On a Strong Notion of Viability for Switched Systems

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Abstract. We propose a strong notion of viability for a set of states of a nonlinear switched system. This notion is defined with respect to a fixed region of the state space and can be interpreted as a condition under with a system can be forced to stay in a given safe set by applying a specific control strategy only when its state is outside the fixed region. When the state of the system is inside the fixed region, the control can be kept constant without the risk of driving the system into unsafe set (the complement of the safe set).
We investigate and give a convenient sufficient condition for strong viability of the complement of the origin for a nonlinear switched system with respect to a fixed region.

Keywords. dynamical system, switched system, viability, global-in-time trajectories, control system.

Key Terms. Mathematical Model, Specification Process, Verification Process

1 Introduction

A subset of the state space of a control system is called viable, if for any initial point in this set there exists a solution of the control system which stays forever in this set. Usual problems associated with viability are checking if a given set is viable, finding a solution (and/or the corresponding control input) which stays forever in this set (viable solution), designing a viable region [2]. Viability was studied in many works on the theory of differential equations and inclusions and the control theory [20, 5, 2, 3, 9, 19, 24, 21, 7, 10, 1, 16, 6]. The corresponding results can be straightforwardly applied to control and verification problems for hybrid (discrete-continuous) systems [11] and other models of cyber-physical systems [22, 4, 17, 23], assuming that viable sets are interpreted as safety regions. However, this interpretation suggests certain natural generalizations of the notion of viability. We propose and investigate one such generalization in this paper.

Let $n \geq 1$ be a natural number, I be a non-empty finite set, and $f_i : \mathbb{R} \to \mathbb{R}^n$, $i \in I$ be an indexed family of vector fields.

Let $T = [0, +\infty)$, $\mathcal{I}$ be the set of all functions from T to I which are piecewise-constant on each compact segment $[a, b] \subset T$, and $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. Consider a switched dynamical system [18] of the form

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t)) \tag{1}$$

where, $\sigma \in \mathcal{I}$, $t \geq 0$.

Assume that for each $i \in I$:

1. f_i is continuous and bounded on $[0, +\infty) \times \mathbb{R}^n$;
2. there exists a number $L > 0$ such that $\|f_i(t, x_1) - f_i(t, x_2)\| \leq L \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^n$, $t \in T$, and $i \in I$ (Lipschitz-continuity).

Under these conditions Caratheodory existence theorem [8] implies that for each $t_0 \in T$ and $x_0 \in \mathbb{R}^n$, and $\sigma \in \mathcal{I}$ the problem

$$\frac{d}{dt} x(t) = f_{\sigma(t)}(t, x(t)) \tag{2}$$

$$x(t_0) = x_0 \tag{3}$$

has a Caratheodory solution defined for all $t \geq t_0$, i.e. a function $t \mapsto x(t; t_0; x_0; u)$ which is absolutely continuous on every segment $[a, b] \subset [t_0, +\infty)$, satisfies the equation (2) a.e. (almost everywhere in the sense of Lebesgue measure), and satisfies (3). Moreover, this solution is unique in the sense that for any function $x : [t_0, t_1) \to \mathbb{R}^n$, which is absolutely continuous on every segment $[a, b] \subset [t_0, t_1)$, satisfies (2) a.e. on $[t_0, t_1)$ and satisfies (3), $x(t) = x(t; t_0; x_0; u)$ holds for $t \in [t_0, t_1)$.

For any $X \subseteq \mathbb{R}^n$ and $x_0 \in X$ denote by $VS(X, x_0)$ (set of viable switchings) the set of all $\sigma \in \mathcal{I}$ such that $x(t; 0; x_0; \sigma) \in X$ for all $t \geq 0$;

If $VS(X, x_0) \neq \emptyset$ for each $x_0 \in X$, then X is a viable set of (1) and functions $t \mapsto x(t; 0; x_0; \sigma)$, $\sigma \in VS(X, x_0)$ are viable solutions for X.

Let $Y \subseteq \mathbb{R}^n$ be a set. Let us say that a set $X \subseteq \mathbb{R}^n$ is Y-*strongly viable*, if for each $x_0 \in X$ there exists $\sigma \in VS(X, x_0)$ such that $\sigma(t)$ is constant on each interval $(t_1, t_2) \subset [0, +\infty)$ such that $x(t; 0; x_0; \sigma) \in Y$ for all $t \in (t_1, t_2)$.

In particular, X is viable if and only if X is $\emptyset$-strongly viable. Thus strong viability is a generalization of viability.

This notion has the following natural interpretation: the state of the system (1) can be forced to stay in a given "safe" set X by applying a specific control strategy (σ) only when its state is outside Y. When the state of the system is inside Y, one can keep the control constant (i.e. do not make any switchings) without the risk of driving the system into the "unsafe" region $\mathbb{R}^n \backslash X$. Then Y can be interpreted as a set of states where "nothing specific needs to be done" to ensure safety of the system and the complement of Y can be interpreted as a set of states upon reaching which "something may need to be done" to ensure safety.

In this paper we will consider the case when X is the complement of the origin (i.e. the origin may be interpreted as a safety hazard) and propose a convenient sufficient condition which can be used to verify that for a given system, X, and Y, X is Y-strongly viable.

To do this we will use the notion of a Nondeterministic Complete Markovian System (NCMS) [14] which is based on the notion of a solution system by O. Hájek [12]. More specifically, we will represent the system (1) using a suitable NCMS and reduce the problem of Y-strong viability of a set X to the problem of the existence of global-in-time trajectories of NCMS which was investigated in [14, 15] and apply a theorem about the right dead-end path in NCMS [15] in order to obtain a condition of Y-strong viability.

To make the paper self-contained, in Section 2 we give the necessary definitions and facts about NCMS. In Section 3 we formulate and prove the main result of the paper.

2 Preliminaries

2.1 Notation

We will use the following notation: $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers, $f : A \to B$ is a total function from a set A to a set B, $f : A \tilde{\to} B$ denotes a partial function from a set A to a set B. We will denote by 2^A the power set of a set A and by $f|_A$ the restriction of a function f to a set A.

If A, B are sets, then B^A will denote the set of all total functions from A to B and ${}^A B$ will denote the set of all partial function from A to B.

For a function $f : A \tilde{\to} B$ the symbol $f(x) \downarrow$ ($f(x) \uparrow$) mean that $f(x)$ is defined, or, respectively, undefined on the argument x.

We will not distinguish the notions of a function and a functional binary relation. When we write that a function $f : A \tilde{\to} B$ is total or surjective, we mean that f is total on the set A specifically ($f(x)$ is defined for all $x \in A$), or, respectively, is onto B (for each $y \in B$ there exists $x \in A$ such that $y = f(x)$).

We will use the following notations for $f : A \tilde{\to} B$: $dom(f) = \{x \mid f(x) \downarrow\}$, i.e. the domain of f (note that in some fields like category theory the domain of a partial function is defined differently), and $range(f) = \{y \mid \exists x \ f(x) \downarrow \wedge \ y = f(x)\}$. We will use the same notation for the domain and range of a binary relation: if $R \subseteq A \times B$, then $dom(R) = \{x \mid \exists y \ (x, y) \in R\}$ and $range(R) = \{y \mid \exists x \ (x, y) \in R\}$.

We will denote by $f(x) \cong g(x)$ the strong equality (where f and g are partial functions): $f(x) \downarrow$ if and only if $g(x) \downarrow$, and $f(x) \downarrow$ implies $f(x) = g(x)$.

We will denote by $f \circ g$ the functional composition: $(f \circ g)(x) \cong f(g(x))$.

For any set X and a value y we will denote by $X \mapsto y$ a constant function defined on X which takes the value y.

Also, we will denote by T the non-negative real time scale $[0, +\infty)$ and assume that T is equipped with a topology induced by the standard topology on $\mathbb{R}$.

The symbols $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ will denote the logical operations of negation, disjunction, conjunction, implication, and equivalence respectively.

2.2 Nondeterministic Complete Markovian Systems (NCMS)

The notion of a NCMS was introduced in [13] for studying the relation between the existence of global and local trajectories of dynamical systems. It is close to the notion of a solution system by O. Hájek [12], however there are some differences between these two notions [14].

Denote by $\mathfrak{T}$ the set of all intervals (connected subsets) in T which have the cardinality greater than one.

Let Q be a set (a state space) and Tr be some set of functions of the form $s : A \to Q$, where $A \in \mathfrak{T}$. The elements of Tr will be called *(partial) trajectories*.

Definition 1. *([13, 14]) A set of trajectories Tr is closed under proper restrictions (CPR), if $s|_A \in Tr$ for each $s \in Tr$ and $A \in \mathfrak{T}$ such that $A \subseteq dom(s)$.*

Definition 2. *([13, 14])*

(1) A trajectory $s_1 \in Tr$ is a subtrajectory of $s_2 \in Tr$ (denoted as $s_1 \sqsubseteq s_2$), if $dom(s_1) \subseteq dom(s_2)$ and $s_1 = s_2|_{dom(s_1)}$.
(2) A trajectory $s_1 \in Tr$ is a proper subtrajectory of $s_2 \in Tr$ (denoted as $s_1 \sqsubset s_2$), if $s_1 \sqsubseteq s_2$ and $s_1 \neq s_2$.
(3) Trajectories $s_1, s_2 \in Tr$ are incomparable, if neither $s_1 \sqsubseteq s_2$, nor $s_2 \sqsubseteq s_1$.

The set $(Tr, \sqsubseteq)$ is a (possibly empty) partially ordered set.

Definition 3. *([13, 14]) A CPR set of trajectories Tr is*

(1) Markovian (Fig. 2), if for each $s_1, s_2 \in Tr$ and $t \in T$ such that $t = \sup dom(s_1) = \inf dom(s_2)$, $s_1(t)\downarrow$, $s_2(t)\downarrow$, and $s_1(t) = s_2(t)$, the following function s belongs to Tr:

$$s(t) = \begin{cases} s_1(t), & t \in dom(s_1) \\ s_2(t), & t \in dom(s_2) \end{cases}$$

(2) complete, if each non-empty chain in $(Tr, \sqsubseteq)$ has a supremum.

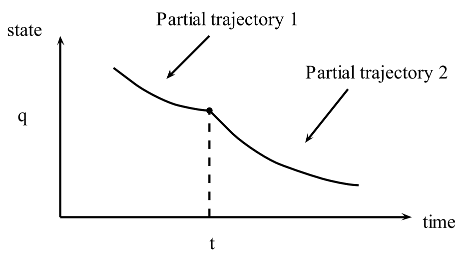

Fig. 1. Markovian property of NCMS. If one trajectory ends and another begins in the state q at time t, then their concatenation is a trajectory.

Definition 4. *([13, 14]) A nondeterministic complete Markovian system (NCMS) is a triple* (T, Q, Tr)*, where* Q *is a set (state space) and* Tr *(trajectories) is a set of functions* $s : T \tilde{\to} Q$ *such that* $dom(s) \in \mathfrak{T}$*, which is CPR, complete, and Markovian.*

An overview of the class of all NCMS can be given using the notion of an *LR representation* [13–15].

Definition 5. *([13, 14]) Let* $s_1, s_2 : T \tilde{\to} Q$*. Then* s_1 *and* s_2 *coincide:*

(1) on a set $A \subseteq T$*, if* $s_1|_A = s_2|_A$ *and* $A \subseteq dom(s_1) \cap dom(s_2)$ *(this is denoted as* $s_1 \doteq_A s_2$*);*
(2) in a left neighborhood of $t \in T$*, if* $t > 0$ *and there exists* $t' \in [0, t)$ *such that* $s_1 \doteq_{(t',t]} s_2$ *(this is denoted as* $s_1 \doteq_{t-} s_2$*);*
(3) in a right neighborhood of $t \in T$*, if there exists* $t' > t$*, such that* $s_1 \doteq_{[t,t')} s_2$ *(this is denoted as* $s_1 \doteq_{t+} s_2$*).*

Let Q be a set. Denote by $ST(Q)$ the set of pairs (s, t) where $s : A \to Q$ for some $A \in \mathfrak{T}$ and $t \in A$.

Definition 6. *([13, 14]) A predicate* $p : ST(Q) \to Bool$ *is*

(1) left-local, if $p(s_1, t) \Leftrightarrow p(s_2, t)$ *whenever* $\{(s_1, t), (s_2, t)\} \subseteq ST(Q)$ *and* $s_1 \doteq_{t-} s_2$ *hold, and, moreover,* $p(s, t)$ *holds whenever* t *is the least element of* $dom(s)$*;*
(2) right-local, if $p(s_1, t) \Leftrightarrow p(s_2, t)$ *whenever* $\{(s_1, t), (s_2, t)\} \subseteq ST(Q)$ *and* $s_1 \doteq_{t+} s_2$ *hold, and, moreover,* $p(s, t)$ *holds whenever* t *is the greatest element of* $dom(s)$*.*

Let $LR(Q)$ be the set of all pairs (l, r), where $l : ST(Q) \to Bool$ is a left-local predicate and $r : ST(Q) \to Bool$ is a right-local predicate.

Definition 7. *([14]) A pair* $(l, r) \in LR(Q)$ *is called a LR representation of a NCMS* $\Sigma = (T, Q, Tr)$*, if*

$$Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A \ l(s, t) \wedge r(s, t))\}.$$

The following theorem gives a representation of NCMS using predicate pairs.

Theorem 1. *([14, Theorem 1])*

(1) Each pair $(l, r) \in LR(Q)$ *is a LR representation of a NCMS with the set of states* Q*.*
(2) Each NCMS has a LR representation.

2.3 Existence global-in-time trajectories of NCMS

The problem of the existence of global trajectories of NCMS was considered in [13, 14] and was reduced to a more tractable problem of the existence of locally defined trajectories. Informally, the method of proving the existence of a global trajectory in NCMS consists of guessing a "region" (subset of trajectories) which presumably contains a global trajectory and has a convenient representation in the form of (another) NCMS and proving that this region indeed contains a global trajectory by finding or guessing certain locally defined trajectories independently in a neighborhood of each time moment.

Below we briefly state the main results about the existence of global trajectories of NCMS described in [15].

Let $\Sigma = (T, Q, Tr)$ be a fixed NCMS.

Definition 8. *([15]) Σ satisfies*

(1) local forward extensibility (LFE) property, if for each $s \in Tr$ of the form $s : [a, b] \to Q$ $(a < b)$ there exists a trajectory $s' : [a, b'] \to Q$ such that $s' \in Tr$, $s \sqsubseteq s'$ and $b' > b$.

(2) global forward extensibility (GFE) property, if for each trajectory s of the form $s : [a, b] \to Q$ there exists a trajectory $s' : [a, +\infty) \to Q$ such that $s \sqsubseteq s'$.

Definition 9. *([15]) A right dead-end path (in Σ) is a trajectory $s : [a, b) \to Q$, where $a, b \in T$, $a < b$, such that there is no $s' : [a, b] \to Q$, $s \in Tr$ such that $s \sqsubset s'$ (i.e. s cannot be extended to a trajectory on $[a, b]$).*

Definition 10. *([15]) An escape from a right dead-end path $s : [a, b) \to Q$ (in Σ) is a trajectory $s' : [c, d) \to Q$ (where $d \in T \cup \{+\infty\}$) or $s' : [c, d] \to Q$ (where $d \in T$) such that $c \in (a, b)$, $d > b$, and $s(c) = s'(c)$. An escape s' is called infinite, if $d = +\infty$.*

Definition 11. *([15]) A right dead-end path $s : [a, b) \to Q$ in Σ is called strongly escapable, if there exists an infinite escape from s.*

Definition 12. *([15])*

(1) A right extensibility measure is a function $f^+ : \mathbb{R} \times \mathbb{R} \tilde{\to} \mathbb{R}$ such that $A = \{(x, y) \in T \times T \mid x \leq y\} \subseteq dom(f^+)$, $f(x, y) \geq 0$ for all $(x, y) \in A$, $f^+|_A$ is strictly decreasing in the first argument and strictly increasing in the second argument, and for each $x \geq 0$, $f^+(x, x) = x$, $\lim_{y \to +\infty} f^+(x, y) = +\infty$.

(2) A right extensibility measure f^+ is called normal, if f^+ is continuous on $\{(x, y) \in T \times T \mid x \leq y\}$ and there exists a function α of class K_∞ (i.e. the function $\alpha : [0, +\infty) \to [0, +\infty)$ is continuous, strictly increasing, and $\alpha(0) = 0$, $\lim_{x \to +\infty} \alpha(x) = +\infty$) such that $\alpha(y) < y$ for all $y > 0$ and the function $y \mapsto f^+(\alpha(y), y)$ is of class K_∞.

An example of a right extensibility measure is $f_1^+(x, y) = 2y - x$.

Let f^+ be a right extensibility measure.

Definition 13. *([15]) A right dead-end path* $s : [a, b) \to Q$ *is called* f^+*-escapable, if there exists an escape* $s' : [c, d] \to Q$ *from* s *such that* $d \geq f^+(c, b)$*.*

Theorem 2. *([15], About right dead-end path) Assume that* f^+ *is a normal right extensibility measure and* Σ *satisfies LFE. Then each right dead-end path is strongly escapable if and only if each right dead-end path is* f^+*-escapable.*

Lemma 1. *([15])* Σ *satisfies GFE if and only if* Σ *satisfies LFE and each right dead-end path is strongly escapable.*

Theorem 3. *([15], Criterion of the existence of global trajectories of NCMS)*

Let (l, r) *be a LR representation of* Σ*. Then* Σ *has a global trajectory if and only if there exists a pair* $(l', r') \in LR(Q)$ *such that*

(1) $l'(s, t) \Rightarrow l(s, t)$ *and* $r'(s, t) \Rightarrow r(s, t)$ *for all* $(s, t) \in ST(Q)$*;*
(2) $\forall t \in [0, \epsilon]\ l'(s, t) \wedge r'(s, t)$ *holds for some* $\epsilon > 0$ *and a function* $s : [0, \epsilon] \to Q$*;*
(3) if (l', r') *is a LR representation of a NCMS* Σ'*, then* Σ' *satisfies GFE.*

3 Main result

Let I, $\mathcal{I}$, and f_i, $i \in I$, and $x(t; t_0; x_0; \sigma)$ be defined as in Section 1. Let $X = \mathbb{R}^n \backslash \{0\}$ and $Y \subset \mathbb{R}^n$ be a set. Let denote $D = \mathbb{R}^n \backslash Y$.

Let us state the main result:

Theorem 4. *Assume that:*

(1) for each $t \in T$ *there exist* $i_1, i_2 \in I$ *such that* $f_{i_1}(t, 0)$ *and* $f_{i_2}(t, 0)$ *are noncollinear;*
(2) $\{0\}$ *is a path-component of* $\{0\} \cup Y$*.*

Then X *is* Y*-strongly viable.*

We will need several lemmas to prove this theorem.

Let us fix an element $x_0^* \in X$.

Let $Q = \mathbb{R}^n \times I$. Denote by $pr_1 : Q \to \mathbb{R}^n$, $pr_2 : Q \to I$ the projections on the first and second component, i.e. $pr_1((x_0, i)) = x_0$ and $pr_2((x_0, i)) = i$.

Let Tr be the set of all functions $s : A \to Q$, where $A \in \mathfrak{T}$, such that the following conditions are satisfied, where $x = pr_1 \circ s$ and $\sigma = pr_2 \circ s$:

1) σ is piecewise-constant on each segment $[a, b] \subseteq A$ $(a < b)$;

2) x is absolutely continuous on each segment $[a, b] \subseteq A$ $(a < b)$ and satisfies the equation $\frac{d}{dt}x(t) = f_i(t, x(t))$ a.e. on A;

3) $x(t) \neq 0$ for all $t \in A$;

4) for each non-maximal $t \in A$ such that $x(t) \notin D$ there exists $t' \in (t, +\infty) \cap A$ such that $\sigma(t'') = \sigma(t)$ for all $t'' \in [t, t')$;

5) for each non-minimal $t \in A$ such that $x(t) \notin D$ there exists $t' \in (0, t) \cap A$ such that $\sigma(t'') = \sigma(t)$ for all $t'' \in (t', t]$;

6) if $0 \in A$, then $x(0) = x_0^*$.

It follows straightforwardly from this definition that $\Sigma(x_0^*) = (T, Q, Tr)$ is a NCMS (i.e. Tr is a CPR, Markovian, and complete set of trajectories).

Let us find a sufficient condition which ensures that Σ has a global trajectory.

Lemma 2. *(1) $\Sigma(x_0^*)$ satisfies the LFE property.*
(2) There exists $s \in Tr$ and $\varepsilon > 0$ such that $dom(s) = [0, \varepsilon]$.

Proof. (1) Let $s : [a, b] \to Q$ be a trajectory, $x = pr_1 \circ s$, and $u = pr_2 \circ s$. Let $\sigma' : [a, +\infty) \to I$ be a function such that $\sigma'(t) = \sigma(t)$, if $t \in [a, b]$ and $\sigma'(t) = \sigma(b)$, if $t > b$. Then $\sigma = \sigma'|_{[a,b]}$, σ' is piecewise-constant on each segment in its domain, and $x(t) = x(t; a; x(a); \sigma')$ for all $t \in [a, b]$. Let $b' = b + 1$ and $x' : [a, b'] \to \mathbb{R}^n$ be a function such that $x'(t) = x(t; a; x(a); \sigma')$ for $t \in [a, b']$. Then $x = x'|_{[a,b]}$. Because $x'(t) \neq 0$ for all $t \in [a, b]$ and x' is continuous, there exists $b'' \in (b, b']$ such that $x'(t) \neq 0$ for all $t \in [a, b'']$. Let $s' : [a, b''] \to Q$ be a function such that $s'(t) = (x'(t), \sigma'(t))$ for all $t \in [a, b'']$. Then it follows immediately that $s' \in Tr$. Besides, $s \sqsubseteq s'$. Thus Σ satisfies LFE.

(2) Let us choose any $i_0 \in I$ and define $x : T \to \mathbb{R}^n$ as $x(t) = x(t; 0; x_0^*; \sigma_0)$ for all $t \in T$, where $\sigma_0(t) = i_0$ for all t. Then x is continuous and $x(0) = x_0^* \neq 0$, so there exists $\varepsilon > 0$ such that $x(t) \neq 0$ for all $t \in [0, \varepsilon]$. Let $s : [0, \varepsilon] \to Q$ be a function $s(t) = (x(t), i_0)$, $t \in [0, \varepsilon]$. Then $s \in Tr$. □

Lemma 3. *Assume that:*

(1) for each $t \in T$ there exist $i_1, i_2 \in I$ such that $f_{i_1}(t, 0)$, $f_{i_2}(t, 0)$ are (nonzero) noncollinear vectors, i.e. $k_1 f_{i_1}(t, 0) + k_2 f_{i_2}(t, 0) \neq 0$ whenever $k_1, k_2 \in \mathbb{R}$ are not both zero;
(2) for each $s \in Tr$ defined on a set of the form $[t_1, t_2)$, if $\lim_{t \to t_2-}(pr_1 \circ s)(t) = 0$, then $pr_1(s(t)) \in D$ for some $t \in [t_1, t_2)$.

Then each right dead-end path in $\Sigma(x_0^*)$ is f_1^+-escapable, where $f_1^+(x, y) = 2y - x$ is a right extensibility measure.

Proof. Let $M' = 1 + \sup\{\|f_i(t', x')\| \,|(t', x') \in T \times \mathbb{R}^n, i \in I\}$. Then $0 < M' < +\infty$, because f is bounded.

Let $s : [a, b) \to Q$ be a right dead-end path and $x = pr_1 \circ s$, $\sigma = pr_2 \circ s$. Let $\sigma' : [a, +\infty) \to I$ be a function such that $\sigma'(t) = \sigma(t)$, if $t \in [a, b)$ and $\sigma'(t) = \sigma(a)$, if $t \geq b$. Then $\sigma = \sigma'|_{[a,b)}$, σ' is Lebesgue-measurable, and $x(t) = x(t; a; x(a); \sigma')$ for all $t \in [a, b)$. Then there exists a limit $x_l = \lim_{t \to b-} x(t) = x(b; a; x(a); \sigma') \in \mathbb{R}^n$.

Firstly, consider the case when $x_l \neq 0$. Then $\|x_l\| > 0$. Let us choose an arbitrary $t_0 \in (a, b)$ such that $b - t_0 < \|x_l\| / (4M')$ and $\|x(t_0) - x_l\| < \|x_l\| / 2$ (this is possible, because $x_l = \lim_{t \to b-} x(t)$). Let $\sigma'' : [t_0, +\infty) \to I$ and $x'' : [t_0, +\infty) \to \mathbb{R}^n$ be functions such that $\sigma''(t) = \sigma(t_0)$ for all $t \geq t_0$ and $x''(t) = x(t; t_0; x(t_0); \sigma'')$ for all $t \geq t_0$. Then $\|x''(t_0)\| = \|x(t_0) - x_l + x_l\| \geq \|x_l\| - \|x(t_0) - x_l\| > \|x_l\| / 2 > 2M'(b - t_0)$. Then for all $t \geq t_0$ we have

$$\|x''(t)\| = \left\| x''(t_0) + \int_{t_0}^{t} f_{\sigma''(t)}(t, x''(t))dt \right\| \geq$$

$$\geq \|x''(t_0)\| - \int_{t_0}^{t} \left\| f_{\sigma''(t)}(t, x''(t)) \right\| dt >$$

$$> 2M'(b - t_0) - M'(t - t_0) = M'(2b - t_0 - t).$$

Let $d = 2b - t_0$. Then $d > t_0$ because $t_0 < b$. Then $x''(t) \neq 0$ for all $t \in [t_0, d]$. Let $s_* : [t_0, d] \to Q$ be a function such that $s_*(t) = (x''(t), \sigma''(t))$ for all $t \in [t_0, d]$. It follows immediately that $s_* \in Tr$. Also, $s_*(t_0) = s(t_0)$ and $d = 2b - t_0 = f_1^+(t_0, b)$. Then s_* is an escape from s and s is f_1^+-escapable.

Now consider the case when $x_l = 0$.

Let us choose $i_1, i_2 \in I$ such that $v_1 = f_{i_1}(b, 0)$ and $v_2 = f_{i_2}(b, 0)$ are noncollinear (this is possible by the assumption 1 of the lemma). Then the function $h(k_1, k_2) = \|k_1 v_1 + k_2 v_2\|$ attains some minimal value $M > 0$ on $\{(k_1, k_2) \in \mathbb{R} \times \mathbb{R} \mid |k_1| + |k_2| = 1\}$. Then for all k_1, k_2 such that $k_1 \neq 0$ or $k_2 \neq 0$,

$$h(k_1, k_2) = (|k_1| + |k_2|) h(k_1(|k_1| + |k_2|)^{-1}, k_2(|k_1| + |k_2|)^{-1}) \geq M(|k_1| + |k_2|).$$

Let $\varepsilon = M/2 > 0$. Because f is continuous, there exists $\delta > 0$ such that for each $j = 1, 2$, $t \in T$, and $x_0 \in \mathbb{R}^n$ such that $|b - t| + \|x_0\| < \delta$ we have $\left\|f_{i_j}(t, x_0) - v_j\right\| = \left\|f_{i_j}(t, x_0) - f_{i_j}(b, 0)\right\| < \varepsilon$. Let $R = \delta/4$, $t_1 = \max\{b - R, a\}$, and $t_2 = b + R$. Then $R > 0$, $a \leq t_1 < b < t_2$ and for all $j = 1, 2$, $t \in [t_1, t_2]$ and x_0 such that $\|x_0\| \leq R$, $\left\|f_{i_j}(t, x_0) - v_j\right\| < \varepsilon$.

Let us choose an arbitrary $c \in (t_1, b)$ such that $b - c < \min\{R/(2M'), R/2\}$. Then $s|_{[c,b)} \in Tr$ by the CPR property and $\lim_{t \to t_2-}(pr_1 \circ s|_{[c,b)})(t) = x_l = 0$, so by the assumption 2 there exists $t_0 \in [c, b)$ such that $pr_1(s(t_0)) = x(t_0) \in D$.

Let $x_1 : [t_0, t_2] \to \mathbb{R}^n$ and $x_2 : [t_0, t_2] \to \mathbb{R}^n$ be functions such that $x_1(t) = x(t; t_0; x(t_0); \sigma_1)$ and $x_2(t) = x(t; t_0; x(t_0); \sigma_2)$ for all $t \in [t_0, t_2]$, where $\sigma_j(t) = i_j$ for all t. Denote $d_j(t) = f_{i_j}(t, x_j(t)) - v_j$ for each $j = 1, 2$ and $t \in [t_0, t_2]$.

Then the following two cases are possible.

a) There exists $j \in \{1, 2\}$ such that $0 \notin range(x_j)$. Let us choose any $d \in (\max\{2b - t_0, t_0\}, t_2)$ (this is possible, because $t_0 < b < t_2$ and $2b - t_0 \leq 2b - c < b + R/2 < b + R = t_2$). Then let $s_* : [t_0, d] \to Q$ be a function such that $s_*(t_0) = s(t_0) = (x(t_0), \sigma(t_0))$ and $s_*(t) = (x_j(t), i_j)$ for all $t \in (t_0, d]$. Because $x_j(t_0) = x(t_0) \in D$ and $x_j(t) \neq 0$ for all $t \in [t_0, t_2] \supset [t_0, d]$, we have that $s_* \in Tr$. Besides, $s_*(t_0) = s(t_0)$ and $d > 2b - t_0 = f_1^+(t_0, b)$, so s_* is an escape from s and s is f_1^+-escapable.

b) $0 \in range(x_1) \cap range(x_2)$. Then because x_1, x_2 are continuous, there exist $t'_j = \min\{t \in [t_0, t_2] \mid x_j(t) = 0\}$ for $j = 1, 2$. Moreover, $t'_j \in (t_0, t_2]$ for $j = 1, 2$, because $x_1(t_0) = x_2(t_0) = x(t_0) \neq 0$.

If we suppose that $\|x_j(t)\| < R$ for each $j = 1, 2$ and $t \in [t_0, t'_j]$, then $\|d_j(t)\| = \left\|f_{i_j}(t, x_j(t)) - v_j\right\| < \varepsilon$ for each $j = 1, 2$ and $t \in [t_0, t'_j]$, whence

$$\|0 - 0\| = \|x_1(t'_1) - x_2(t'_2)\| =$$

$$= \left\| x(t_0) + \int_{t_0}^{t'_1} f_{i_1}(t, x_1(t)) dt - x(t_0) - \int_{t_0}^{t'_2} f_{i_2}(t, x_2(t)) dt \right\| =$$

$$= \left\| \int_{t_0}^{t'_1} v_1 + d_1(t) dt - \int_{t_0}^{t'_2} v_2 + d_2(t) dt \right\| =$$

$$= \left\| v_1(t_1' - t_0) - v_2(t_2' - t_0) + \int_{t_0}^{t_1'} d_1(t)dt - \int_{t_0}^{t_2'} d_2(t)dt \right\| \geq$$

$$\geq \|v_1(t_1' - t_0) - v_2(t_2' - t_0)\| - \int_{t_0}^{t_1'} \|d_1(t)\| \, dt - \int_{t_0}^{t_2'} \|d_2(t)\| \, dt \geq$$

$$\geq M(|t_1' - t_0| + |t_2' - t_0|) - \varepsilon(t_1' - t_0) - \varepsilon(t_2' - t_0) = \frac{M}{2}(t_1' - t_0 + t_2' - t_0) > 0.$$

We have a contradiction, so there exists $j \in \{1, 2\}$ and $t'' \in [t_0, t_j']$ such that $\|x_j(t'')\| \geq R$. This implies that

$$R \leq \|x_j(t'')\| = \|x_j(t_j') - x_j(t'')\| = \left\| \int_{t''}^{t_j'} f_{i_j}(t, x_j(t))dt \right\| \leq M'(t_j' - t'').$$

Then $t_j' - t_0 \geq t_j' - t'' \geq R/M' > 2(b - c) \geq 2(b - t_0)$, so $t_j' > 2b - t_0$. Let us choose any $d \in (\max\{2b - t_0, t_0\}, t_j')$. Let $s_* : [t_0, d] \to Q$ be a function such that $s_*(t_0) = s(t_0) = (x(t_0), \sigma(t_0))$ and $s_*(t) = (x_j(t), i_j)$ for all $t \in (t_0, d]$. Because $x_j(t_0) = x(t_0) \in D$ and $x_j(t) \neq 0$ for all $t \in [t_0, t_j') \supset [t_0, d]$, we have $s_* \in Tr$. Besides, $s_*(t_0) = s(t_0)$ and $d > 2b - t_0 = f_1^+(t_0, b)$, so s_* is an escape from s and s is f_1^+-escapable. □

Lemma 4. *Assume that:*

(1) for each $t \in T$ there exist $i_1, i_2 \in I$ such that $f_{i_1}(t, 0)$ and $f_{i_2}(t, 0)$ are noncollinear;
(2) $\{0\}$ is a path-component of $\{0\} \cup Y$.

Then $\Sigma(x_0^*)$ has a global trajectory.

Proof. Let us show that the assumption 2 of Lemma 3 holds. Let $s \in Tr$, $dom(s) = [t_1, t_2)$ $(t_1 < t_2)$, $\lim_{t \to t_2 -}(pr_1 \circ s)(t) = 0$. Denote $x = pr_1 \circ s$. Suppose that $x(t) \notin D$ for all $t \in [t_1, t_2)$. Let $\gamma : [0, 1] \to \{0\} \cup (\mathbb{R}^n \backslash D)$ be a function such that $\gamma(\varepsilon) = x(t_1 + \varepsilon(t_2 - t_1))$, if $\varepsilon \in [0, 1)$ and $\gamma(1) = 0$. Then γ is continuous, so there is a path from $\gamma(0) = x(t_1) \neq 0$ to 0 in $\{0\} \cup (\mathbb{R}^n \backslash D) = \{0\} \cup Y$ (considered as a topological subspace of $\mathbb{R}^n$). This contradicts the assumption that $\{0\}$ is a path-component of $\{0\} \cup Y$. Thus $x(t) \in D$ for some $t \in [t_1, t_2)$.

The assumption 1 of Lemma 3 also holds, so by Lemma 2, Lemma 3, Lemma 1, Theorem 2, Σ satisfies GFE. Besides, by Lemma 2 there exists $s \in Tr$ with $dom(s) = [0, \varepsilon]$ for some $\varepsilon > 0$, so by the GFE property, Σ has a global trajectory. □

Proof (of Theorem 4). Follows straightforwardly from Lemma 4, because the statement of Lemma 4 holds for any $x_0^* \in X$.

4 Conclusion

We have proposed the notion of an Y-strongly viable set X for nonlinear switched systems. This notion follows naturally from interpretation of viable sets as safety regions. We have considered the case when X is the complement of the origin (i.e. the origin may be interpreted as a safety hazard) and proposed a convenient sufficient condition which can be used to verify that for a given system, X, and Y, X is Y-strongly viable. In the forthcoming papers we plan to investigate other cases give the corresponding conditions.

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