^*, y^*, z^*),\\ \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk} x_i^* y_j^* \ \ge\ f_3(x^*, y^*, z^*). \end{cases}$$

The rest of the constraints are satisfied because of $x \in D_m$, $y \in D_n$ and $z \in D_s$. It meant that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a feasible point. Choose any $x \in D_m$, $y \in D_n$, $z \in D_s$ and multiply (6)-(8) by x_i, y_j and z_k respectively. If we have sum up these inequalities, we obtain

$$f_1(x, y, z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk} x_i y_j z_k \ \le\ p,$$

$$f_2(x,y,z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}x_iy_jz_k \ \le \ q,$$

$$f_3(x,y,z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}x_iy_jz_k \ \le \ t.$$

Hence, we get

$$F(x,y,z,p,q,t) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(a_{ijk}+b_{ijk}+c_{ijk})x_iy_jz_k - p - q - t \ \le \ 0$$

for all $x \in D_m,\ y \in D_n$ and $z \in D_s$.
But with $p^* = f_1(x^*,y^*,z^*),\ q^* = f_2(x^*,y^*,z^*)$, and $t^* = f_3(x^*,y^*,z^*)$, we have $F(x^*,y^*,z^*,p^*,q^*,t^*) = 0$. Hence, the point $(x^*,y^*,z^*,p^*,q^*,t^*)$ is a solution to the problem (5)-(9).
Sufficiency: Now we have to show reverse, namely, that any solution of problem (5)-(9) must be a Nash equilibrium point. Let $(\bar{x},\bar{y},\bar{z},\bar{p},\bar{q},\bar{t})$ be any solution of problem (5)-(9). Let (x^*,y^*,z^*) be a Nash equilibrium point for the game, and set $p^* = f_1(x^*,y^*,z^*),\ q^* = f_2(x^*,y^*,z^*)$, and $t^* = f_3(x^*,y^*,z^*)$. We will show that $(\bar{x},\bar{y},\bar{z})$ must be a Nash equilibrium of the game. Since $(\bar{x},\bar{y},\bar{z},\bar{p},\bar{q},\bar{t})$ is a feasible point, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{y}_j\bar{z}_k \ \le \ \bar{p}, \quad i = 1,\ldots,m, \tag{10}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{z}_k \ \le \ \bar{q}, \quad j = 1,\ldots,n, \tag{11}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk}\bar{x}_i\bar{y}_j \ \le \ \bar{t}, \quad k = 1,\ldots,s. \tag{12}$$

Hence, we receive

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \ \le \ \bar{p},$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \ \le \ \bar{q},$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \ \le \ \bar{t}.$$

Adding these inequalities, we obtain

$$F(\bar{x},\bar{y},\bar{z},\bar{p},\bar{q},\bar{t}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left[a_{ijk}+b_{ijk}+c_{ijk}\right]\bar{x}_i\bar{y}_j\bar{z}_k - \bar{p} - \bar{q} - \bar{t} \ \le \ 0. \tag{13}$$

We know that at a Nash equilibrium $F(x^*, y^*, z^*, p^*, q^*, t^*) = 0$. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ is also a solution, $F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ be equal to zero:

$$\begin{aligned} F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t}) &= (\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k - \bar{p}) + \\ &+ (\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k - \bar{q}) + \\ &+ (\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k - \bar{t}) = 0. \end{aligned} \tag{14}$$

Consequently,

$$\begin{cases} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k = \bar{p}, \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k = \bar{q}, \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k = \bar{t}. \end{cases}$$

Since a point $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ is feasible, we can write the constraints (10)-(12) as follows:

$$\begin{cases} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \le \sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{y}_j\bar{z}_k, & i = 1, \ldots, m, \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \le \sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{z}_k, & j = 1, \ldots, n, \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \le \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk}\bar{x}_i\bar{y}_j, & k = 1, \ldots, s. \end{cases}$$

Now taking into account the above results, by Theorem 1 we conclude that $(\bar{x}, \bar{y}, \bar{z})$ is a Nash equilibrium point which completes the proof.

2 The Curvilinear Global Search Algorithm

In order to solve problem (5)-(9), we use curvilinear search algorithm introduced in [4]. This algorithm allows us to search for the minimum value of the function along the scanning domain curve. The algorithm was originally developed for solving box-constrained optimization problems, therefore, we convert our problem from the constrained to unconstrained form using penalty function techniques. For each equality constraint $g(x) = 0$, we construct a simple penalty function $\hat{g}(x) = g^2(x)$. For each inequality constraint $q(x) \le 0$, we also construct the corresponding penalty function as follows:

$$\hat{q}(x) = \begin{cases} 0, & \text{if } q(x) \le 0, \\ q^2(x), & \text{if } q(x) > 0. \end{cases}$$

Thus, we have the following box-constrained optimization problem:

$$\hat{f}(x) = f(x) + \frac{\gamma}{2}\sum_i \hat{g}_i(x) + \frac{\gamma}{2}\sum_j \hat{q}_j(x) \to \min_X,$$

$$X = \left\{x \in \mathbb{R}^n | \underline{x_i} \le x_i \le \overline{x_i},\ i = 1, ..., n\right\}.$$

where γ is a penalty parameter, $\underline{x}$ and $\overline{x}$ - are upper and below bounds. For original x, y and z variables the constraint is the box $[0, 1]$; for p, q and t box constraints are $[0, \overline{p}]$,

$[0, \overline{q}]$ and $[0, \overline{t}]$. Values of $\overline{p}$, $\overline{q}$ and $\overline{t}$ are chosen from some intervals. An initial value of a penalty parameter γ is chosen not too large (something about 1000) and after finding some local minimums we increase it for searching another local minimum.

The proposed algorithm starts from some initial point $x^1 \in X$. At each $k-$th iteration the algorithm performs randomly "drop" of two auxiliary points $\tilde{x}^1$ and $\tilde{x}^2$ and generating a curve (parabola) which passes through all three points x^k, $\tilde{x}^1$ and $\tilde{x}^2$. Then we solve one-dimensional minimization problem along this curve. If a solution to this problem is better than x^k, we use it as a new minimum point, otherwise, we start a new iteration from the previous point. Details are presented in Algorithm 1:

Input: $x^1 \in X$ – initial (start) point; $K > 0$ – iterations count;
Output: Global minimum point x^* and $f^* = f(x^*)$;

for $k \leftarrow 1$ **to** K **do**
  $f^k \leftarrow f(x^k)$;
  for $i \leftarrow 1$ **to** 2 **do**
    generate stochastic point $\tilde{x}^i \in X$;
    if $f(\tilde{x}^i) < f^k$ **then**
      $x^{k+1} \leftarrow \tilde{x}^i$;
      go to the next iteration;
    end
  end
  $\alpha^k \leftarrow \underset{\alpha \in [-1,1]}{\text{argmin}} f(\hat{x}(\alpha))$,
  where
    $\hat{x}(\alpha) = \text{Proj}_X \left(\alpha^2 \left((\tilde{x}^1 + \tilde{x}^2)/2 - x^k\right) + \alpha/2 \left(\tilde{x}^2 - \tilde{x}^1\right) + x^k\right)$;
    $\text{Proj}_X(z)$ - projection of point z onto set X.
  `// note that` $\hat{x}(-1) = \hat{x}^1$, $\hat{x}(1) = \hat{x}^2$, $\hat{x}(0) = x^k$
  if $f(\hat{x}(\alpha^k)) < f^k$ **then**
    $x^{k+1} = \hat{x}(\alpha^k)$;
  else
    $x^{k+1} = x^k$;
  end
end
$x^* \leftarrow x^k$;
$f^* \leftarrow f(x^k)$

Algorithm 1: The Curvilinear Global Search

3 Computational Experiments

The proposed method was implemented in C language and tested on compatibility with using the GNU Compiler Collection (GCC, versions: 4.8.4, 4.9.3, 5.2.1), clang (versions: 3.4.2, 3.5.2, 3.6.1, 3.7.0) and Intel C Compiler (ICC, version 15.0.3) on both GNU/Linux, Microsoft Windows and Mac OS X operating systems.

The results of numerical experiments presented below were obtained on a personal computer with the following characteristics:

- Ubuntu server 14.04, x86_64
- Intel Core i5-2500K, 16 Gb RAM
- used compiler — gcc-5.2.1,
  build flags: `-O2 -DNDEBUG`

Three problems of type (5)−(9) have been solved numerically for dimensions 2×2×2 (Problems 1–3) and 5 × 6 × 4 (Problem 4). In all cases, Nash equilibrium points were found successfully. These problems were:

Problem 1. Let $a_{111} = 2$, $a_{112} = 3$, $a_{121} = -1$, $a_{122} = 0$, $a_{211} = 1$, $a_{212} = -2$, $a_{221} = 4$, $a_{222} = 3$, $b_{111} = 1$, $b_{112} = 2$, $b_{121} = 0$, $b_{122} = -1$, $b_{211} = -1$, $b_{212} = 0$, $b_{221} = 2$, $b_{222} = 1$, and $c_{111} = 3$, $c_{112} = 2$, $c_{121} = 1$, $c_{122} = -3$, $c_{211} = 0$, $c_{212} = 2$, $c_{221} = -1$, $c_{222} = 2$.
Then problem (5)–(9) can be written as:

$$F(x, y, z, p, q, t) = 6x_1y_1z_1 + 7x_1y_1z_2 - 4x_1y_2z_2 + 5x_2y_2z_1+$$

$$+6x_2y_2z_2 - p - q - t \to \max .$$

$$\begin{cases} 2y_1z_1 + 3y_1z_2 - y_2z_1 - p & \le 0, \\ y_1z_1 - 2y_1z_2 + 4y_2z_1 + 3y_2z_2 - p & \le 0, \\ x_1z_1 + 2x_1z_2 - x_2z_1 - q & \le 0, \\ -x_1z_2 + 2x_2z_1 + x_2z_2 - q & \le 0, \\ 3x_1y_1 + x_1y_2 - x_2y_2 - t & \le 0, \\ 2x_1y_1 - 3x_1y_2 + 2x_2y_1 + 2x_2y_2 - t & \le 0, \\ x_1 + x_2 & = 1, \\ y_1 + y_2 & = 1, \\ z_1 + z_2 & = 1, \\ x_1 \ge 0,\ x_2 \ge 0,\ y_1 \ge 0,\ y_2 \ge 0, & \\ z_1 \ge 0,\ z_2 \ge 0,\ p \ge 0,\ q \ge 0,\ t \ge 0. & \end{cases}$$

Nash equilibrium points are:

F^*	x^*	y^*	z^*	p	q	t
0	(0; 1)	(0; 1)	(0; 1)	3	1	2
0	(1; 0)	(1; 0)	(1; 0)	2	1	3
$2.08 \cdot 10^{-8}$	(0.5191; 0.4809)	(0.5888; 0.4112)	(0.5382; 0.4618)	1.2281	0.5	0.9327
$3.37 \cdot 10^{-8}$	(0.75; 0.25)	(0.8333; 0.1667)	(1.0; 0.0)	1.5	0.5	1.9583

Problem 2. Let $a_{111} = 5$, $a_{112} = 3$, $a_{121} = 6$, $a_{122} = 7$, $a_{211} = 0$, $a_{212} = 8$, $a_{221} = 2$, $a_{222} = 1$, $b_{111} = 2$, $b_{112} = 4$, $b_{121} = -1$, $b_{122} = 0$, $b_{211} = 3$, $b_{212} = 5$, $b_{221} = 4$, $b_{222} = 9$, and $c_{111} = 2$, $c_{112} = 0$, $c_{121} = -4$, $c_{122} = -1$, $c_{211} = -2$, $c_{212} = 6$, $c_{221} = 8$, $c_{222} = 9$.

Nash equilibrium points are:

F^*	x^*	y^*	z^*	p	q	t
0	(1; 0)	(1; 0)	(1; 0)	5	2	2
$2.6 \cdot 10^{-8}$	(0.5; 0.5)	(0.5454; 0.4545)	(0; 1)	4.8181	4.5	3.4545
$9.9 \cdot 10^{-8}$	(0.8; 0.2)	(1; 0)	(0.5; 0.5)	4.0	3.2	1.2

Problem 3. Let $a_{111} = 3$, $a_{112} = 2$, $a_{121} = 1$, $a_{122} = 5$, $a_{211} = 8$, $a_{212} = 4$, $a_{221} = 1$, $a_{222} = 3$, $b_{111} = 3$, $b_{112} = 2$, $b_{121} = 4$, $b_{122} = 0$, $b_{211} = 1$, $b_{212} = 8$, $b_{221} = 6$, $b_{222} = 6$, and $c_{111} = 3$, $c_{112} = 1$, $c_{121} = 9$, $c_{122} = 2$, $c_{211} = 4$, $c_{212} = 7$, $c_{221} = 2$, $c_{222} = 3$.

Nash equilibrium points are:

F^*	x^*	y^*	z^*	p	q	t
0	(1; 0)	(0; 1)	(1; 0)	1	4	9
0	(0; 1)	(1; 0)	(0; 1)	4	8	7
0	(0.5; 0.5)	(0; 1)	(1; 0)	1	5	5.5
$-1.33 \cdot 10^{-15}$	(0.7; 0.3)	(0; 1)	(1; 0)	1	4.6	6.9

Problem 4. We have considered competitions of 3 companies sharing the bread market of city Ulaanbataar where each company maximizes own profit depending on its manufacturing strategies. The problem was formulated as the three-person game with profit matrices $A = \{a_{ijk}\}$, $B = \{b_{ijk}\}$, $C = \{c_{ijk}\}$, $i = \overline{1,5}$, $j = \overline{1,6}$, $k = \overline{1,4}$. The matrix data can be downloaded from [1]. In this case the problem had 18 variables with 18 constraints. The solution of the problem found by the proposed algorithm was:

$$\begin{cases} F^* = 0, \\ x^* = (0,0,0,0,1), \\ y^* = (0,0,0,0,0,1), \\ z^* = (1,0,0,0), \\ p^* = 65, \\ q^* = 160, \\ t^* = 53. \end{cases}$$

It means that first and second companies must follow their 5-th and 6-th production strategies while third company applies for its 1-st production strategy. Companies's maximum profits were 65, 160 and 53 respectively.

Conclusion

We examine non-zero sum three-person matrix game from a view of point of global optimization. Finding a Nash equilibrium point of the game reduces to a global optimization problem. Based on generalization of Mills's theorem [10] (1960), we derive a sufficient condition for Nash equilibrium points for the game. To find the equilibrium points we apply the curvilinear algorithm. The proposed algorithm found Nash equilibrium points in considered problems. The algorithm was tested also for solving a real-world problem which arises in Mongolian industry.

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