Space-Time Viewpoints for Concurrent Processes Represented by Relational Structures*

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1 Introduction

In contrast to the standard physical theories which model systems by the continuum, Petri proposed a combinatorial representation of a spacetime — concurrent structures (with occurrence nets as a special case thereof) in which notions corresponding to the relativistic concepts of world line and causal cone can be defined by means of the concurrency and causal dependence relations, repectively. As a result, the density notion of the continuum model is replaced by several properties — so called concurrency axioms (including K-density, N-density, etc.). K-density is based on the idea that at any time instant, any sequential subprocess of a concurrent structure must be in some state or changing its state. N-density can be viewed as a sort of local density. Furthermore, it has turned out that concurrency axioms allow avoiding inconsistency between syntactic and semantic representations of processes and, thereby, to exclude unreasonable processes represented by the concurrent structures.

Petri's occurrence nets model system behaviors by occurrences of local states (also called conditions) and of events which are partially ordered. The partial order is interpreted as a kind of causal dependency relation. Also, occurrence nets are endowed with a symmetric, but in general non-transitive, concurrency relation — absence of the causality. Poset models do not discriminate between conditions and events. The power and limitations of concurrency axioms in the context of occurrence nets [4, 5] and posets [6, 9, 20] have been widely studied to allow better understanding the connections of causality and concurrency relations between systems events. In contrast to these treatments, the authors of the paper [16] have dealt with causality and concurrency on cyclic processes represented by net models which do not require an underlying partial order of causality. The paper [21] has studied the interrelations between concurrency axioms in the setting of prime event structures (occurrence nets with forward hereditary (w.r.t. causality) conflicts), where the nondeterministic aspects of concurrent processes are explicitly described. In the more recent papers [1, 2], algebraic and orthomodular lattices (the elements of the lattices are the closed subsets w.r.t. a closure operator, defined starting from the concurrency relation) have been generated from occurrence nets with and

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without forward conflicts. Also, an alternative characterization of K-density is given on the basis of a relation between maximal sets of pairwise causally related elements and closed sets.

It is known that some aspects of concurrent behavior (e.g., the specification of priorities, error recovery, inhibitor nets, treatment of simultaneity, etc.) are to some extent problematic to be dealt with using only partially ordered causality based models. One way to cope with the problems is to utilize the model of a so called relational structure — a set of elements (systems events) with a number of different kind relations on it. The authors of the papers [11–14] have proposed and carefully studied a subclass of the model where general causal concurrent behavior is represented by a pair of relations instead just one, as in the standard partial order approach. Depending on the assumptions and goals for the chosen model of concurrency, the pair of the relations are interpreted in two versions: either as partially ordered causality and irreflexive weak causality (not in general a partial order) or as a symmetric and irreflexive mutex relation (non-simultaneity) and irreflexive weak causality (herewith, the relations may not be completely distinct). The approaches allow modeling and studying concurrency at different levels of consideration: from abstract behavioral observations — concurrent histories (consisting of step sequence executions) to system level models such as elementary Petri nets and their generalizations with inhibitor arcs and mutex arcs.

In this paper, we intend to get a better understanding of the space-time nature of concurrency axioms, by establishing their interrelations and revealing their algebraic lattice views, in the context of a subclass of relational structures with completely distinct, irreflexive relations on countable sets of elements.

2 Preliminaries

Introduce some notions and notations which will be useful throughout the text.

Sets and relations. Given a set X and a relation $R \subseteq X \times X$,

- R is *cyclic* iff there exists a sequence of distinct elements $x_1, \ldots, x_k \in X$ ($k > 1$) such that $x_j \; R \; x_{j+1}$ ($1 \leq j \leq k-1$) and $x_k \; R \; x_1$,
- R is *acyclic* iff it is not cyclic,
- R is *antisymmetric* iff $(x \; R \; x') \wedge (x \neq x') \Rightarrow \neg(x' \; R \; x)$, for all $x, x' \in X$,
- R is *transitive* iff $(x \; R \; x') \wedge (x' \; R \; x'') \Rightarrow (x \; R \; x'')$, for all $x, x', x'' \in X$,
- R is *irreflexive* iff $\neg(x \; R \; x)$, for all $x \in R$,
- $R^\alpha = R \cup \mathrm{id}$ (the reflexive closure of R),
- $R^\beta = R \cup R^{-1}$ (the symmetric closure of R),
- $R^\gamma = R^\beta \cup \mathrm{id}$ (the reflexive and symmetric closure of R),
- $R^\delta = (R \setminus \mathrm{id}) \setminus (R \setminus \mathrm{id})^2$ (the irreflexive, intransitive relation), if R is a transitive relation, and $R^\delta = R$, otherwise,

Notice that if a relation R is irreflexive and transitive, then it is acyclic and antisymmetric, i.e. a (strict) partial order, and, moreover, R^δ is the immediate predecessor relation.

Given elements $x, x_1, x_2 \in X$, subsets $A \subseteq X' \subseteq X$, and a relation $R \subseteq X \times X$,

- $[x_1 \; R \; x_2] = \{x \in X \mid x_1 \; R^\alpha \; x \; R^\alpha \; x_2\}$,
- ${}^R x = \{x' \in X \mid x' \; R^\delta \; x\}$, $x^R = \{x' \in X \mid x \; R^\delta \; x'\}$,

- ${}^{R}A = \{x' \in X \mid \exists x \in A : (x' \, R^{\alpha} \, x)\}$, $A^{R} = \{x' \in X \mid \exists x \in A : (x \, R^{\alpha} \, x')\}$,
- ${}_{R}A = \{x' \in X \mid \forall x \in A : (x' \, R \, x)\}$, $A_{R} = \{x' \in X \mid \forall x \in A : (x \, R \, x')\}$,
- A is a *(maximal) R-clique of* X' iff A is a (maximal) set containing only pairwise $(R \cup \mathrm{id}|_{X'})$-related elements of X',
- A is an *R-closed set* of X' iff $A_{RR} = (A_R)_R = A$. The family of all the R-closed sets of X' is denoted by $RCl(X')$.

Partially ordered sets and lattices. A partially ordered set (poset) is a set together with a partial order $\leq$, i.e. a binary relation which is reflexive, transitive and antisymmetric. The powerset $\mathscr{P}(X)$ of a set X together with inclusion $\subseteq$ is a poset.

A mapping $\mathcal{C} : \mathscr{P}(X) \to \mathscr{P}(X)$ is a *closure operator on a set* X, if for all $A, B \subseteq X$ it holds:

1. $A \subseteq \mathcal{C}(A)$ ($\mathcal{C}$ is extensive),
2. $A \subseteq B \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(B)$ ($\mathcal{C}$ is monotonic),
3. $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$ ($\mathcal{C}$ is idempotent).

Let $\mathcal{A} \subseteq \mathcal{X} \subseteq \mathscr{P}(X)$, and $\mathcal{P} = (\mathcal{X}, \subseteq)$ be a poset. Then,

- $R \in \mathcal{X}$ is an *upper bound* (*lower bound*) of $\mathcal{A}$ if $A \subseteq R$ ($R \subseteq A$) for all $A \in \mathcal{A}$. $T \in \mathcal{X}$ is called the *least upper bound (l.u.b.)* of $\mathcal{A}$ if it is an upper bound, and $T \subseteq R$, for all upper bounds R of $\mathcal{A}$; T is the *greatest lower bound (g.l.b.)* of $\mathcal{A}$ if it is a lower bound, and $R \subseteq T$, for all lower bounds R of $\mathcal{A}$,
- $\mathcal{P}$ is a *lattice* iff every two elements in $\mathcal{X}$ have both a g.l.b. (denoted $\wedge$) and a l.u.b. (denoted $\vee$),
- a lattice $\mathcal{P}$ is *complete* iff every subset of $\mathcal{X}$ has both a l.u.b. and a g.l.b.,
- a complete lattice $\mathcal{P}$ is *algebraic* iff for all $x \in \mathcal{X}$ it holds $x = \bigvee\{k \in K(\mathcal{P}) \mid k \subseteq x\}$, where $K(\mathcal{P}) \subseteq \mathcal{X}$ denotes the set of *compact* elements of $\mathcal{P}$. An element $k \in \mathcal{X}$ is said to be *compact* in the lattice $\mathcal{P}$ iff, for every $\mathcal{S} \subseteq \mathcal{X}$ such that $k \subseteq \bigvee \mathcal{S}$, it holds that $k \subseteq \bigvee \mathcal{T}$, for some finite $\mathcal{T} \subseteq \mathcal{S}$.

3 Relational Structures

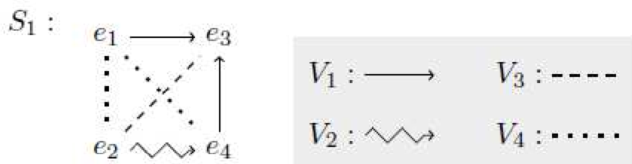

Fig. 1.

In this section, we define a slight modification of the model of relational structures whose subclasses are put forward and studied in the papers [11–14], as a suitable model of structurally complex concurrent behaviors.

Definition 1. *A* relational structure *is a tuple $S = (E, V_1, \ldots, V_n)$ ($n \geq 1$), where*

- *E is a countable set of elements,*
- *$V_1, \ldots, V_n \subseteq E \times E$ are irreflexive relations such that*
  - $\bigcup_{1 \leq i \leq n} V_i^{\beta} = (E \times E) \setminus \mathrm{id}$,
  - *$V_i^{\beta} \cap V_j^{\beta} = \emptyset$, for all $1 \leq i \neq j \leq n$.*

Clearly, Winskel's event structures with forward hereditary conflict [17] and Boudol and Castellani's event structures with forward and backward non-hereditary conflict [8] can be seen as relational structures with three relations, one of them is transitive (causality), and the two other are symmetric (concurrency and conflict).

Example 1. A simple example of a relational structure with four relations is shown in Fig. 1. Assume that V_1 is an irreflexive and transitive relation (partial order), V_2 is an asymmetric relation, and V_3 and V_4 are irreflexive and symmetric relations. We can interpret the relation V_1 as causality dependence, V_2 as asymmetric conflict [19, 7], V_3 as synchronous concurrency (simultaneity), and V_4 as asynchronous concurrency (independence).

From now on, we shall use P, Q and R to denote the relations on E of the form $\bigcup_{V \in \mathcal{V}} V$, where $\mathcal{V} \subseteq \{V_i, V_j \mid 1 \leq i, j \leq n$, V_i and V_j possess the same relation properties$\}$.

Consider the definitions of auxiliary properties of relational structures which will be useful in further considerations. We shall call a relational structure S

- *P-finite* iff any P-clique of E is finite,
- *P-degree-finite* iff $|\ {}^{P}e\ \cup\ e^{P}| < \infty$, for all $e \in E$,
- *P-degree-restricted* iff ${}^{P}e_1\ \cap\ {}^{P}e_2 \neq \emptyset \Rightarrow |{}^{P}e_1| = |{}^{P}e_2| = 1$, for all $e_1, e_2 \in E$,
- *P-discrete* iff $|\ [e_1\ P\ e_2] \cap E'\ | < \infty$, for all $e_1, e_2 \in E$ and P-cliques E' of E,
- *P-interval-finite* iff $|[e_1\ P\ e_2]| < \infty$, for all $e_1, e_2 \in E$,
- *$\bigtriangledown_{PQR}$-free* iff in any maximal $(P \cup Q \cup R)$-clique of E, there are no distinct elements e_1, e_2, and e_3 such that $e_1\ P\ e_2\ Q\ e_3\ R\ e_1$.

From now on we shall consider only P-discrete relational structures, whenever P is a transitive relation, and call them simply relational structures.

4 Concurrency Axioms

The notion of K-density and other concurrency axioms introduced by Petri [18] first for non-branching occurrence nets allow one to get better understanding the interaction of causality and concurrency. In [15], an analog of K-density under the name L-density has been put forward on the so called "sequential" nets with causality and symmetric hereditary conflict relations. Another analog under the name of R-density in the context of concurrent and conflct substructures of event structures has been dealt with in the paper [21]. More recently, the authors of [2] have adapted K-density to occurrence nets with symmetric hereditary conflicts, and rename it B-density. Our aim in this section is to give generalized definitions of a hierarchy of density properties and to study their interrelations, in the setting of relational structures.

Define concurrency axioms as properties of relational structures.

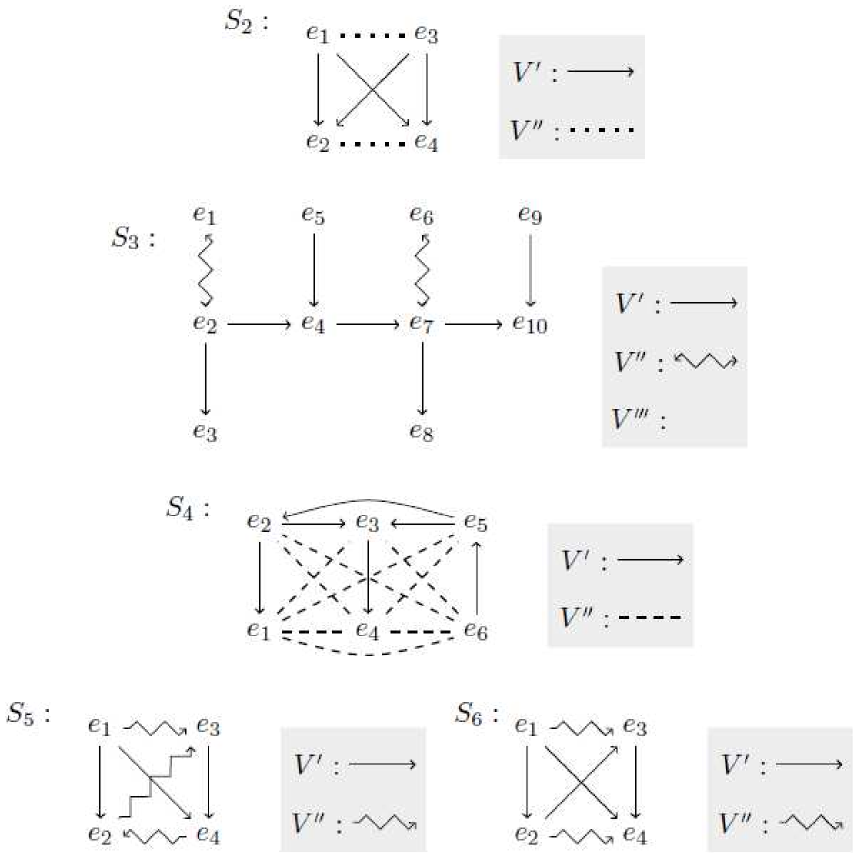

Fig. 2.

Definition 2. *Given a relational structure S and a maximal $(P \cup Q)$-clique $\widetilde{E}$ of E,*

- $\widetilde{E}$ *is* K_{PQ}-dense *iff for any maximal P-clique E' of $\widetilde{E}$ and for any maximal Q-clique E'' of $\widetilde{E}$, $E' \cap E''$ is a (unique) maximal $(P \cap Q)$-clique of $\widetilde{E}$,*
- $\widetilde{E}$ *is* K_{PQ}-crossing *iff for any maximal P-clique E' of $\widetilde{E}$ and for any maximal Q-clique E'' of $\widetilde{E}$, $E' \cap {}^{P}E'' \neq \emptyset$ and $E' \cap E''^{P} \neq \emptyset$,*
- $\widetilde{E}$ *is* $\bowtie_{PQ}$-dense *iff whenever $(e_0\ P\ e_1\ Q\ e_2)$ and $(e_0\ Q\ e_3\ P\ e_2)$, then $(e_0\ P^{\delta}\ e_2) \Longrightarrow (e_3\ P\ e_1)$, for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$,*
- $\widetilde{E}$ *is* $\bowtie^{\beta}_{PQ}$-dense *iff whenever $(e_0\ P^{\beta}\ e_1\ Q^{\beta}\ e_2)$ and $(e_0\ Q^{\beta}\ e_3\ P^{\beta}\ e_2)$, then $(e_0\ P^{\delta\beta}\ e_2) \Longrightarrow (e_3\ P^{\beta}\ e_1)$, for all distinct elements $e_0, e_1, e_2, e_3 \in \widetilde{E}$,*
- *S is* K_{PQ}-dense *(*K_{PQ}-crossing, $\bowtie_{PQ}$-dense, $\bowtie^{\beta}_{PQ}$-dense, *respectively) iff any maximal w.r.t. E clique $\widetilde{E}$ of $(P \cup Q)$ is K_{PQ}-dense (K_{PQ}-crossing, $\bowtie_{PQ}$-dense, $\bowtie^{\beta}_{PQ}$-dense, respectively).*

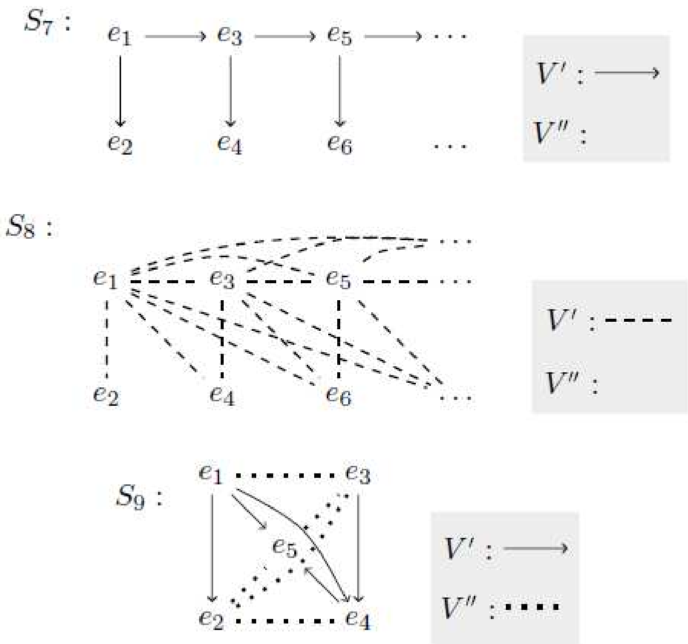

Fig. 3.

The following results state the relationships between the properties defined so far.

Proposition 1. *Let S be a relational structure with distinct relations P and Q, and let P be a transitive or symmetric relation and Q a symmetric relation. Then,*

$$S \text{ is } \bowtie^{\beta}_{PQ}\text{-dense} \iff S \text{ is } \bowtie_{PQ}\text{-dense.}$$

Example 2. Consider the relational structures S_2–S_6 shown in Fig. 2. It is easy to see that $S_2 = (E_2, V', V'')$, with the transitive or symmetric relation V' and the symmetric relation V'', is $\bowtie^{\beta}_{V'V''}$- and $\bowtie_{V'V''}$-dense. On the other hand, the relational structure $S_3 = (E_3, V', V'', V''')$, with the transitive or symmetric relation V' and the symmetric relation V''', is neither $\bowtie_{V'V'''}$- nor $\bowtie^{\beta}_{V'V'''}$-dense because in the maximal $(V' \cup V''')$-clique $\{e_2, e_3, e_4, e_5, e_7, e_8, e_9, e_{10}\}$ of E_3 there are distinct elements e_2, e_3, e_4, e_5 such that ($e_2\ V'\ e_3\ V'''\ e_4$), ($e_2\ V'''\ e_5\ V'\ e_4$), and ($e_2\ (V')^{\delta}\ e_4$) but $\neg(e_5\ V'\ e_3)$. The relational structure $S_4 = (E_4, V', V'')$ with the non-transitive and non-symmetric relation V' and the symmetric relation V'', is $\bowtie_{V'V''}$-dense but not $\bowtie^{\beta}_{V'V''}$-dense. Indeed, in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_6\}$ of E_4 there are elements e_1, e_2, e_3, e_4 such that ($e_2\ (V')^{\beta}\ e_1\ (V'')^{\beta}\ e_3$), ($e_2\ (V'')^{\beta}\ e_4\ (V')^{\beta}\ e_3$), and ($e_2\ ((V')^{\delta})^{\beta}\ e_3$) but $\neg(e_4\ (V')^{\beta}\ e_1)$. The same holds for the relational structure $S_5 = (E_5, V', V'')$ with the transitive or symmetric relation V' and the asymmetric relation V''. Truly, in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_4\}$ of E_5 there are distinct elements e_1, e_2, e_3, e_4 such that ($e_1\ (V')^{\beta}\ e_2\ (V'')^{\beta}\ e_4$), ($e_1\ (V'')^{\beta}\ e_3\ (V')^{\beta}\ e_4$), and

$(e_1\ ((V')^\delta)^\beta\ e_4)$ but $\neg(e_3\ (V')^\beta\ e_2)$. The relational structure $S_6 = (E_6, V', V'')$ with the non-transitive and non-symmetric relation V' is $\bowtie^\beta_{V'V''}$-dense but not $\bowtie_{V'V''}$-dense. In fact, in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_4\}$ of E there are distinct elements e_1, e_2, e_3, e_4 such that $(e_1\ V'\ e_2\ V''\ e_4)$, $(e_1\ V''\ e_3\ V'\ e_4)$, and $(e_1\ V'^\delta\ e_4)$ but $\neg(e_3\ V'\ e_2)$.

Proposition 2. *Let S be a $\bowtie^\beta_{PQ}$-dense relational structure with distinct relations P and Q, and let P be a transitive relation. Then,*

$$S \text{ is } K_{PQ}\text{-dense} \iff S \text{ is } K_{PQ}\text{-crossing.}$$

Example 3. First, consider the relational structures $S_2 = (E_2, V', V'')$ and $S_3 = (E_3, V', V'', V''')$ shown in Fig. 2. We know that S_2, with the transitive relation V' and the symmetric relation V'', is $\bowtie^\beta_{V'V''}$-dense (see Example 2). It is easy to check that S_2 is $K_{V'V''}$-dense and $K_{V'V''}$-crossing. On the other hand, the relational structure S_3, with the transitive relation V' and the symmetric relation V'', is not $\bowtie^\beta_{V'V'''}$-dense (see Example 2) It is easy to verify that S_3 is $K_{V'V'''}$-crossing. However, S_3 is not $K_{V'V'''}$-dense because in the maximal $(V' \cup V''')$-clique $\widetilde{E} = \{e_2, e_3, e_4, e_5, e_7, e_8, e_9, e_{10}\}$ of E_3 the intersection of the maximal V'-clique $\{e_2, e_4, e_7, e_{10}\}$ of $\widetilde{E}$ with the maximal V'''-clique $\{e_3, e_5\}$ of $\widetilde{E}$ is empty. Next, contemplate the relational structures $S_7 = (E_7, V', V'')$ and $S_8 = (E_8, V', V'')$ depicted in Fig. 3. The relational structure S_7 with the transitive relation V' and the symmetric relation V'', is $\bowtie^\beta_{PQ}$-dense but neither $K_{V'V''}$-dense nor $K_{V'V''}$-crossing since in the maximal $(V' \cup V'')$-clique $\widetilde{E} = \{e_1, e_2, \ldots\}$ of E_7 the intersection of the maximal V'-clique $E' = \{e_{2\cdot k+1} \mid k \geq 0\}$ of $\widetilde{E}$ with the maximal V''-clique $E'' = \{e_{2\cdot k} \mid k \geq 1\}$ of $\widetilde{E}$ is empty, and, moreover, the intersection of E' with $E''^{V'}$ is also empty, because $E''^{V'} = E''$. Further, the relational structure S_8, with the non-transitive relation V' and the symmetric relation V'', is $\bowtie^\beta_{V'V''}$-dense and $K_{V'V''}$-crossing but not $K_{V'V''}$-dense because in the maximal $(V' \cup V'')$-clique $\widetilde{E} = \{e_1, e_2, \ldots\}$ of E_8 the intersection of the maximal V'-clique $\{e_{2\cdot k+1} \mid k \geq 0\}$ of $\widetilde{E}$ with the maximal V''-clique $\{e_{2\cdot k} \mid k \geq 1\}$ of $\widetilde{E}$ is empty.

Theorem 1. *Let S be a P- or Q-finite relational structure with distinct relations P and Q, and let P be a transitive relation. Then,*

$$S \text{ is } K_{PQ}\text{-dense} \iff S \text{ is } \bowtie^\beta_{PQ}\text{-dense.}$$

Example 4. First, again consider the relational structures $S_2 = (E_2, V', V'')$ and $S_3 = (E_3, V', V'', V''')$ shown in Fig. 2. We know that S_2, with the transitive relation V' and the symmetric relation V'', is $\bowtie^\beta_{V'V''}$-dense (see Example 2) and $K_{V'V''}$-dense (see Example 3). Notice that S_2 is V'- and V''-finite. On the other hand, the relational structure S_3, with the transitive relation V' and the symmetric relation V'', is neither $\bowtie^\beta_{V'V'''}$-dense (see Example 2) nor $K_{V'V''}$-dense (see Example 3). Moreover, S_3 is V'- and V''-finite. Next, contemplate the relational structures $S_8 = (E_8, V'V'')$ and $S_9 = (E_9, V', V'')$ depicted in Fig. 3. We know from Example 3 that S_8, with the transitive relation V' and the symmetric relation V'', is $\bowtie^\beta_{V'V''}$-dense but not $K_{V'V''}$-dense. At the same time, S_8 is neither V'- nor V''-finite. The relational structure S_9, with the non-transitive relation V' and the symmetric relation V'', is $K_{V'V''}$-dense but not $\bowtie^{\beta}_{V'V''}$-dense because in the maximal $(V' \cup V'')$-clique $\{e_1, \ldots, e_5\}$ of E_9, there are distinct elements e_1, e_2, e_3, e_4 such that $e_1\ V'\ e_2\ V''\ e_4$, $e_1\ V''\ e_3\ V'\ e_4$, and $e_1\ V'\ e_4$ but $\neg(e_3\ V'\ e_2)$.

Theorem 2. *Given a $\bigtriangledown_{PQR}$-free and P- or Q- or R-finite relational structure S with distinct relations P, Q, and R,*

$$S \textit{ is } K_{PQ}\textit{-dense} \iff S \textit{ is } K_{\widetilde{P}\widetilde{Q}}\textit{-dense},$$

where $\widetilde{P} = (P \cup R)$ and $\widetilde{Q} = (Q \cup R)$.

Example 5. Consider the relational structures S_{10}–S_{13}, with the transitive relation V' and the symmetric relations V'' and V''', shown in Fig. 4. It is easy to check that $S_{10} = (E_{10}, V', V'', V''')$ is $K_{V''V'''}$-dense, $\bigtriangledown_{V'V''V'''}$-free, and $K_{\widetilde{V}''\widetilde{V}'''}$-dense, where $\widetilde{V}'' = V' \cup V''$ and $\widetilde{V}''' = V' \cup V'''$. Clearly, S_9 is V'-, V''- and V'''-finite. It is not difficult to see that the relational structure $S_{11} = (E_{11}, V', V'', V''')$ is $K_{V''V'''}$-dense, and V''- and V'''-finite. However, S_{11} is neither $\bigtriangledown_{V'V''V'''}$-free, because $e_1\ V'\ e_4\ V''\ e_5\ V'''\ e_1$, nor $K_{\widetilde{V}''\widetilde{V}'''}$-dense, because the intersection of the maximal $\widetilde{V}''$-clique $\{e_3, e_4, e_7\}$ of $\widetilde{E}$ and the maximal $\widetilde{V}'''$-clique $\{e_1, e_3, e_6\}$ of $\widetilde{E}$ is not a maximal V'-clique of $\widetilde{E}$ where $\widetilde{V}'' = V' \cup V''$, $\widetilde{V}''' = V' \cup V'''$, and $\widetilde{E} = \{e_1, \ldots, e_7\}$ is the only maximal $\widetilde{V}'' \cup \widetilde{V}'''$-clique of E_{11}. The relational structure $S_{12} = (E_{12}, V', V'', V''')$ is $\bigtriangledown_{V'V''V'''}$-free but neither $K_{V'V'''}$-dense, because the intersection of the maximal V'-clique $\{e_1, e_4\}$ of $\widetilde{E}'$ with the maximal V'''-clique $\{e_2, e_3\}$ of $\widetilde{E}'$ is empty, nor $K_{\widetilde{V}'\widetilde{V}'''}$-dense, because the intersection of maximal $\widetilde{V}'$-clique $\{e_1, e_4, e_5\}$ of $\widetilde{E}''$ and maximal $\widetilde{V}'''$-clique $\{e_2, e_3, e_5\}$ of $\widetilde{E}''$ is not a maximal V''-clique of $\widetilde{E}''$, where $\widetilde{E}' = \{e_1, e_2, e_3, e_4\}$ is the maximal $(V' \cup V''')$-clique of E_{12}, $\widetilde{E}' = \{e_1, e_2, e_3, e_4, e_5\}$ is the maximal $(\widetilde{V}' \cup \widetilde{V}''')$-clique of E_{12}, $\widetilde{V}' = V' \cup V''$, and $\widetilde{V}''' = V'' \cup V'''$. We can see that $S_{13} = (E_{13}, V', V'', V''')$ is $K_{\widetilde{V}'\widetilde{V}''}$-dense but neither V'- nor V''- nor V'''-finite, where $\widetilde{V}' = V' \cup V'''$ and $\widetilde{V}'' = V'' \cup V'''$. Clearly, the maximal $V'V''$-clique $\{b_1, b_2, b_3, \ldots, c_1, c_2, c_3, \ldots\}$ of E_{13} is not $K_{V'V''}$-dense. Hence, S_{13} is not $K_{V'V''}$-dense.

5 Concurrency Axioms and Algebraic Lattices of Closed Sets

In the papers [1, 2], it has been demonstrated that K-density of occurrence nets with and without forward conflicts guarantees that the lattices whose elements are the closed subsets of a closure operator, defined starting from the concurrency relation of the models, are algebraic. In this section, we first show that the above results can be exended to the model of relational structures, where a closure operator can be defined from any symmetric relation, and then formulate a necessary condition of the algebraicity of the lattices of the closed sets. Before doing so, we need to introduce the following concepts.

Definition 3. *Given a relational structure S and a maximal $(P \cup Q)$-clique $\widetilde{E}$ of E,*

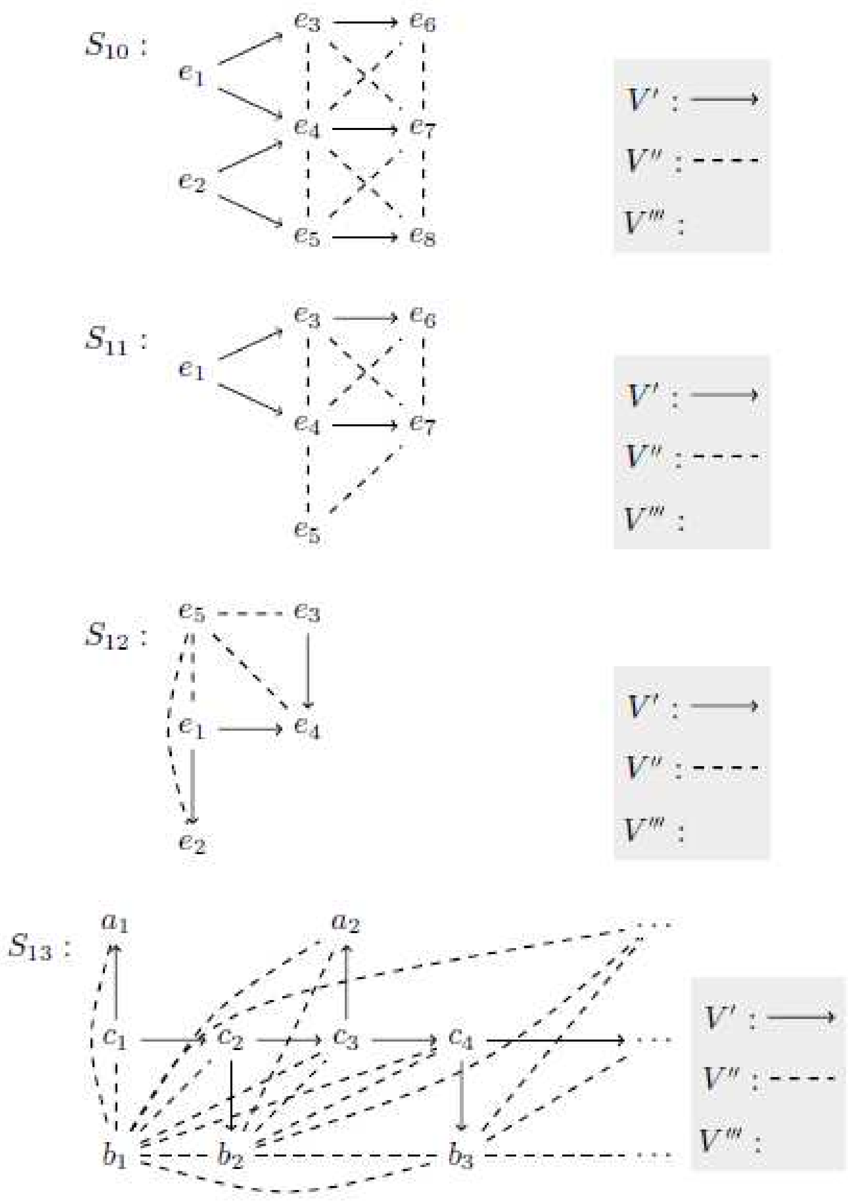

Fig. 4.

- $\widetilde{E}$ *is* PQ-encountering *iff for any maximal* P*-clique* E' *of* $\widetilde{E}$ *and for any maximal* Q*-clique* E'' *of* $\widetilde{E}$*,* $E' \cap E'''_{QQ} \neq \emptyset$*, for some finite* $E''' \subseteq E''$*,*

- $\widetilde{E}$ *is* weak K_{PQ}-dense *iff for any maximal P-clique E' of $\widetilde{E}$ and for any maximal Q-clique E'' of $\widetilde{E}$, $(E' \cap E'')$ is a (unique) $(P \cap Q)$-clique of $\widetilde{E}$, or $A \not\subseteq E'$, for any Q-closed set A of $\widetilde{E}$,*
- $\widetilde{E}$ *is* PQ-algebraic *iff $(QCl(\widetilde{E}), \subseteq)$ is an algebraic lattice,*
- *S is* PQ-encountering *(*weak K_{PQ}-dense*,* PQ-algebraic*, respectively) iff any maximal $(P \cup Q)$-clique $\widetilde{E}$ of E is PQ-encountering (weak K_{PQ}-dense, PQ-algebraic, respectively).*

Fig. 5.

The following fact will be helpful to obtain weak K_{PQ}-density as a necessary condition of PQ-algebraicity.

Theorem 3. *Given a $\bowtie_{PQ}$-dense relational structure S with a transitive relation P and a symmetric relation Q,*

$$S \text{ is } K_{PQ}\text{-dense} \iff S \text{ is } PQ\text{-encountering.}$$

Example 6. Consider the relational structures $S_{14} = (E_{14}, V', V'')$ and $S_{15} = (E_{15}, V', V'')$ with the transitive relation V' and the symmetric relation V'', shown in Fig. 5. One can easily check that S_{14} is $V'V''$-encountering, $\bowtie_{V'V''}$- and $K_{V'V''}$-dense. On the other hand, S_{15} is $V'V''$-encountering and, obviously, not $\bowtie_{V'V''}$-dense. Moreover, as the maximal V'-clique $\{e_1, e_4\}$ and the maximal V''-clique $\{e_2, e_3\}$ of E_{15} are disjoint, S_{14} is not $K_{V'V''}$-dense. We know from Example 3 that the relational structure S_7 depicted in Fig. 3 is $\bowtie_{V'V''}$-dense but not $K_{V'V''}$-dense. Furthermore, S_7 is not $V'V''$-encountering, because for the maximal V'-clique $E' = \{e_1, e_3, e_5, \ldots\}$ and the maximal V''-clique $E'' = \{e_2, e_3, e_6, \ldots\}$ of E_7, we have $E' \cap E'''_{V''V''} = E' \cap E''' = \emptyset$ for any finite $E''' \subseteq E''$.

Finally, the following theorem describes interconnections between the properties of K_{PQ}-density and weak K_{PQ}-density, and PQ-algebraicity of a relational structure.

Theorem 4. *Given a P-degree-restricted, P-degree-finite and P-interval-finite relational structure S with a transitive relation P and a symmetric relation Q,*

(i) S is K_{PQ}-dense $\Longrightarrow$ S is PQ-algebraic,

(ii) S is weak K_{PQ}-dense $\Longleftarrow$ S is PQ-algebraic.

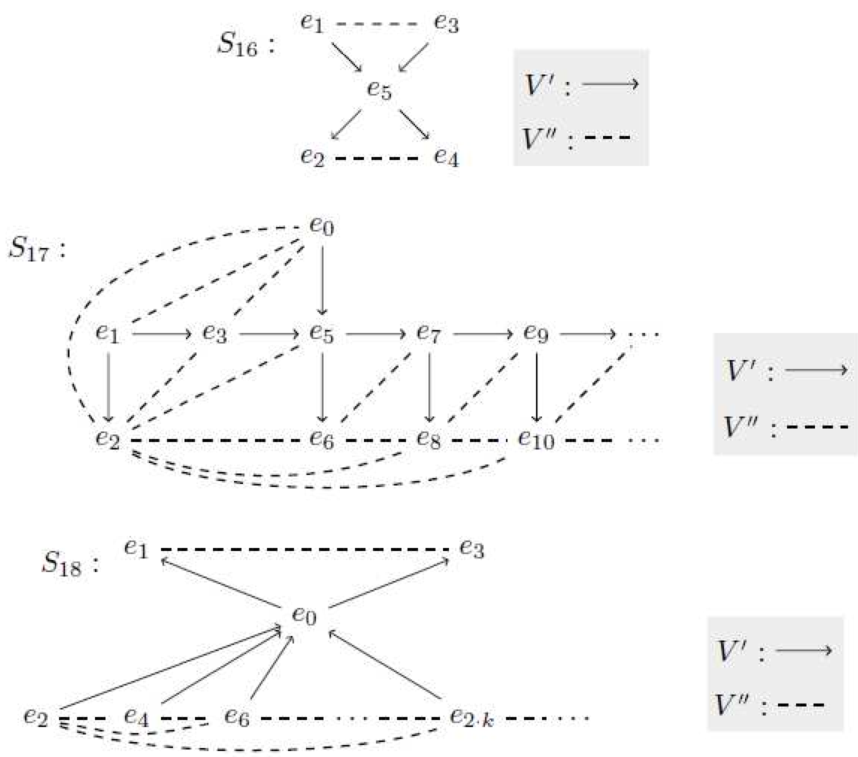

Fig. 6.

Example 7. Contemplate the V'-degree-restricted relational structures S_{16}–S_{18}, with the transitive relation V' and the symmetric relation V'', depicted in Fig. 6. The relational structure $S_{16} = (E_{16}, V', V'')$ is, obviously, V'-degree-finite, V'-interval-finite and $K_{V'V''}$-dense. Since it is finite, it is $V'V''$-algebraic. Next, consider the V'-degree-finite relational structure $S_{17} = (E_{17}, V', V'')$ with the maximal V'-clique $E' = \{e_{2\cdot i+1} \mid i \geq 0\}$ of E_{17} and the maximal V''-clique $E'' = \{e_{2\cdot j} \mid j \geq 1 \wedge j \neq 2\}$ of E_{17}. Since $E' \cap E'' = \emptyset$, S_{17} is not $K_{V'V''}$-dense. Moreover, S_{17} is not weak $K_{V'V''}$-dense, because $A = \{e_3\}$ is a V''-closed set of E_{17} such that $A \subseteq E'$. Furthermore, A is not compact in the lattice $(V''Cl(E_{17}), \subseteq)$. This implies that $\emptyset$ is the only compact element less than A. As $\bigvee \emptyset = \emptyset \neq A$, S_{17} is not $V'V''$-algebraic. Finally, consider the non-V'-degree-finite relational structure S_{18}. One can easily check that S_{18} is $K_{V'V''}$-dense and V'-interval-finite. As the V''-closed set $A = \{e_1\}$ is not compact in the lattice $(V''Cl(E_{18}), \subseteq)$, S_{18} is not $V'V''$-algebraic.

References

1. L. Bernardinello, L. Pomello, S. Rombola. Closure operators and lattices derived from concurrency in posets and occurrence nets. Fundamenta Informaticae 105(3) (2010) 211-235
2. L. Bernardinello, C. Ferigato, S. Haar, L. Pomello. Closed sets in occurrence nets with conflicts. Fundamenta Informaticae 133(4) (2014) 323-344
3. L. Bernardinello, L. Pomello, S. Rombola. Orthomodular algebraic lattices related to combinatorial posets. CEUR-WS.OR 1231 (2014) 241-245
4. E. Best. The relative strenghth of K-density. Lecture Notes in Computer Science 84 (1980) 261-276
5. E. Best. A theorem on the characteristics of non-sequential processes. Fundamenta Informaticae 3 (1980) 77-94
6. E. Best, C. Fernandez. Nonsequential processes — a Petri net view. EATCS Monographs on Theoretical Computer Science 13, Springer-Verlag (1988)
7. P. Baldan, A. Corradini, U. Montanari. Contextual Petri nets, asymmetric event structures, and processes. Information and Computation 171(1) (2001) 1-49
8. G. Boudol, I. Castellani. Concurrency and atomicity. Theoretical Computer Science 59 (1988) 25-84
9. C. Fernandez. Non-sequential processes. Lecture Notes in Computer Science, Vol. 254 (1986) 95-116
10. C. Fernandez, P.S. Thiagarajan. A lattice theoretic view of K-density. Arbeitspapiere der GMD, N 76 (1983)
11. R. Janicki, M. Koutny. Invariants and paradigms of concurrency theory. Lecture Notes in Computer Science 506 (1991) 59-74
12. G. Juhas, R. Lorenz, S. Mauser. Synchronous + Concurrent = Earlier Than + Not Later Than. In: Proceedings of ACSD06, IEEE Press, New York (2006) 261-272
13. J. Kleijn, M. Koutny. Process semantics of general inhibitor nets. Information and Computation 190 (2004) 18-69.
14. J. Kleijn, M. Koutny. Mutex Causality in Processes and Traces of General Elementary Nets. Fundamenta Informaticae 122(1-2) (2013) 119-146
15. V. Kotov, L. Cherkasova. On structural properties of generalized processes. Lecture Notes in Computer Science 188 (1984) 288-306
16. O. Kummer, M.-O. Stehr. Petri's axioms of concurrency a selection of recent results. Lecture Notes in Computer Science, Vol. 1248, 1997, 195-214.
17. M. Nielsen, G. Plotkin, G. Winskel, G. Petri nets, event structures and domains. Theoretical Computer Science 13 (1981) 85-108
18. C.A. Petri. Concurrency theory. Lecture Notes in Computer Science 254 (1987) 4-24
19. G. M. Pinna, A. Poigné. On the nature of events: another perspective in concurrency. Theoretical Computer Science 138 (1995) 425-454
20. H. Plünnecke. K-density, N-density and finiteness properties. Lecture Notes in Computer Science 188 (1984) 392-412
21. I. Virbitskaite. Some characteristics of nondeterministic processes. Parallel Processing Letters 3(1) (1993) 99-106