How many times do we need an assumption to prove a tautology in Minimal logic: An example on the compression power of Classical reasoning

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Abstract In this article we present a class of formulas φ_n, $n \in Nat$, that need at least 2^n assumption occurrences to be proved in a normal proof in Natural Deduction for purely implicational minimal propositional logic. In purely implicational classical propositional logic, with Peirce's rule, each φ_n is proved with only one assumption occurrence in Natural Deduction in a normal proof. Besides that, the formulas φ_n have exponentially sized proofs in cut-free Sequent Calculus. In fact 2^n is the lower-bound for normal proofs in ND and cut-free Sequent proofs. We briefly discuss the consequences of the existence of this class of formulas for designing automatic proof-procedures based on these deductive systems.

1 Introduction

Providing proofs for propositional tautologies seems to be a hard task. Huge proofs are such that their size is super-polynomial with regard to the size of their conclusions. Knowing that there is a classical propositional logic tautology having only huge proofs is related to know whether $NP = CoNP$ or not (see [2]). Intuitionistic logic is PSPACE-complete ([11]) and Richard Statman (see [16]) showed that purely implicational minimal logic ($\mathbf{M}_{\rightarrow}$) is PSPACE-complete too. We showed in [8] that, if a propositional logic has a Natural Deduction (ND) with the subformula property then it is in PSPACE. This follows from the fact that $\mathbf{M}_{\rightarrow}$polynomially encodes any propositional logic that has such ND system. Thus, the existence of huge proofs for a more general class of propositional logics is related to the existence of huge proofs in $\mathbf{M}_{\rightarrow}$that amounts to know whether $PSPACE = NP$ or not. The relations between these computational complexity classes and the existence of huge proofs involve arbitrary proof systems, indeed. For example, $NP = PSPACE$ is the case, if and only if, for any $\mathbf{M}_{\rightarrow}$tautology there is a proof system that produces a polynomially sized proof of this tautology.

Dealing with arbitrary and general proof systems is quite hard, and is obviously out of scope of this article. However, studying particular proof systems for key logics, like $\mathbf{M}_{\rightarrow}$or classical logic, can shed some light on practical aspects of implementing propositional theorem provers from the efficiency and economy of storage point of view. $\mathbf{M}_{\rightarrow}$carries almost all the proof-theoretical and logical information to produce polynomially bounded proofs in well-behaved[1] propositional logics. Thus we can conclude that focusing investigations on $\mathbf{M}_{\rightarrow}$is worth of noticing.

There are many proof systems for $\mathbf{M}_{\rightarrow}$. The most well-known are structural/analytic proof systems. Well-known systems are the Sequent Calculus ([4], Natural Deduction ([4] and [13]) and Tableaux ([1,15]) based. These systems, mainly the first and the third kind, are quite good in providing means to produce proofs automatically. The backward chaining procedure, for example, if applied to a Sequent Calculus based proof system provides an automatic way to produce proofs. The problem with these proof procedures is when a decision on which rule to apply has to be made and how to deal with non-provable formulas when it is the case. With respect to this feature of dealing with invalid formulas, the literature on both systems, Sequent Calculus and Tableaux, provides methods that either produce a proof or a counter-model uniformly in a unique proof-procedure. Besides that, since CoPSPACE=PSPACE, providing a counter-model in $\mathbf{M}_{\rightarrow}$is so hard as to provide a proof. We know that $\mathbf{M}_{\rightarrow}$has finite model property and that the size of the counter-model can be super-polynomial with respect to the formula. It is interesting to investigate how this is related to the size of proofs in $\mathbf{M}_{\rightarrow}$, or at least to have a concrete evidence that huge proofs may be the case. Most well-known huge proofs in the literature are considered inside Classical Logic. They are so in Classical as well as in Minimal logic. Our intention is not only show huge proofs in $\mathbf{M}_{\rightarrow}$. To do that, we could use the polynomial translations reported in [16] or [8] to generate a formula of the Pigeon-Hole principle by translating from the full Minimal Logic into $\mathbf{M}_{\rightarrow}$. We know, from [9], that this formula has only super-polynomially sized proofs in Resolution, and hence in cut-free Sequent Calculus and the same happens to the translation to $\mathbf{M}_{\rightarrow}$in cut-free Sequent Calculus and Natural Deduction. It is quite hard to detect from these translations why they are huge in $\mathbf{M}_{\rightarrow}$, since there is nothing specific to $\mathbf{M}_{\rightarrow}$. We believe that directly focusing on $\mathbf{M}_{\rightarrow}$is a promising path, since $\mathbf{M}_{\rightarrow}$has less combinatorial alternatives, less logical constants, less alternative deductive system. The genesis of huge proofs in $\mathbf{M}_{\rightarrow}$is interesting and may shed some new

[1] With sub-formula property

light in propositional logic complexity. This is strongly emphasized by the fact that the formulas shown in this article does not have huge proofs when considered the use of Classical Reasoning, performed by Peirce's rule. This article has the purpose of showing, by means of these formulas, how the use of Classical Logic can improve the size of proofs obtained by an automatic proof procedure of the kind that is able to generated normal and cut-free derivations. In section 3 we introduce the class of formulas and in section 4 we show that they have exponentially sized normal proofs in the usual Natural Deduction for $\mathbf{M}_{\rightarrow}$. In the same section we also show that this is a lower bound in $\mathbf{M}_{\rightarrow}$. In classical propositional logic, these formulas have linear-sized proofs as it is shown in section 3.

All the formal propositional proofs/derivations in this article are presented in Prawitz-style Natural Deduction. The size of these normal proofs/derivations is polynomially simulated by cut-free Sequent Calculus and/or Tableaux. Thus, the lower bound shown here also applies to them.

2 The purely implicational minimal logic

The (purely) implicational minimal logic $\mathbf{M}_{\rightarrow}$is the fragment of minimal logic containing only the logical constant $\rightarrow$. Its semantics is the intuitionistic Kripke semantics restricted to $\rightarrow$ only. Given propositional language $\mathcal{L}$, a $\mathbf{M}_{\rightarrow}$model is a structure $\langle U, \preceq, \mathcal{V}\rangle$, where U is a non-empty set (worlds), $\preceq$ is a partial order relation on U and $\mathcal{V}$ is a function from U into the power set of $\mathcal{L}$, such that if $i, j \in U$ and $i \preceq j$ then $\mathcal{V}(i) \subseteq \mathcal{V}(j)$. Given a model, the satisfaction relationship $\models$ between worlds, in the model, and formulas is defined as:

- $\langle U, \preceq, \mathcal{V}\rangle \models_i p$, $p \in \mathcal{L}$, iff, $p \in \mathcal{V}(i)$
- $\langle U, \preceq, \mathcal{V}\rangle \models_i \alpha_1 \rightarrow \alpha_2$, iff, for every $j \in U$, such that $i \preceq j$, if $\langle U, \preceq, \mathcal{V}\rangle \models_j \alpha_1$ then $\langle U, \preceq, \mathcal{V}\rangle \models_j \alpha_2$.

Obs: In (full) minimal logic, $\bot$ has no special meaning, so there is no item declaring that $\langle U, \preceq, \mathcal{V}\rangle \not\models_i \bot$. We remind that $\mathbf{M}_{\rightarrow}$does not have the $\bot$ in its language.

As usual a formula α is valid in a model $\mathcal{M}$, namely $\mathcal{M} \models \alpha$, if and only if, it is satisfiable in every world i of the model, namely $\forall i \in U \mathcal{M} \models_i \alpha$. A formula is a $\mathbf{M}_{\rightarrow}$tautology, if and only if, it is valid in every model. A formula is satisfiable in $\mathbf{M}_{\rightarrow}$if it is valid in a model $\mathcal{M}$ of $\mathbf{M}_{\rightarrow}$. The problem of knowing whether a formula is satisfiable or not is trivial in $\mathbf{M}_{\rightarrow}$. Every formula is satisfiable in the model $\langle \{\star\}, \preceq, \mathcal{V}\rangle$, where $\star$ is the only world, and $p \in \mathcal{V}(\star)$, for every p. Thus, SAT is not an interesting problem in $\mathbf{M}_{\rightarrow}$. The same cannot be told about knowing whether a formula is a $\mathbf{M}_{\rightarrow}$tautology or not.

It is known that Prawitz Natural Deduction system for minimal logic with only the $\rightarrow$-rules ($\rightarrow$-Elim and $\rightarrow$-Intro below) is sound and complete for the $\mathbf{M}_{\rightarrow}$Kripke semantics. As a consequence of this, Gentzen's LJ system (see [17]) containing only right and left $\rightarrow$-rules is also sound and complete. As it is well-known one of these rules is not invertible[2]. A naive proof-procedure based on backward chaining for $\mathbf{M}_{\rightarrow}$, based only on this usual Gentzen sequent calculus is not possible.

$$\dfrac{\begin{array}{c}[\alpha]\\ \vdots\\ \beta\end{array}}{\alpha \rightarrow \beta}\rightarrow\text{-Intro} \qquad \dfrac{\alpha \qquad \alpha \rightarrow \beta}{\beta}\rightarrow\text{-Elim}$$

3 Needing exponentially many assumptions

In [3] we can find a discussion on the fact that when proving theorems in a logic weaker than classical logic, the need of using an assumption more than once has a strong influence on how complex is the proof procedure and consequently the decision procedure for this logic. There, we can find the formula $((((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow B) \rightarrow B$. Considering the proof systems of ND and CS mentioned in the previous section, this formula needs to use the assumption $((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow B$ at least twice in order to be proved in $\mathbf{M}_{\rightarrow}$. Inspired by this example, we can define a class of formulas with no bounds on the use of assumptions. This shows that limiting the use of assumptions in an automatic proof-procedure for $\mathbf{M}_{\rightarrow}$is not an alternative that ensures completeness. In the sequel we define the class of formulas. Below you find a normal proof of $((((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow B) \rightarrow B$. Note that it cannot be proved with less than 2 use of assumptions $(((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow B$.

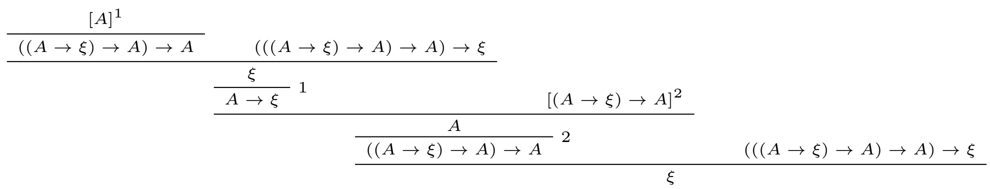
The following formula combines two instances of the formula mentioned above in order to have a formula that needs 4 times an assumption.

$$((((A \rightarrow \xi) \rightarrow A) \rightarrow A) \rightarrow \xi) \rightarrow C \tag{1}$$

where $\xi = (((D \rightarrow C) \rightarrow D) \rightarrow D) \rightarrow C$.

[2] A rule is invertible, iff, whenever the premises are valid the conclusion is valid and whenever any premise is invalid the conclusion is also invalid

In figure 1 we show a normal derivation of this formula 1 above. We can see that it has 4 assumptions of $((A \rightarrow \xi) \rightarrow A) \rightarrow A) \rightarrow \xi)$. They are from the two assumption occurrences in the derivation Σ shown below, that is used twice in the proof in figure 1



We can see how to use this pattern such that if it is repeated n-times we define a formula φ_n, such that, any normal proof of φ_n has to use an assumption at least 2^n times, see section 4. Before we proceed with φ_n definition, we have to show that the need for repeating assumptions is not the case for classical propositional logic.

Consider now that the logic is the purely implicational *classical* logic instead of the purely implicational minimal logic. That is, we consider the $\rightarrow$ introduction and elimination rules, plus the classical absurdity rule, or the Peirce's rule: from $C \rightarrow D \vdash C$ then infer C. Taking into account the version with Peirce's rule, we provide the proof of the formula 1 with only use of assumption, as shown in figure 2. This comes from the fact that $(((D \rightarrow C) \rightarrow D) \rightarrow D)$ in an instance of the implicational form of Peirce's rule, so it is provable. From this proof and $\xi = (((D \rightarrow C) \rightarrow D) \rightarrow D) \rightarrow C$ we prove C. ξ itself is provable by means of a proof of the Peirce's formula $((A \rightarrow \xi) \rightarrow A) \rightarrow A)$ and the $(((A \rightarrow \xi) \rightarrow A) \rightarrow A) \rightarrow \xi$ discharged to proof the desired formula. The purely implicational classical logic is not the focus of this article, in [12] and [7] we can find a detailed presentation of the purely implicational classical logic with some proof-theoretic results. Our discussion on the classical setting has the purpose of showing how the use of classical logic can, in some cases, turns proofs smaller.

4 No bounds for occurrence assumptions in $\mathbf{M}_{\rightarrow}$

In this section we prove that for each n there is a formula φ_n, such that, any normal proof of φ_n has at least 2^n occurrence assumptions of the same formula, that are all of them discharged in only one introduction rule. The following proposition 1 shows that 2^n is an upper bound by showing the normal proof that uses 2^n assumptions for proving φ_n. Theorem 1 shows

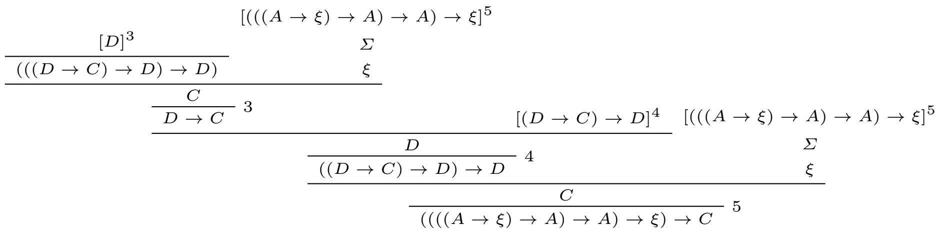


Figure 1. Proof of the formula in purely implicational minimal logic

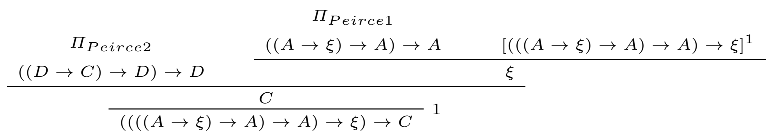


Figure 2. Proof of the formula in purely implicational *classical* logic

that there is no normal proof for any of the φ_n, in $\mathbf{M}_{\rightarrow}$, with less than 2^n assumptions discharged. In the sequel we define φ_n. As it was already said in section 3, φ_n arises from an iteration process derived from the previous examples.

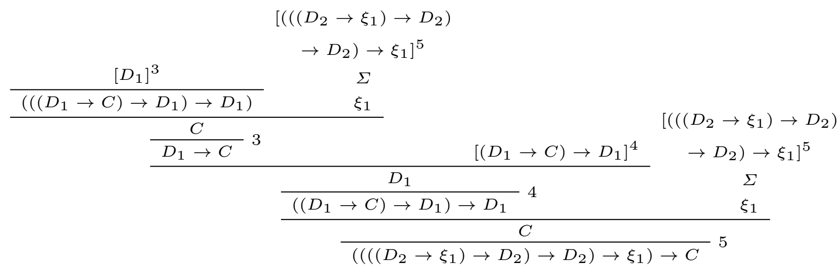
Definition 1. *Let* $\chi[X,Y] = (((X \rightarrow Y) \rightarrow X) \rightarrow X) \rightarrow Y$*. Using* $\chi[X,Y]$ *we define recursively a family of formulas. Consider the propositional letters* C *and* D_i*,* $i > 0$*. Let* ξ_i*,* $i > 0$*, be the formula recursively defined as:*

$$\xi_1 = \chi[D_1, C] \tag{2}$$
$$\xi_{i+1} = \chi[D_{i+1}, \xi_i] \tag{3}$$

Using this family of formulas we define the formula φ_n*,* $n > 0$*, such that, for any* $i \geq 0$*:*

$$\varphi_{i+1} = \xi_{i+1} \rightarrow C$$

We can observe that $\varphi_1 = \xi_1 \rightarrow C$ can be proved by using proof Σ, replacing ξ for C and A for D_1, and applying an $\rightarrow$-introduction as the last rule. The obtained proof has 2 occurrence assumptions of the formula ξ_1. The proof of φ_2 is the proof shown in figure 1, replacing ξ by ξ_1, A by D_2 and D by D_1, resulting in the proof shown below.



The following lemma will be used in the proof of proposition 1.

Lemma 1. *In the formula* ξ_i*,* $i > 0$*, if we simultaneously replace* C *by* ξ_1*, and for each* $k > 0$*,* D_k *by* D_{k+1}*, the resulting formula is* $\chi[D_{i+1}, \xi_i]$*.*

Proof. This lemma is proved by induction on i. For ξ_1 we observe that replacing C by ξ_1 and D_1 by D_2 in ξ_1, the resulting formula is $\chi[D_2, \xi_1]$. Assuming that for $i > 0$, replacing of C by ξ_1 and, for each $k = 1, i$, simultaneously replacing D_i by D_{i+1} in ξ_i, yields $\chi[D_{i+1}, \xi_i]$. Observing that $\xi_{i+1} = \chi[D_{i+1}, \xi_i]$ and by inductive hypothesis, simultaneous replacing of C by ξ_1 and D_k by D_{k+1} in ξ_i, $k = 1, i$, yields ξ_{i+1}. As D_{i+1} does not occur in ξ_i, finally replacing D_{i+1} by D_{i+2} in $\xi_{i+1} = \chi[D_{i+1}, \xi_{i+1}]$ yields $\chi[D_{i+2}, \xi_{i+1}]$. This proves the inductive step.

Another observation is that substitutions as the above shown in the lemma, if applied in a derivation Π in $\mathbf{M}_{\rightarrow}$, do imply that the resulting tree is a valid derivation too. This fact is justified by observing that the replacements are always on atomic formulas and the rules of $\mathbf{M}_{\rightarrow}$do not have provisos to be unsatisfied as consequence of these replacements. Thus,we have the following fact.

Fact 1 *If* Π *is a derivation of* α *from* $\gamma_1, \ldots, \gamma_l$ *and a substitution* $\mathcal{S}$ *(of atomic formulas only) is applied to* Π *then* $\mathcal{S}(\Pi)$ *is a derivation of* $\mathcal{S}(\alpha)$ *from* $\mathcal{S}(\gamma_1), \ldots, \mathcal{S}(\gamma_l)$*. Besides that, if* Π *is normal then* $\mathcal{S}(\Pi)$ *is normal too.*

As φ_1 has two (2^1) occurrences of the same assumption and φ_2 has four (2^2) occurrences of the same assumptions, we have the following result.

Proposition 1. *For any* $n > 0$*, there is a normal proof of* φ_n *having* 2^n *occurrences of the same assumptions, that are discharged by the last rule of the proof.*

Proof. The proof proceeds by induction. The basis $n = 1$ is the proof Σ shown inside proof below. Assuming that φ_i, $i > 0$ has a normal proof Π_{φ_i} having 2^i occurrences of ξ_i discharged by its last inference rule. Thus, we have a normal derivation Π of C from 2^i occurrences of ξ_i, remembering that $\varphi_i = \xi_i \rightarrow C$. We argue that if we simultaneously replace C by ξ_1, and for each $k = 1, i$, replace D_k by D_{k+1}, we will have, by lemma 1 and fact 1, a normal derivation of ξ_1 from 2^i occurrences of $\chi[D_{i+1}, \xi_i]$. Let us call this derivation $\Pi^\star$. The following derivation (see figure 3) is a derivation of C from $((((D_{i+1} \rightarrow \xi_i) \rightarrow D_{i+1}) \rightarrow D_{i+1}) \rightarrow \xi_i) \rightarrow C$, i.e., it is a derivation of C from ξ_{i+1}, and hence, by an $\rightarrow$-introduction of we have a normal derivation of φ_{i+1} discharging $2^i + 2^i = 2^{i+1}$ assumptions of the formula ξ_{i+1}

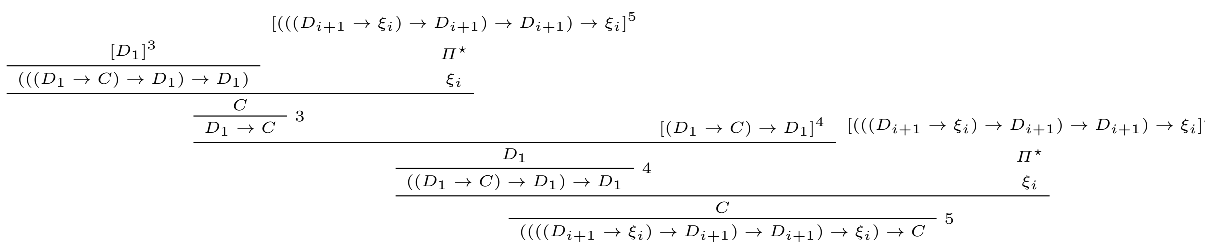


Figure 3. Proof of φ_{i+1} in $\mathbf{M}_{\rightarrow}$with 2^{i+1} discharged assumptions of ξ_{i+1}

Q.E.D.

The following proposition provides 2^i as the lower bound for number of assumption occurrences of a sole formula in proving φ_i by means of normal proofs in $\mathbf{M}_{\rightarrow}$.

Theorem 1. *Any normal proof of φ_i in $\mathbf{M}_{\rightarrow}$has at least 2^i assumption occurrences of ξ_i.*

Proof. We prove that for any i, there is no normal proof of φ_i with less than 2^i assumption occurrences of ξ_i. We first observe that φ_1, i.e., $((((D_1 \rightarrow C) \rightarrow D_1) \rightarrow D_1) \rightarrow C) \rightarrow C$ is not provable with only one occurrence of $\xi_1 = (((D_1 \rightarrow C) \rightarrow D_1) \rightarrow D_1) \rightarrow C)$. If this was the case we would have, from an analysis of the form of the normal proof of C from ξ_1, that $((D_1 \rightarrow C) \rightarrow D_1) \rightarrow D_1$ would be provable in $\mathbf{M}_{\rightarrow}$, and this cannot be the case since this formula is only classically valid. A Kripke model with two worlds such that in the first world neither C nor D_1 holds and in second D_1 holds but not C falsifies $(((D_1 \to C) \to D_1) \to D_1) \to C$.

Consider that there are normal proofs of φ_i with less than 2^i assumption occurrences of ξ_i. So there is the least k ($k > 0$), such that, φ_k has a normal proof with less than 2^k assumption occurrences of ξ_k. Let Σ_k be such proof. Since $\varphi_k = \xi_k \to C$, this proof is as follows. We remember that ξ_k is the only open assumption in Σ_k.

$$\dfrac{\begin{array}{c}[\xi_k]^l \\ \Sigma_k \\ C\end{array}}{\xi_k \to C}\, l$$

Since $\xi_k = \chi[D_k, \xi_{k-1}] = (((D_k \to \xi_{k-1}) \to D_k) \to D_k) \to \xi_{k-1}$, it has to be major premise of an $\to$-elim rule. If this is not the case then ξ_k is minor premise of a $\to$-elim rule having a major premise of the form $\xi_k \to \beta$. This formula on its turn has to be sub-formula of the open assumption of this branch, for the derivation is normal and $\xi_k \to \beta$ can be only conclusion of an application of an $\to$-elim rule. Since the only open assumption in Σ_k is ξ_k itself, the case of ξ_k as minor premise is not possible. Thus, as ξ_k is major premise, Σ_k is of the following form, remembering how is ξ_k, showed in the first line of this paragraph.

$$\dfrac{\dfrac{\begin{array}{c}\Sigma' \\ (((D_k \to \xi_{k-1}) \to D_k) \to D_k)\end{array} \qquad [(((D_k \to \xi_{k-1}) \to D_k) \to D_k) \to \xi_{k-1}]^l}{\begin{array}{c}\xi_{k-1} \\ \Sigma_k \\ C\end{array}}}{\xi_k \to C}\, l$$

Note that Σ' is a sub-derivation of Σ_k and it may have ξ_k as open assumption too, but this is not necessary. If we remove every sub-derivation like Σ' from Σ_k we end up with a proof as following:

$$\dfrac{\begin{array}{c}[\xi_{k-1}]^l \\ \Sigma_{k-1} \\ C\end{array}}{\xi_{k-1} \to C}\, l$$

The proof above is a proof of φ_{k-1} with less than 2^{k-1} assumption occurrences of ξ_{k-1} discharged by the last rule. This contradicts the fact that k is the least number holding this property.

Q.E.D.

5 A brief discussion on counter-model construction in Sequent Calculus for $\mathbf{M}_{\rightarrow}$

Consider the following (incomplete) sequent calculus for $\mathbf{M}_{\rightarrow}$.

$$\frac{}{\Xi, p \Rightarrow p, [\Delta]}\text{ Axiom}$$

$$\frac{\Xi, \gamma_1 \Rightarrow \gamma_2, [\Delta]}{\Xi \Rightarrow \gamma_1 \rightarrow \gamma_2, [\Delta]}\rightarrow\text{-right}$$

$$\frac{\Xi \Rightarrow \alpha, [\gamma, \Delta] \qquad \Xi, \beta \Rightarrow \gamma}{\Xi, \alpha \rightarrow \beta \Rightarrow \gamma, [\Delta]}\rightarrow\text{-left}$$

The formulas in the right-hand side of the sequent and between the brackets are used only for counter-model construction. The main idea is that a sequent of the form $\Xi, p \Rightarrow q, [\Delta]$ having all members of $\Xi \cup \Delta$ as propositional letters and $\{\Xi, p\} \cap \{q, \Delta\} = \emptyset$ is falsified in a Kripke model with two worlds[3]. From this case and using the invertible (if any premise is not valid the conclusion is too) rules of the system it is possible to build a polynomially sized Kripke model for the conclusion of the tree. Remember that in this case we do not have a proof. As already said, this system is incomplete, for it is unable to prove any of the formulas belonging to the class we presented here. The mentioned formulas are only provable if the correct version of $\rightarrow$-left rule is used, as the following, instead of the above.

$$\frac{\Xi, \alpha \rightarrow \beta \Rightarrow \alpha, [\gamma, \Delta] \qquad \Xi, \alpha \rightarrow \beta, \beta \Rightarrow \gamma}{\Xi, \alpha \rightarrow \beta \Rightarrow \gamma, [\Delta]}\rightarrow\text{-left}$$

In this case a counter-model generation is not so obvious, since a loop-detecting mechanism is need.We apologize the lack of a deeper technical discussion due to lack of space. We can, however offer a more intuitive reason. If there were a bound on the use of repeated formulas, we could have used both versions of the $\rightarrow$-left rule for a counter-model generation. Of course, for every formula there is a bound, for example the formula $((((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow B) \rightarrow B$ the bound is 2. What we have shown is that there is no fixed bound for every formula. In fact, if such a fixed bound existed we would have that every $\mathbf{M}_{\rightarrow}$formula would have a polynomially sized search-space to find either proofs or counter-models and this is counter-intuitive.

[3] We cannot provide the details here due to lack of space

6 Conclusion

Our contribution is in the context that $\mathbf{M}_{\to}$is the hardest and most representative propositional logic to define efficient proof-procedures. We show an example alerting for the fact that allowing unlimited use of assumptions is worth for any complete proof-procedure. This example runs in $\mathbf{M}_{\to}$. We are not aware of a similar example for classical logic. In this case classical propositional logic would be more efficient than $\mathbf{M}_{\to}$if such example does not existed. Propositional logic complexity has a lot of conjectures, starting with the relations between the main complexity classes. This article has the sole purpose of providing an example where the exponential grow of proofs has nothing to do with disjunction and combinatorial principles like the Pigeon-Hole[4]. We provided such example.

Developers of theorem provers have to be aware of many aspects of the logic in order to design a efficient system. A system that saves memory and it is fast. Of course, dealing with PSPACE-complete problems is not a so easy task. Any information that can guide the designer is of help. Knowing that the number of copies of a formulas in a proof can be a “bottleneck” for saving memory, an obvious solution would be the use of references instead of copies when representing proofs. The number of references is exponential, but references to formulas are smaller than formulas in most of the cases. This approach points out to the use of graphs (digraphs in fact) for representing proofs. There are a lot of developments done in this direction reported in the literature. Most of them are more semantically than implementation driven. Proof-nets (see [6]) represents an approach that defends the use of graphs as the most adequate representation for proofs. We agree with that and we add a practical motivation for considering digraphs instead of trees for representing proofs (see [14])

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[4] The pigeon-hole principle was used to provide a super-polynomial lower bound for Robinson’s (propositional) Resolution

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