Weak Completion Semantics and its Applications in Human Reasoning

Steffen Hölldobler

International Center for Computational Logic,TU Dresden, 01062 Dresden, Germany
sh@iccl.tu-dresden.de

Abstract. I present a logic programming approach based on the weak completions semantics to model human reasoning tasks, and apply the approach to model the suppression task, the selection task as well as the belief-bias effect, to compute preferred mental models of spatial reasoning tasks and to evaluate indicative as well as counterfactual conditionals.

1 Introduction

Observing the performance of humans in cognitive tasks like the suppression [3] or the selection task [31] it is apparent that human reasoning cannot be adequately modeled by classical two-valued logic. Whereas there have been many approaches to develop a normative model for human reasoning which are not based on logic like the mental model theory [22] or probabilistic approaches [15], Keith Stenning and Michiel von Lambalgen have developed a logic-based approach [30] where, in a first step, they reason towards an appropriate representation of some aspects of the world as logic program and, in a second step, reason with respect to the least model of the program. Their approach is based on the three-valued (strong) Kripke-Kleene logic [23], is non-monotonic, and utilizes some form of completion as well as abduction. Most interestingly, the results developed within the fields of logic programming and computational logic within the last decades could not be immediately applied to adequately model human reasoning tasks but rather some modifications were needed. As a consequence, theorems, propositions and lemmas formally proven for a theory without these modification cannot be readily applied but their proofs must be adapted as well.

Unfortunately, some of the formal results stated in [30] are not correct. Somewhat surprisingly, we were able to show in [19] that the results do hold if the Kripke-Kleene logic is replaced by the three-valued Łukasiewicz logic [25]. We have called our approach *weak completion semantics* (WCS) because in the completion of a program, undefined relations are not identified with falsehood but rather are left *unknown*. Whereas our original emphasis was on obtaining formally correct results, WCS has been applied to many different human reasoning tasks in the meantime: the suppression task, the abstract as well as the social selection task, the belief-bias effect, the computation of preferred mental models in spational reasoning tasks as well as the evaluation of conditionals.

This paper gives an overview on WCS as well as its applications to human reasoning tasks.

2 Weak Completion Semantics

2.1 Logic Programs

We assume the reader to be familiar with logic programming, but we repeat basic notions and notations. A *(logic) program* is a finite set of (program) clauses of the form $A \leftarrow \top$, $A \leftarrow \bot$ or $A \leftarrow B_1 \wedge \ldots \wedge B_n$, $n > 0$ where A is an atom, B_i, $1 \leq i \leq n$, are literals and $\top$ and $\bot$ denote truth and falsehood, resp. A is called *head* and $\top$, $\bot$ as well as $B_1 \wedge \ldots \wedge B_n$ are called *body* of the corresponding clause. Clauses of the form $A \leftarrow \top$ and $A \leftarrow \bot$[1] are called *positive* and *negative facts*, resp. In this paper, $\mathcal{P}$ denotes a program, A a ground atom and F a formula. We assume that each non-propositional program contains at least one constant symbol. We also assume for each program that the underlying alphabet consists precisely of the symbols mentioned in the program, if not indicated differently. When writing sets of literals we omit curly brackets if a set has only one element.

$g\mathcal{P}$ denotes the set of all ground instances of clauses occurring in $\mathcal{P}$. A ground atom A is *defined* in $g\mathcal{P}$ iff $g\mathcal{P}$ contains a clause whose head is A; otherwise A is said to be *undefined*. $\mathit{def}(\mathcal{S}, \mathcal{P}) = \{A \leftarrow \mathit{body} \in g\mathcal{P} \mid A \in \mathcal{S} \vee \neg A \in \mathcal{S}\}$ is called *definition* of $\mathcal{S}$ in $\mathcal{P}$, where $\mathcal{S}$ is a set of ground literals. Such a set $\mathcal{S}$ is said to be *consistent* iff it does not contain a pair of complementary literals.

A *level mapping* for $\mathcal{P}$ is a function ℓ which assigns to each atom occurring in $g\mathcal{P}$ a natural number. Let $\ell(\neg A) = \ell(A)$. $\mathcal{P}$ is *acyclic* iff there exists a level mapping ℓ such that for each $A \leftarrow L_1 \wedge \ldots \wedge L_n \in g\mathcal{P}$ we find that $\ell(A) > \ell(L_i)$, $1 \leq i \leq n$.

2.2 Weak Completion

For a given $\mathcal{P}$, consider the following transformation: (1) For each defined atom A, replace all clauses of the form $A \leftarrow \mathit{body}_1, \ldots, A \leftarrow \mathit{body}_m$ occurring in $g\mathcal{P}$ by $A \leftarrow \mathit{body}_1 \vee \ldots \vee \mathit{body}_m$. (2) Replace all occurrences of $\leftarrow$ by $\leftrightarrow$. The obtained ground program is called *weak completion* of $\mathcal{P}$ or $wc\mathcal{P}$.[2]

2.3 Łukasiewicz Logic

An *interpretation* is a mapping from the set of formulas into the set of truth values. A *model* for F is an interpretation which maps F to *true*. We consider the three-valued Łukasiewicz (or Ł-) logic [25] (see Table 1) and represent each interpretation I by $\langle I^\top, I^\bot \rangle$, where $I^\top = \{A \mid I(A) = \top\}$, $I^\bot = \{A \mid I(A) = \bot\}$, $I^\top \cap I^\bot = \emptyset$, and each ground atom $A \notin I^\top \cup I^\bot$ is mapped to U. Hence, under the empty interpretation $\langle \emptyset, \emptyset \rangle$ all ground atoms are *unknown*. Let $\langle I^\top, I^\bot \rangle$ and $\langle J^\top, J^\bot \rangle$ be two interpretations. We define

$$\langle I^\top, I^\bot \rangle \subseteq \langle J^\top, J^\bot \rangle \text{ iff } I^\top \subseteq J^\top \text{ and } I^\bot \subseteq J^\bot,$$
$$\langle I^\top, I^\bot \rangle \cup \langle J^\top, J^\bot \rangle = \langle I^\top \cup J^\top, I^\bot \cup J^\bot \rangle.$$

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[1] Under WCS a clause of the form $A \leftarrow \bot$ is turned into $A \leftrightarrow \bot$ provided that it is the only clause in the definition of A.

[2] Note that undefined atoms are not identified with $\bot$ as in the completion of $\mathcal{P}$ [5].

F	$\neg F$
$\top$	$\bot$
$\bot$	$\top$
U	U

$\wedge$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	U	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	$\top$	U	U
$\bot$	$\top$	U	$\bot$

$\leftarrow$	$\top$	U	$\bot$
$\top$	$\top$	$\top$	$\top$
U	U	$\top$	$\top$
$\bot$	$\bot$	U	$\top$

$\leftrightarrow$	$\top$	U	$\bot$
$\top$	$\top$	U	$\bot$
U	U	$\top$	U
$\bot$	$\bot$	U	$\top$

Table 1. Truth tables for the Ł-semantics, where we have used $\top$, $\bot$ and U instead of *true*, *false* and *unknown*, resp., in order to shorten the presentation.

Theorem 1. *(Model Intersection Property) For each program $\mathcal{P}$, the intersection of all Ł-models of $\mathcal{P}$ is an Ł-model of $\mathcal{P}$.*

This result was formally proven in [19] for programs not containing negative facts, but it holds also for programs with negative facts.

2.4 A Semantic Operator

The following operator was introduced by Stenning and van Lambalgen [30], where they also showed that it admits a least fixed proint: $\Phi_{\mathcal{P}}(\langle I^\top, I^\bot\rangle) = \langle J^\top, J^\bot\rangle$, where

$$\begin{aligned} J^\top &= \{A \mid A \leftarrow \mathit{body} \in g\mathcal{P} \text{ and } \mathit{body} \text{ is } \mathit{true} \text{ under } \langle I^\top, I^\bot\rangle\}, \\ J^\bot &= \{A \mid \mathit{def}(A,\mathcal{P}) \neq \emptyset \text{ and} \\ &\qquad \mathit{body} \text{ is } \mathit{false} \text{ under } \langle I^\top, I^\bot\rangle \text{ for all } A \leftarrow \mathit{body} \in \mathit{def}(A,\mathcal{P})\}. \end{aligned}$$

The $\Phi_{\mathcal{P}}$ operator differs from the semantic operator defined by Fitting in [13] in the additional condition $\mathit{def}(A,\mathcal{P}) \neq \emptyset$ required in the definition of $J^\bot$. This condition states that A must be defined in order to be mapped to *false*, whereas in the (strong) Kripke-Kleene-semantics considered by Fitting an atom is mapped to *false* if it is undefined. This reflects precisely the difference between the weak completion and the completion semantics. The (strong) Kripke-Kleene-semantics was also applied in [30]. However, as shown in [19] this semantics is not only the cause for a technical bug in one theorem of [30], but it does also lead to a non-adequate model of some human reasoning tasks. Both, the technical bug as well as the non-adequate modeling, can be avoided by using WCS.

Theorem 2. *The least fixed point of $\Phi_{\mathcal{P}}$ is the least Ł-model of the weak completion of $\mathcal{P}$.* [19]

In the remainder of this paper, $\mathcal{M}_{\mathcal{P}}$ denotes the least Ł-model of $wc\mathcal{P}$.

2.5 Contraction

It was Fitting's idea [14] to apply metric methods to compute least fixed points of semantic operators and, in particular, he showed that for so-called *acceptable*[3]

[3] Please see [14] for a definition of acceptable programs. The class of acyclic programs is a proper subset of the class of acceptable programs.

programs the semantic operator defined in [13] is a contraction.[4] Consequently, Banach's contraction mapping theorem [2] can be applied to compute the least fixed point of the semantic operator.

As shown in [18], $\Phi_{\mathcal{P}}$ may not be a contraction if $\mathcal{P}$ is acceptable. But the following weaker result holds for programs not containing any cycles.

Theorem 3. *If $\mathcal{P}$ is an acyclic program, then $\Phi_{\mathcal{P}}$ is a contraction.* [18]

As a consequence, the computation of the least fixed point of $\Phi_{\mathcal{P}}$ can be initialized with an arbitrary interpretation.

2.6 A Connectionist Realization

Within the CORE-method [1, 17] semantic operators of logic programs are computed by feed-forward connectionist networks, where the input and the output layer represent interpretations. By connecting the output with the input layer, the networks are turned into recurrent ones and can now be applied to compute the least fixed points of the semantic operators.

Theorem 4. *For each datalog program $\mathcal{P}$ there exists a recurrent connectionist network which will converge to a stable state representing $\mathcal{M}_{\mathcal{P}}$ if initialized with the empty interpretation.*

The theorem was proven in [20] for propositional programs but extends to datalog programs. From the discussion in the previous paragraph we conclude that the network may be initialized by some interpretation if $\Phi_{\mathcal{P}}$ is a contraction.

2.7 Weak Completion Semantics

The *weak completion semantics* (WCS) is the approach to consider weakly completed logic programs and to reason with respect to the least Ł-models of these programs. We write $\mathcal{P} \models_{wcs} F$ iff formula F holds in $\mathcal{M}_{\mathcal{P}}$. WCS is non-monotonic.

2.8 Relation to Well-Founded Semantics

WCS is related to the well-founded semantics (WFS) as follows: Let $\mathcal{P}^+ = \mathcal{P} \setminus \{A \leftarrow \bot \mid A \leftarrow \bot \in \mathcal{P}\}$ and u be a new nullary relation symbol not occurring in $\mathcal{P}$. Furthermore, let $\mathcal{P}^* = \mathcal{P}^+ \cup \{B \leftarrow u \mid \mathit{def}(B, \mathcal{P}) = \emptyset\} \cup \{u \leftarrow \neg u\}$.

Theorem 5. *If $\mathcal{P}$ is a program which does not contain a positive loop, then $\mathcal{M}_{\mathcal{P}}$ and the well-founded model for $\mathcal{P}^*$ coincide.* [11]

[4] A mapping $f : \mathcal{M} \rightarrow \mathcal{M}$ on a metric space $(\mathcal{M}, d)$ is a *contraction* iff there exists a $k \in (0, 1)$ such that for all $x, y \in \mathcal{M}$ we find $d(f(x), f(y)) \leq k \times d(x, y)$.

2.9 Abduction

An *abductive framework* consists of a logic program $\mathcal{P}$, a set of *abducibles* $\mathcal{A}_\mathcal{P} = \{A \leftarrow \top \mid \mathit{def}(A, \mathcal{P}) = \emptyset\} \cup \{A \leftarrow \bot \mid \mathit{def}(A, \mathcal{P}) = \emptyset\}$, a set of *integrity constraints* $\mathcal{IC}$, i.e., expressions of the form $\bot \leftarrow B_1 \wedge \ldots \wedge B_n$, and the entailment relation $\models_{wcs}$; it is denoted by $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$.

By Theorem 1, each program and, in particular, each finite set of positive and negative ground facts has an Ł-model. For the latter, this can be obtained by mapping all heads occurring in this set to *true*. Thus, in the following definition, explanations as well as the union of a program and an explanation are satisfiable.

An *observation* $\mathcal{O}$ is a set of ground literals; it is *explainable* in the framework $\langle \mathcal{P}, \mathcal{A}_\mathcal{P}, \mathcal{IC}, \models_{wcs} \rangle$ iff there exists a minimal $\mathcal{E} \subseteq \mathcal{A}_\mathcal{P}$ called *explanation* such that $\mathcal{M}_{\mathcal{P} \cup \mathcal{E}}$ satisfies $\mathcal{IC}$ and $\mathcal{P} \cup \mathcal{E} \models_{wcs} L$ for each $L \in \mathcal{O}$. F *follows creduluously from* $\mathcal{P}$ *and* $\mathcal{O}$ iff there exists an explananation $\mathcal{E}$ such that $\mathcal{P} \cup \mathcal{E} \models_{wcs} F$. F *follows skeptically from* $\mathcal{P}$ *and* $\mathcal{O}$ iff for all explanantions $\mathcal{E}$ we find $\mathcal{P} \cup \mathcal{E} \models_{wcs} F$.

2.10 Revision

Let $\mathcal{S}$ be a finite and consistent set of ground literals in

$$\mathit{rev}(\mathcal{P}, \mathcal{S}) = (\mathcal{P} \setminus \mathit{def}(\mathcal{S}, \mathcal{P})) \cup \{A \leftarrow \top \mid A \in \mathcal{S}\} \cup \{A \leftarrow \bot \mid \neg A \in \mathcal{S}\},$$

where A denotes an atom. $\mathit{rev}(\mathcal{P}, \mathcal{S})$ is called the *revision of* $\mathcal{P}$ *with respect to* $\mathcal{S}$. The following result was formally proven in [7].

Proposition 6. *1. rev is non-monotonic,*
i.e., there exist $\mathcal{P}$, $\mathcal{S}$ *and* F *such that* $\mathcal{P} \models_{wcs} F$ *and* $\mathit{rev}(\mathcal{P}, \mathcal{S}) \not\models_{wcs} F$.
2. If $\mathcal{M}_\mathcal{P}(L) = U$ *for all* $L \in \mathcal{S}$*, then rev is monotonic.*
3. $\mathcal{M}_{\mathit{rev}(\mathcal{P},\mathcal{S})}(\mathcal{S}) = \top$.

3 Applications

3.1 The Suppression Task

Ruth Byrne has shown in [3] that graduate students with no previous exposure to formal logic did suppress previously drawn conclusions when additional information became available. Table 2 shows the abbreviations that will be used in this subsection, whereas Table 3 gives an account of the findings of [3]. Interestingly, in some instances the previously drawn conclusions were valid (cases AE and ACE in Table 3) whereas in other instances the conclusions were invalid (cases AL and ABL in Table 3) with respect to classical two-valued logic.

Following [30] conditionals are encoded by licences for implications using *abnormality* predicates. In the case AE no abnormalities concerning the library are known. However, in the case ACE it becomes known that one can visit the library only if it is open and, thus, not being open becomes an abnormality for the first implication. Likewise, one may argue that there must be a reason for studying in the library. In the case ACE the only reason for studying in

A	*If she has an essay to finish, then she will study late in the library.*
B	*If she has a textbook to read, then she will study late in the library.*
C	*If the library stays open, she will study late in the library.*
E	*She has an essay to finish.*
$\overline{E}$	*She does not have an essay to finish.*
L	*She will study late in the library.*
$\overline{L}$	*She will not study late in the library.*

Table 2. The suppression task [3] and used abbreviations. Subjects received conditionals A, B or C and facts E, $\overline{E}$, L or $\overline{L}$ and had to draw inferences.

Cond.	Fact	Exp. Findings	Cond.	Fact	Exp. Findings
A	E	96% conclude L	A	L	53% conclude E
$A\ B$	E	96% conclude L	$A\ B$	L	16% conclude E
$A\ C$	E	38% conclude L	$A\ C$	L	55% conclude E
A	$\overline{E}$	46% conclude $\overline{L}$	A	$\overline{L}$	69% conclude $\overline{E}$
$A\ B$	$\overline{E}$	4% conclude $\overline{L}$	$A\ B$	$\overline{L}$	69% conclude $\overline{E}$
$A\ C$	$\overline{E}$	63% conclude $\overline{L}$	$A\ C$	$\overline{L}$	44% conclude $\overline{E}$

Table 3. The drawn conclusions in the experiment of Byrne. The different cases will be denoted by the word obtained by concatenating the conditionals and the fact like AE or AL for the cases in the first row of the table.

the library is to finish an essay and, consequently, not having to finish an essay becomes an abnormality for the second implication. Alltogether, for the cases AE and ACE we obtain the programs

$$\begin{aligned} \mathcal{P}_{AE} &= \{\ell \leftarrow e \wedge \neg ab_1,\ e \leftarrow \top,\ ab_1 \leftarrow \bot\}, \\ \mathcal{P}_{ACE} &= \{\ell \leftarrow e \wedge \neg ab_1,\ e \leftarrow \top,\ ab_1 \leftarrow \neg o,\ \ell \leftarrow o \wedge \neg ab_2,\ ab_2 \leftarrow \neg e\} \end{aligned}$$

with $\mathcal{M}_{\mathcal{P}_{AE}} = \langle\{e, \ell\}, \{ab_1\}\rangle$ and $\mathcal{M}_{\mathcal{P}_{ACE}} = \langle\{e\}, \{ab_2\}\rangle$, where ℓ, e, o and ab denote that *she will study late in the library*, *she has an essay to finish*, *the library stays open* and *abnormality*, resp. Hence, $\mathcal{M}_{\mathcal{P}_{AE}}(\ell) = \top$ and $\mathcal{M}_{\mathcal{P}_{ACE}}(\ell) =$ U. Thus, WCS can model the suppression of a previously drawn conclusion.

For the examples in the second column of Table 3 abduction is needed. E.g., for the case ABL we obtain the program

$$\mathcal{P}_{AB} = \{\ell \leftarrow e \wedge \neg ab_1,\ ab_1 \leftarrow \bot,\ \ell \leftarrow t \wedge \neg ab_3,\ ab_3 \leftarrow \bot\}$$

with $\mathcal{M}_{\mathcal{P}_{AB}} = \langle\emptyset, \{ab_1, ab_3\}\rangle$, where t denotes that *she has a textbook to read.* The observation $\mathcal{O} = \ell$ can be explained by $\mathcal{E}_1 = \{e \leftarrow \top\}$ and $\mathcal{E}_2 = \{t \leftarrow \top\}$. In order to adequately model Byrne's selection task, we have to be skeptical as otherwise–being credoluous–we would conclude that *she has an essay to finish.*

A complete account of Byrne's selection task under WCS is given in [10, 21].

D	F	3	7
89%	16%	62%	25%

beer	coke	22yrs	16yrs
95%	0.025%	0.025%	80%

Table 4. The results of the abstract and social case of the selection task, where the first row gives the symbol(s) on the cards and the second row shows the percentage of participants which turned it.

$\mathcal{O}$	$\mathcal{E}$	$\mathcal{M}_{\mathcal{P}_{ac} \cup \mathcal{E}}$	turn
D	$\{D \leftarrow \top\}$	$\langle \{D, 3\}, ab_1 \rangle$	yes
F	$\{F \leftarrow \top\}$	$\langle F, ab_1 \rangle$	*no*
3	$\{D \leftarrow \top\}$	$\langle \{D, 3\}, ab_1 \rangle$	*yes*
7	$\{7 \leftarrow \top\}$	$\langle 7, ab_1 \rangle$	*no*

Table 5. The computational logic approach for the abstract case of the selection task.

3.2 The Selection Task

In the original (abstract) selection task [31] participants were given the conditional *if there is a* D *on one side of the card, then there is* 3 *on the other side* and four cards on a table showing the letters D and F as well as the numbers 3 and 7. Furthermore, they know that each card has a letter on one side and a number on the other side. Which cards must be turned to prove that the conditional holds?

Griggs and Cox [16] adapted the abstract task to a social case. Consider the conditional *if a person is drinking beer, then the person must be over 19 years of age* and again consider four cards, where one side shows the person's age and on the other side shows the person's drink: *beer*, *coke*, *22yrs* and *16yrs*. Which drinks and persons must be checked to prove that the conditional holds?

When confronted with both tasks, participants reacted quite differently as shown in Table 4. Moreover, if the conditionals are modeled as implications in classical two-valued logic, then some of the drawn conclusions are not valid.

The Abstract Case This case is artificial and there is no common sense knowledge about the conditional. Let D, F, 3, and 7 be propositional variables denoting that the corresponding symbol or number is on one side of a card. Following [24], we assume that the given conditional is viewed as a belief and represented as a clause in

$$\mathcal{P}_{ac} = \{3 \leftarrow D \wedge \neg ab_1,\ ab_1 \leftarrow \bot\},$$

where the negative fact was added as there are no known abnormalities. We obtain $\mathcal{M}_{\mathcal{P}_{ac}} = \langle \emptyset, ab_1 \rangle$ and find that this model does not explain any symbol on the cards. Let $\mathcal{A}_{ac} = \{D \leftarrow \top,\ D \leftarrow \bot,\ F \leftarrow \top,\ F \leftarrow \bot,\ 7 \leftarrow \top,\ 7 \leftarrow \bot\}$ in the abductive framework $\langle \mathcal{P}_{ac}, \mathcal{A}_{ac}, \emptyset, \models_{wcs} \rangle$. Table 5 shows the explanations for the cards with respect to this framework.

In case D was observed, the least model maps also 3 to $\top$. In order to be sure that this corresponds to the real situation, we need to check if 3 is *true*.

case	$\mathcal{P}_{sc}$	$\mathcal{M}_{\mathcal{P}_{sc}}$	$\models_{wcs} o \leftarrow b \wedge \neg ab_2$	*turn*
beer	$\{ab_2 \leftarrow \bot, b \leftarrow \top\}$	$\langle b, ab_2 \rangle$	*no*	*yes*
coke	$\{ab_2 \leftarrow \bot, b \leftarrow \bot\}$	$\langle \emptyset, \{b, ab_2\} \rangle$	*yes*	*no*
22yrs	$\{ab_2 \leftarrow \bot, o \leftarrow \top\}$	$\langle o, ab_2 \rangle$	*yes*	*no*
16yrs	$\{ab_2 \leftarrow \bot, o \leftarrow \bot\}$	$\langle \emptyset, \{o, ab_2\} \rangle$	*no*	*yes*

Table 6. The computational logic approach for the social case of the selection task.

Therefore, the card showing D is turned. Likewise, in case 3 is observed, D is also mapped to $\top$, which can only be confirmed if the card is turned.

The Social Case In this case most humans are quite familiar with the conditional as it is a standard law. They are also aware–it is common sense knowledge–that there are no exceptions or abnormalities. Let o represent a person being older than 19 years and b a person drinking beer. The conditional can be represented by $o \leftarrow b \wedge \neg ab_2$ and is viewed as a social constraint which must follow logically from the given facts. Table 6 shows the four different cases.

One should observe that in the case *16yrs* the least model of the weak completion of $\mathcal{P}_{sc}$, i.e. $\langle \emptyset, \{o, ab_2\} \rangle$, assigns U to b and, consequently, to both, $b \wedge \neg ab_2$ and $o \leftarrow b \wedge \neg ab_2$, as well. Overall, in the cases *beer* and *16yrs* the social constraint is not entailed by the least Ł-model of the weak completion of the program. Hence, we need to check these cases out and, hopefully, find that the beer drinker is older than 19 and that the 16 years old is not drinking beer.

A complete account of the selection task under WCS is given in [6].

3.3 The Belief-Bias Effect

Evans et. al. [12] made a psychological study showing possibly conflicting processes in human reasoning. Participants were confronted with syllogisms and had to decide whether they are logically valid. Consider the following syllogism:

No addictive things are inexpensive. (PREMISE1)
Some cigarettes are inexpensive. (PREMISE2)
Therefore, some addictive things are not cigarettes. (CONCLUSION)

The conclusion does not follow from the premises in classical logic: If there are inexpensive cigarettes but no addictive things, then the premises are *true*, but the conclusion is *false*. Nevertheless, most participants considered the syllogism to be valid. Evans et. al. explained the answers by an unduly influence of the participants' own beliefs.

Before we can model this line of reasoning under WCS, we need to tackle the problem that the head of a program clause must be an atom, whereas the conclusion of the rule *if something is inexpensive, then it is not addictive*[5] is a

[5] (PREMISE1) can be formalized in many syntactically different, but semantically equivalent ways in classical logic. We have selected a form which allows WCS to adequately model the belief-bias effect.

negated atom. If the relation symbol *add* is used to denote addiction, then this technical problem can be overcome by introducing a new relation symbol add', specifying by means of the clause

$$add(X) \leftarrow \neg add'(X) \tag{1}$$

that add' is the negation of *add* under WCS and requiring by means of the integrity constraint

$$\mathcal{IC}_{add} = \{\bot \leftarrow add(X) \wedge \neg add'(X)\}$$

that *add* and add' cannot be simultaneously true.

We can now encode (PREMISE1) following Stenning and van Lambalgen's idea to represent conditionals by licences for implications [30]:

$$add'(X) \leftarrow inex(X) \wedge \neg ab_1(X), \qquad ab_1(X) \leftarrow \bot. \tag{2}$$

As for (PREMISE2), Evans et. al. have argued that it includes two pieces of information. Firstly, there exists something, say a, which is a cigarette:

$$cig(a) \leftarrow \top. \tag{3}$$

Secondly, it contains the following belief that humans seem to have:

Cigarettes are inexpensive. (BIAS1)

This belief implies (PREMISE2) and biases the process of reasoning towards a representation such that we obtain:

$$inex(X) \leftarrow cig(X) \wedge \neg ab_2(X), \qquad ab_2(X) \leftarrow \bot. \tag{4}$$

Additionally, it is assumed that there is a second piece of background knowledge, viz. it is commonly known that

Cigarettes are addictive, (BIAS2)

which in the context of (1) and (2) can be specified by stating that cigarettes are abnormalities regarding add':

$$ab_1(X) \leftarrow cig(X). \tag{5}$$

Alltogether, let $\mathcal{P}_{add}$ be the program consisting of the clauses (1)-(5). Because (CONCLUSION) is about an object which is not necessarily a we need to add another constant, say b, to the alphabet underlying $\mathcal{P}_{add}$. We obtain

$$\mathcal{M}_{\mathcal{P}_{add}} = \langle \{cig(a), inex(a), ab_1(a), add(a)\}, \{ab_2(a), ab_2(b), add'(a)\}\rangle.$$

Turning to (CONCLUSION) we consider its first part as the observation $\mathcal{O} = add(b)$ which needs to be explained with respect to the abductive framework

$$\langle \mathcal{P}_{add}, \{cig(b) \leftarrow \top, cig(b) \leftarrow \bot\}, \mathcal{IC}_{add}, \models_{wcs}\rangle.$$

We find two minimal explanations $\mathcal{E}_\bot = \{cig(b) \leftarrow \bot\}$ and $\mathcal{E}_\top = \{cig(b) \leftarrow \top\}$ leading to the minimal models

$$\begin{aligned}
\mathcal{M}_{\mathcal{P}_{add} \cup \mathcal{E}_\bot} &= \langle\ \{cig(a), inex(a), ab_1(a), add(a), add(b)\}, \\
&\quad\ \ \{ab_2(a), ab_2(b), add'(a), cig(b), inex(b), ab_1(b), add'(b)\}\ \rangle, \\
\mathcal{M}_{\mathcal{P}_{add} \cup \mathcal{E}_\top} &= \langle\ \{cig(a), inex(a), ab_1(a), add(a), cig(b), inex(b), ab_1(b), add(b)\}, \\
&\quad\ \ \{ab_2(a), ab_2(b), add'(a), add'(b)\}\ \rangle,
\end{aligned}$$

respectively. Because under $\mathcal{M}_{\mathcal{P}_{add} \cup \mathcal{E}_\top}$ all known addictive objects (a and b) are cigarettes and under $\mathcal{M}_{\mathcal{P}_{add} \cup \mathcal{E}_\bot}$ the addictive object b is not a cigarette, (Conclusion) follows creduluously, but not skeptically.

On the other hand, the two explanations $\mathcal{E}_\bot$ and $\mathcal{E}_\top$ do not seem to be equally likely given (Premise1) and (Bias1). Rather, $\mathcal{E}_\bot$ seems to be the main explanation whereas $\mathcal{E}_\top$ seems to be the exceptional case. Pereira and Pinto [26] have introduced so-called *inspection points* which allow to distinguish between main and exceptional explanations in an abductive framework. Formally, they introduce a meta-predicate *inspect* and require that if $inspect(A) \leftarrow \top$ or $inspect(A) \leftarrow \bot$ are elements of an explanation $\mathcal{E}$ for some literal or observation L, then either $A \leftarrow \top$ or $A \leftarrow \bot$ must be in $\mathcal{E}$ as well and, moreover, $A \leftarrow \bot$ or $A \leftarrow \top$ must be elements of explanations for some literal or observation $L' \neq L$, where A is a ground atom.

With the help of inspection points, the program $\mathcal{P}_{add}$ can be rewritten to

$$\mathcal{P}'_{add} = (\mathcal{P}_{add} \setminus \{ab_1(X) \leftarrow cig(X)\}) \cup \{ab_1(X) \leftarrow inspect(cig(X))\}$$

and the explanation $\mathcal{O} = add(b)$ is to be explained with respect to the abductive framework $\langle \mathcal{P}'_{add}, \mathcal{A}'_{add}, \mathcal{IC}_{add}, \models_{wcs}\rangle$, where

$$\begin{aligned}
\mathcal{A}'_{add} = \{\ & cig(b) \leftarrow \top,\ cig(b) \leftarrow \bot, \\
& inspect(cig(b)) \leftarrow \top,\ inspect(cig(b)) \leftarrow \bot, \\
& inspect(cig(a)) \leftarrow \top,\ inspect(cig(a)) \leftarrow \bot\ \}.
\end{aligned}$$

Now, $\mathcal{E}_\bot$ is the only explanation for $add(b)$ and, hence, (Conclusion) follows skeptically in the revised approach.

More details about our model of the belief-bias effect and abduction using inspection points can be found in [27, 28].

3.4 Spatial Reasoning

Consider the following *spatial reasoning problem*. Suppose it is known that *a ferrari is left of a porsche*, *a beetle is right of the porsche*, *the porsche is left of a hummer*, and *the hummer is left of a dodge. Is the beetle left of the hummer?*

The *mental model theory* [22] is based on the idea that humans construct so-called *mental models*, which in case of a spatial reasoning problem is understood as the presentation of the spatial arrangements between objects that correspond to the premises. In the example, there are three mental models:

ferrari porsche beetle hummer dodge
ferrari porsche hummer beetle dodge
ferrari porsche hummer dodge beetle

Hence, the answer to the above mentioned question depends on the construction of the mental models.

In the *preferred model theory* [29] it is assumed that humans do not construct all mental models, but rather a single, *preferred* one, and that reasoning is performed with respect to the preferred mental model. The preferred mental model is believed to be constructed by considering the premises one by one in the order of their occurrence and to place objects directly next to each other or, if this impossible, in the next available space. For the example, the preferred mental model is constructed as follows:

ferrari porsche
ferrari porsche beetle
ferrari porsche beetle hummer
ferrari porsche beetle hummer dodge

Hence, according to the preferred model theory, *the beetle is left of the hummer.*

In [8] we have specified a logic program $\mathcal{P}$ taking into account the premises of a spatial reasoning problem such that $\mathcal{M}_\mathcal{P}$ corresponds to the preferred mental model. Moreover, within the computation of $\mathcal{M}_\mathcal{P}$ as the least fixed point of $\Phi_\mathcal{P}$, the preferred mental model is constructed step by step as in [29].

3.5 Conditionals

Conditionals are statements of the form *if condition then consequence*. In this paper we distinguish between indicative and subjunctive (or counterfactual) conditionals. *Indicative conditionals* are conditionals whose condition is either *true* or *unknown*; the consequence is asserted to be *true* if the condition is *true*. On the contrary, the condition of a *subjunctive* or *counterfactual conditional* is either *false* or *unknown*; in the counterfactual circumstance of the condition being *true*, the consequence is asserted to be *true*.[6] We assume that the condition and the consequence of a conditional are finite and consistent sets of literals.

Conditionals are evaluated with respect to some background information specified as a program and a set of integrity constraints. More specifically, as the weak completion of each program admits a least Ł-model, conditionals are evaluated under the least Ł-model of a program. In the reminder of this section let $\mathcal{P}$ be a program, $\mathcal{IC}$ be a finite set of integrity constraints, and $\mathcal{M}_\mathcal{P}$ be the least Ł-model of $wc\mathcal{P}$ such that $\mathcal{M}_\mathcal{P}$ satisfies $\mathcal{IC}$.

[6] In the literature the case of a condition being *unknown* is usually not explicitely considered; there also seems to be no standard definition for indicative and counterfactual conditionals.

In this setting we propose to evaluate a conditional $cond(\mathcal{C}, \mathcal{D})$ as follows, where $\mathcal{C}$ and $\mathcal{D}$ are finite and consistent sets of literals:

1. If $\mathcal{M}_\mathcal{P}(\mathcal{C}) = \top$ and $\mathcal{M}_\mathcal{P}(\mathcal{D}) = \top$, then $cond(\mathcal{C}, \mathcal{D})$ is *true*.
2. If $\mathcal{M}_\mathcal{P}(\mathcal{C}) = \top$ and $\mathcal{M}_\mathcal{P}(\mathcal{D}) = \bot$, then $cond(\mathcal{C}, \mathcal{D})$ is *false*.
3. If $\mathcal{M}_\mathcal{P}(\mathcal{C}) = \top$ and $\mathcal{M}_\mathcal{P}(\mathcal{D}) = \mathrm{U}$, then $cond(\mathcal{C}, \mathcal{D})$ is *unknown*.
4. If $\mathcal{M}_\mathcal{P}(\mathcal{C}) = \bot$, then evaluate $cond(\mathcal{C}, \mathcal{D})$ with respect to $\mathcal{M}_{rev(\mathcal{P},\mathcal{S})}$, where $\mathcal{S} = \{L \in \mathcal{C} \mid \mathcal{M}_\mathcal{P}(L) = \bot\}$.
5. If $\mathcal{M}_\mathcal{P}(\mathcal{C}) = \mathrm{U}$, then evaluate $cond(\mathcal{C}, \mathcal{D})$ with respect to $\mathcal{M}_{\mathcal{P}'}$, where
   - $\mathcal{P}' = rev(\mathcal{P}, \mathcal{S}) \cup \mathcal{E}$,
   - $\mathcal{S}$ is a smallest subset of $\mathcal{C}$ and $\mathcal{E} \subseteq \mathcal{A}_{rev(\mathcal{P},\mathcal{S})}$ is a minimal explanation for $\mathcal{C} \setminus \mathcal{S}$ such that $\mathcal{M}_{\mathcal{P}'}(\mathcal{C}) = \top$.

In words, if the condition of a conditional is *true*, then the conditional is an indicative one and is evaluated as implication in Ł-logic. If the condition is *false*, then the conditional is a counterfactual conditional. In this case, i.e., in case 4, non-monotonic revision is applied to the program in order to reverse the truth value of those literals, which are mapped to *false*.

The main novel contribution concerns the final case 5. If the condition $\mathcal{C}$ of a conditional is *unknown*, then we propose to split $\mathcal{C}$ into two disjoint subsets $\mathcal{S}$ and $\mathcal{C} \setminus \mathcal{S}$, where the former is treated by revision and the latter by abduction. In case $\mathcal{C}$ contains some literals which are *true* and some which are *unknown* under $\mathcal{M}_\mathcal{P}$, then the former will be part of $\mathcal{C} \setminus \mathcal{S}$ because the empty explanation explains them. As we assume $\mathcal{S}$ to be minimal this approach is called *minimal revision followed by abduction* (MRFA). Furthermore, because revision as well as abduction are only applied to literals which are assigned to *unknown*, case 5 is monotonic.

As an example consider the *forest fire scenario* taken from [4]: The conditional $cond(\neg dl, \neg ff)$, *if there had not been so many dry leaves on the forest floor, then the forest fire would not have occurred*, is to be evaluated with respect to

$$\mathcal{P}_{ff} = \{ff \leftarrow l \wedge \neg ab_1,\ l \leftarrow \top,\ ab_1 \leftarrow \neg dl,\ dl \leftarrow \top\},$$

which states that *lightning* (l) *causes a forest fire* (ff) *if nothing abnormal* (ab_1)*, is taking place, lightning happened, the absence of dry leaves* (dl) *is an abnormality, and dry leaves are present.* We obtain $\mathcal{M}_{\mathcal{P}_{ff}} = \langle\{dl, l, ff\}, \{ab_1\}\rangle$ and find that the condition $\neg dl$ is *false*. Hence, we are dealing with a counterfactual conditional. Following Step 4 we obtain $\mathcal{S} = \{\neg dl\}$,

$$rev(\mathcal{P}_{ff}, \neg dl) = \{ff \leftarrow l \wedge \neg ab_1,\ l \leftarrow \top,\ ab_1 \leftarrow \neg dl,\ dl \leftarrow \bot\}$$

and $\mathcal{M}_{rev(\mathcal{P}_{ff},\neg dl)} = \langle\{l, ab_1\}, \{dl, ff\}\rangle$. Because ff is mapped to *false* under this model, the conditional is *true*.

Let us extend the example by adding *arson (a) causes a forest fire*:

$$\mathcal{P}_{ffa} = \mathcal{P}_{ff} \cup \{ff \leftarrow a \wedge \neg ab_2,\ ab_2 \leftarrow \bot\}.$$

We find $\mathcal{M}_{\mathcal{P}_{ffa}} = \langle\{dl, l, ff\}, \{ab_1, ab_2\}\rangle$ and $\mathcal{M}_{rev(\mathcal{P}_{ffa},\neg dl)} = \langle\{l, ab_1\}, \{dl, ab_2\}\rangle$. Under this model ff is *unknown* and, consequently, $cond(\neg dl, \neg ff)$ is *unknown* as well.

As final example consider $\mathcal{P}_{\mathit{ffa}}$ and the conditional $cond(\{\mathit{ff}, \neg dl\}, a)$: *if a forest fire occurred and there had not been so many dry leaves on the forest floor, then arson must have caused the fire.* Because the condition $\{\mathit{ff}, \neg dl\}$ is *false* under $\mathcal{M}_{\mathcal{P}_{\mathit{ffa}}}$ we follow Step 4 and obtain $\mathcal{S} = \{\neg dl\}$,

$$rev(\mathcal{P}_{\mathit{ffa}}, \neg dl) = (\mathcal{P}_{\mathit{ffa}} \setminus \{dl \leftarrow \top\}) \cup \{dl \leftarrow \bot\}$$

and $\mathcal{M}_{rev(\mathcal{P}_{\mathit{ffa}}, \neg dl)} = \langle\{l, ab_1\}, \{dl, ab_2\}\rangle$. One should observe that ff as well as the condition $\{\mathit{ff}, \neg dl\}$ are *unknown* under this model. Hence, we follow Step 5, consider the abductive framework

$$\langle rev(\mathcal{P}_{\mathit{ffa}}, \neg dl), \{a \leftarrow \top, a \leftarrow \bot\}, \emptyset, \models_{wcs}\rangle$$

and learn that $\{\mathit{ff}, \neg dl\}$ can be explained by $\{a \leftarrow \top\}$. Hence, by MRFA we obtain as final program $rev(\mathcal{P}_{\mathit{ffa}}, \neg dl) \cup \{a \leftarrow \top\}$ and find

$$\mathcal{M}_{rev(\mathcal{P}_{\mathit{ffa}}, \neg dl) \cup \{a \leftarrow \top\}} = \langle\{l, ab_1, \mathit{ff}, a\}, \{dl, ab_2\}\rangle.$$

Because a is mapped to *true* under this model, the conditional is *true* as well.

More details about the evaluation of conditionals under WCS can be found in [7, 9].

4 Conclusion

I have presented the weak completion semantics (WCS) and have demonstrated how various human reasoning tasks can be adequately modeled under WCS. To the best of my knowledge, WCS is the computational logic based approach which can handle most human reasoning tasks within a single framework. For example, [30] discusses only the selection task in detail and mentions the selection task, whereas [24] discusses the selection task in detail and mentions the suppression task.

But there are many open questions. I only claim that conditionals are adequately evaluated as shown in Section 3.5; this claim must be thoroughly tested. We may also consider scenarios, where abduction needs to be applied to satisfy the consequent of a conditional. The connectionist model reported in 2.6 does not yet include abduction and we are unaware of any connectionist realization of sceptical abduction.

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The suppression task was the running example throughout the development of WCS involving *Carroline*, *Tobias*, *Christoph*, *Emma* and *Marco Ragni*. The solution for the selection task was developed with *Emma* and *Marco*. The approach to model spatial reasoning problems is a revised version of the ideas first developed by Raphael Höps in his bachelor thesis under the supervision of *Emma*; many thanks to *Marco* who introduced us to this problem. *Emma* and *Luís* proposed the solution for the belief bias effect. The procedure to evaluate conditionals is the result of many discussions with *Emma*, *Luís* and *Bob Kowalski*. Finally, I like to thank the referees of the paper for many helpful comments.

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