Towards a Logic of Dilation

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Abstract. We investigate the notion of dilation of a propositional theory based on neighbourhoods in a generalized approximation space. We take both a semantic and a syntactic approach in order to define a suitable notion of theory dilation in the context of approximate reasoning on the one hand, and a generalized notion of forgetting in propositional logic on the other hand. We place our work in the context of existing theories of approximation spaces and forgetting, and show that neighbourhoods obtained by combining collective and selective dilation provide a suitable semantic framework within which to reason computationally with uncertainty in a classical setting.

Keywords: dilation; indiscernibility; rough sets; forgetting

1 Introduction

The formalization of resemblance between objects (or sets of objects) in rough set theory [38, 40, 41] is based on the notion of an approximation space, defined by a partition on, or covering of, object attributes. Knowledge about objects is granular, with the partition of equivalence classes or covering of neighbourhoods providing the only discerning measure between objects and determining the granularity of object descriptions.

A related problem is that of information approximation in formal logic. Given a knowledge base $\mathcal{K}$, we ask when a sentence α follows from $\mathcal{K}$, subject to some degree of tolerance on a given subsignature $\mathcal{S}$. This yields a parameterized supra-classical consequence relation. On the one hand, the (atomic features described by) elements of $\mathcal{S}$ may be considered *collectively* irrelevant. This case has been studied in depth in various guises, such as forgetting [33] and rough set theory [40]. On the other hand, each element of $\mathcal{S}$ may be deemed *individually* irrelevant. This case has also been studied in rough set data analysis [53]. Variation between these two extremes gives rise to an increasingly refined granularity of neighbourhoods, applicable to a range of problems in knowledge representation, approximate pattern matching and information retrieval, yet remaining discrete and qualitative in nature.

In this paper we first present a detailed account of selective propositional theory dilation and its syntactic counterpart of selective forgetting. We then generalize and internalize the notion of dilation, which gives rise to a logic of theory dilation. Semantically, our approach is based on neighbourhoods in a generalized approximation space; computationally, it is based on a generalization of the forgetting operator allowing for both *collective* and *selective* dilation.

Similarity and tolerance relations have been studied extensively in knowledge representation, notably in knowledge base merging [3], where they are used as measures to assist in dealing with inconsistencies [28, 43], in approximate reasoning about indiscernible information [42], and as an alternative to nonmonotonic reasoning [16, 17, 30]. More generally, they have been studied in the context of vagueness or incompleteness in information theory [34, 41, 48]. Fuzzy rough information theory has also been studied extensively in this context [45, 49]. (Pawlak and Skowron [41] give a broad overview of rough set theory and applications, including an extensive list of references.)

Although we only consider the propositional case in the present paper, our aim is to extend the work to more expressive knowledge representation formalisms such as modal logics [13], as well as description logics [1] and fragments thereof, by building upon existing modal perspectives on generalized rough sets [32, 50].

The remainder of the present text is structured as follows. In Section 2 we set up the notation used in the paper, and give some background on rough set theory [40, 41] and forgetting [33]. We then show how selective indiscernibility can be defined semantically (Section 3), and syntactically (Section 4). This provides us with the handle needed to define a logic to express dilation at the object level, which we present in Section 5. We then present an outline of the generalization of our work to the infinite case (Section 6), and conclude with a discussion and directions for future investigation.

2 Background

2.1 Formal Preliminaries

In what follows, we assume a classical propositional language $\mathcal{L}$ built up from at most denumerably many variables (or atoms) $\mathcal{P}$, the special constants $\bot$ and $\top$, and the usual connectives. We shall use $p, q, \ldots$ (possibly with subscripts) as meta-variables for the atomic propositions, and $\alpha, \beta, \ldots$ to denote the sentences of our language. Given $\alpha \in \mathcal{L}$, $\mathit{atm}(\alpha)$ denotes the set of propositional variables occurring in α. A *literal* is an atom or the negation of an atom and is denoted by ℓ. Given $\alpha \in \mathcal{L}$, $\mathit{lit}(\alpha)$ denotes the set of literals occurring in α. A *term* (or *diagrammatic sentence*) is a conjunction of literals and is denoted by π. A term π' is a *subterm* of π if $\mathit{lit}(\pi') \subseteq \mathit{lit}(\pi)$.

We denote propositional valuations (or interpretations, or worlds) by $u, v, \ldots : \mathcal{P} \longrightarrow \{0, 1\}$, with 0 denoting falsity and 1 truth. We shall sometimes represent valuations as sequences of 0s and 1s, and with the obvious implicit ordering of atoms. Thus, for the logic generated from p and q, the valuation in which p is true and q is false will be represented as 10. For a given $\alpha \in \mathcal{L}$, $\mathit{Mod}(\alpha) := \{v \mid v \Vdash \alpha\}$ denotes the set of all *models* of α, where $v \Vdash \alpha$ denotes the standard classical satisfaction of α by v. We say that α is *valid* (denoted $\models \alpha$) if $v \Vdash \alpha$ for every valuation v.

A *theory* is a (possibly infinite) set of sentences $\mathcal{T} \subseteq \mathcal{L}$. A *knowledge base* $\mathcal{K}$ is a finite theory. Our primary focus in this paper is on knowledge bases, although some of the definitions and results are also applicable to infinite theories (cf. Section 6). We make the restriction to knowledge bases explicit in all cases. The set of variables occurring in $\mathcal{T}$ is denoted $\mathit{atm}(\mathcal{T})$. $\mathcal{T}$ entails α, written $\mathcal{T} \models \alpha$, if and only if $\mathit{Mod}(\mathcal{T}) \subseteq \mathit{Mod}(\alpha)$. Theories $\mathcal{T}$ and $\mathcal{T}'$ are logically equivalent, written $\mathcal{T} \equiv \mathcal{T}'$, if and only if $\mathit{Mod}(\mathcal{T}) = \mathit{Mod}(\mathcal{T}')$.

2.2 Indiscernibility

The basic building block of rough set theory is the notion of an *approximation space* of the form $\langle \mathcal{U}, \Theta \rangle$, where $\mathcal{U}$ is a set of objects and Θ an equivalence relation on $\mathcal{U}$. The intuition is that elements of $\mathcal{U}$ can only be distinguished up to their respective equivalence classes, while objects from the same equivalence class are indistinguishable.

Relaxation of the symmetry and transitivity conditions on Θ has been studied in a number of different contexts, amongst which is the study of *tolerance relations*, which may fail transitivity [36, 42, 44]. Our building blocks are now *tolerance spaces* of the form $\langle \mathcal{U}, \Omega \rangle$, where $\mathcal{U}$ is a set and Ω is a reflexive and symmetric *tolerance* relation on $\mathcal{U}$. As before, the intuition is that elements of $\mathcal{U}$ can only be differentiated if they are not Ω-related to each other. The set $\Omega(x) := \{y \mid (x, y) \in \Omega\}$ is referred to as the *neighbourhood* of x [48].

With each tolerance space $\langle \mathcal{U}, \Omega \rangle$ we associate two operators: If $X \subseteq \mathcal{U}$, then $\overline{X} := \{x \in \mathcal{U} \mid \Omega(x) \cap X \neq \emptyset\}$ is the *dilation* of X, and its *erosion* is $\underline{X} := \{x \in \mathcal{U} \mid \Omega(x) \subseteq X\}$. If Ω is also transitive, the dilation and erosion operators coincide with the upper and lower approximation operators of an approximation space.

From now on, let $\mathcal{U}$ denote the set of all propositional valuations. It then immediately follows that, for any $\mathcal{S} \subseteq \mathcal{P}$ and

$$\Theta_{\mathcal{S}} := \{(u, v) \mid u \Vdash p \text{ if and only if } v \Vdash p, \text{ for all } p \in \mathcal{P} \setminus \mathcal{S}\},$$

$\langle \mathcal{U}, \Theta_{\mathcal{S}} \rangle$ is an approximation space.

Not every tolerance (or approximation) space defines some theory $\mathcal{T}$. In Section 3 we describe a class of tolerance spaces that do arise syntactically from some notion of indiscernibility amongst atoms. We next present some background on an elegant way to capture indiscernibility syntactically.

2.3 Forgetting and Irrelevance

Lin and Reiter [33] introduced the notion of forgetting a set of predicates in a first-order theory in the context of cognitive robotics. We present a propositional version of forgetting here, which has since been studied extensively, especially in the context of modal logics [15, 51] and description logics [47]. Intuitively, the result of forgetting a set of atoms $\mathcal{S}$ should be weaker than the original knowledge base, but still entail the same set of sentences that are irrelevant to the signature $\mathcal{S}$. As pointed out by Lin and Reiter, the notions of forgetting, irrelevance, and equivalence of interpretations are strongly related.

Definition 1. *Let $\mathcal{K}$ be a knowledge base and $\mathcal{S} \subseteq \mathcal{P}$. Let $\langle \mathcal{U}, \Theta_{\mathcal{S}} \rangle$ be the approximation space with $\Theta_{\mathcal{S}} := \{(u, v) \mid u \Vdash p$ if and only if $v \Vdash p$, for all $p \in \mathcal{P} \setminus \mathcal{S}\}$. A knowledge base $\mathcal{K}'$ is a result of (conjunctively) forgetting about $\mathcal{S}$ in $\mathcal{K}$ if and only if*

$$Mod(\mathcal{K}') = \{w \mid Mod(\mathcal{K}) \cap \Theta_{\mathcal{S}}(w) \neq \emptyset\}.$$

Hence the $\mathcal{K}'$-models are all the worlds that are indistinguishable from some $\mathcal{K}$-model with respect to $\mathcal{S}$.

Theorem 1 (Lin & Reiter [33]). *The result of forgetting always exists and is unique modulo logical equivalence.*

This result allows us to write $\mathit{Forget}_{\mathcal{S}}(\mathcal{K})$ for the semantically unique result of forgetting all the atoms in $\mathcal{S}$ in the knowledge base $\mathcal{K}$. If $\mathcal{K}$ is a singleton set, say $\mathcal{K} = \{\alpha\}$, we write $\mathit{Forget}_{\mathcal{S}}(\alpha)$ as shorthand for $\mathit{Forget}_{\mathcal{S}}(\{\alpha\})$, some set of sentences that has the intended set of models.

Definition 2. *Let $\mathcal{T} \subseteq \mathcal{L}$ and $\mathcal{S} \subseteq \mathcal{P}$. $\mathcal{S}$ is (collectively) irrelevant to $\mathcal{T}$ if and only if there exists a theory $\mathcal{T}'$, logically equivalent to $\mathcal{T}$, such that $\mathit{atm}(\mathcal{T}') \cap \mathcal{S} = \emptyset$.*

Theorem 2 (Lin & Reiter [33]). *Let $\mathcal{K}$ be a knowledge base and $\mathcal{S} \subseteq \mathcal{P}$. Then*

$$\mathit{Forget}_{\mathcal{S}}(\mathcal{K}) \equiv \{\alpha \mid \mathcal{K} \models \alpha \text{ and } \mathcal{S} \text{ is irrelevant to } \alpha\}.$$

A closely related notion to irrelevance is that of *essential atoms* [39]. We say that an atom p is *essential* to a theory $\mathcal{T}$ if and only if $p \in \mathit{atm}(\mathcal{T}')$ for every $\mathcal{T}'$ such that $\mathcal{T} \equiv \mathcal{T}'$. For instance, p is essential to $\mathcal{T} = \{\neg p, \neg p \vee q\}$. Given $\mathcal{T}$, $\mathit{atm}!(\mathcal{T})$ denotes the set of essential atoms of $\mathcal{T}$. (If $\mathcal{T}$ is not contingent, i.e., $\mathcal{T}$ is tautological or contradictory, then $\mathit{atm}!(\mathcal{T}) = \emptyset$.)

Given a theory $\mathcal{T}$, let $\mathcal{T}^! := \{\alpha \mid \mathcal{T} \models \alpha \text{ and } \mathit{atm}(\alpha) \subseteq \mathit{atm}!(\mathcal{T})\}$. Clearly, $\mathit{atm}(\mathcal{T}^!) = \mathit{atm}!(\mathcal{T}^!)$. Moreover, for every $\alpha \in \mathcal{L}$ such that $\mathcal{T} \equiv \alpha$, $\mathit{atm}!(\mathcal{T}) = \mathit{atm}!(\alpha)$ and $\mathcal{T}^! = \{\alpha\}^!$.

Theorem 3 (Least Atom-Set Theorem [39]). *Let $\mathcal{T} \subseteq \mathcal{L}$. Then $\mathcal{T} \equiv \mathcal{T}^!$, and for every α such that $\mathcal{T} \equiv \{\alpha\}$, $\mathit{atm}(\mathcal{T}^!) \subseteq \mathit{atm}(\alpha)$.*

A proof of this theorem is given by Makinson [35]. Essentially, it establishes that, for every theory $\mathcal{T}$, there is a unique least set of elementary atoms such that $\mathcal{T}$ may be expressed equivalently using only atoms from that set. Hence, $\mathcal{T} \equiv \mathcal{T}^!$.

In general, theories may be infinite, and the semantic characterization of forgetting of Definition 1 may therefore not be applicable. (See Section 6 for the infinite case.) If $\mathcal{T}$ is finite, there are only finitely many atoms essential to $\mathcal{T}$, and only these atoms affect the result of forgetting $\mathcal{S}$ in $\mathcal{T}$.

3 Selective Indiscernibility

We have seen that any set of atoms naturally yields an indiscernibility relation on valuations which is an equivalence relation. By weakening the transitivity condition, we now replace the equivalence relation with a range of increasingly fine-grained neighbourhood relations.

Definition 3. *Let Ω be a reflexive and symmetric binary relation on $\mathcal{U}$, and let Ω^0 be the identity relation on $\mathcal{U}$. For $n \geq 1$, the n-transitive closure of Ω is the smallest relation Ω^n such that*

(i) $\Omega \subseteq \Omega^n$, *and*
(ii) *if* $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n) \in \Omega$, *then* $(x_0, x_n) \in \Omega^n$.

Equivalently, $\Omega^n := \bigcup\{\Psi^* \mid \Psi \subseteq \Omega \text{ and } |\Psi| = n\}$, where Ψ^* is the transitive closure of Ψ. From a morphological perspective [4], Ω^n is a structuring element, which can also be defined as a *neighbourhood function* $\Omega^n : \mathcal{U} \longrightarrow 2^{\mathcal{U}}$ based on the Manhattan distance measure, with $\Omega^n(x) = \{y \mid (x, y) \in \Omega^n\}$ the image of x under the relation Ω^n.

The n-transitive closure of a reflexive and symmetric binary relation provides a mechanism to describe neighbourhood-based rough sets whose upper and lower bounds are determined by a tolerance relation that exhibits only a limited degree of transitivity. As we shall see in Section 4, it also provides a semantic characterization for an operation akin to forgetting.

We next show how a theory and a set of atoms give rise to a range of tolerance spaces. First, we consider all functions $\mathcal{C}_\mathcal{S}$ picking out n elements from a given set $\mathcal{S}$:

Definition 4. *Let $\mathcal{S} \subseteq \mathcal{P}$, $\mathcal{C}_\mathcal{S} : \{0, \ldots, |\mathcal{S}|\} \longrightarrow 2^{\mathcal{S}}$ is a function such that $\mathcal{C}_\mathcal{S}(n) \mapsto \mathcal{C}^n_\mathcal{S}$, where $\mathcal{C}^n_\mathcal{S} \subseteq \mathcal{S}$ and $|\mathcal{C}^n_\mathcal{S}| = n$.*

Lemma 1. *Let $\mathcal{S} \subseteq \mathcal{P}$ and $0 \leq n \leq |\mathcal{S}|$. For this fixed $\mathcal{S}$ and n, let*

$$\Omega^n := \bigcup_{\mathcal{C}^n_\mathcal{S} \subseteq \mathcal{S}} \{(u, v) \mid u \Vdash p \textit{ if and only if } v \Vdash p, \textit{ for all } p \in \mathcal{P} \setminus \mathcal{C}^n_\mathcal{S}\}.$$

Then $\langle \mathcal{U}, \Omega^n \rangle$ is a tolerance space. Moreover, if $n = 0$, Ω^0 is the identity relation on $\mathcal{U}$, and for $n \geq 1$, Ω^n is the n-transitive closure of

$$\Omega := \{(u, v) \mid \textit{ for some } q \in \mathcal{S}, u \Vdash p \textit{ if and only if } v \Vdash p, \textit{ for all } p \in \mathcal{P} \setminus \{q\}\}.$$

Corollary 1. *If $n = |\mathcal{S}|$, then Ω^n is an equivalence relation.*

Example 1. Let $\mathcal{P} = \{p, q, r, s\}$ and let $\mathcal{S} = \{p, q, r\}$. Then $\mathcal{C}^0_\mathcal{S} = \emptyset$, $\mathcal{C}^1_\mathcal{S}$ is one of the singleton sets $\{p\}$, $\{q\}$ or $\{r\}$, $\mathcal{C}^2_\mathcal{S}$ is one of the sets $\{p, q\}$, $\{p, r\}$ or $\{q, r\}$, and $\mathcal{C}^3_\mathcal{S} = \{p, q, r\}$. Ω^0 is the identity relation on $\mathcal{U}$, Ω^1 is the set of all those pairs of valuations that agree everywhere except on at most one element of $\mathcal{S}$, Ω^2 is the set of all those pairs of valuations that agree everywhere except on at most two elements of $\mathcal{S}$, and Ω^3 is the set of all those pairs of valuations that agree on the proposition s. Note that Ω^0 and Ω^3 are equivalence relations, but that Ω^1 and Ω^2 are not.

4 Selective Forgetting

Forgetting, as traditionally studied in the literature, is *conjunctive*, in the sense that all the elements of a given set of atoms are forgotten, or deemed irrelevant. Syntactically, forgetting $\mathcal{S}$ from $\mathcal{K}$ yields a knowledge base in which *none* of the atoms from $\mathcal{S}$ occur. Semantically, forgetting $\mathcal{S}$ yields an approximation space with indiscernibility relation $\Theta_\mathcal{S}$ as outlined in Section 2.2.

We now define the more general problem of forgetting any n atoms from $\mathcal{S}$ in $\mathcal{K}$, and show that this describes the syntactic counterpart of tolerance spaces in the same way as standard forgetting is the syntactic counterpart of approximation spaces. The term 'forgetting' is not really appropriate here, since our aim is not to forget atoms but rather to *dilate* theories semantically, but we retain it because of the computational link with forgetting that we shall establish.

Definition 5. *Let $\mathcal{K}$ be a knowledge base, $\mathcal{S} \subseteq \mathcal{P}$ and $0 \leq n \leq |\mathcal{S}|$. Let $\langle \mathcal{U}, \Omega^n \rangle$ be the tolerance space with*

$$\Omega^n = \bigcup_{\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}} \{(u, v) \mid u \Vdash p \textit{ if and only if } v \Vdash p, \textit{ for all } p \in \mathcal{P} \setminus \mathcal{C}^n_{\mathcal{S}}\}.$$

A knowledge base $\mathcal{K}'$ is a result of selective forgetting at most n atoms from $\mathcal{S}$ in $\mathcal{K}$ iff

$$\mathit{Mod}(\mathcal{K}') = \{w \mid \mathit{Mod}(\mathcal{K}) \cap \Omega^n(w) \neq \emptyset\},$$

where $\Omega^n(w)$ denotes the image of w under Ω^n.

We are now ready to state the most important result of this section.

Theorem 4. *The result of selective forgetting always exists and is unique modulo logical equivalence.*

Proof. We know that, for a fixed $\mathcal{S}$ and $0 \leq n \leq |\mathcal{S}|$, $\mathit{Mod}(\mathit{Forget}_{\mathcal{C}^n_{\mathcal{S}}}(\mathcal{K})) = \{w \mid \mathit{Mod}(\mathcal{K}) \cap \Theta_{\mathcal{C}^n_{\mathcal{S}}}(w) \neq \emptyset\}$ from Definition 1. On the lefthand side of this equality, letting $\mathcal{C}^n_{\mathcal{S}}$ range over all possible choices from $\mathcal{S}$, we obtain $\bigvee_{\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}} \bigwedge \mathit{Forget}_{\mathcal{C}^n_{\mathcal{S}}}(\mathcal{K})$ syntactically, while on the righthand side, we obtain $\{w \mid \mathit{Mod}(\mathcal{K}) \cap \Omega^n(w) \neq \emptyset\}$ for some $\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}$ semantically. Therefore, let $\mathcal{K}' := \{\bigvee_{\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}} \bigwedge \mathit{Forget}_{\mathcal{C}^n_{\mathcal{S}}}(\mathcal{K})\}$. Then it follows that $\mathit{Mod}(\mathcal{K}') = \{w \mid \mathit{Mod}(\mathcal{K}) \cap \Omega^n(w) \neq \emptyset\}$. $\square$

This result allows us to write $\mathit{SForget}^n_{\mathcal{S}}(\mathcal{K})$ for the (semantically unique) result of selective forgetting any n atoms from $\mathcal{S}$ in $\mathcal{K}$. It then follows from Lemma 1 that $\mathit{SForget}^n_{\mathcal{S}}(\mathcal{K})$ arises from a tolerance space $\langle \mathcal{U}, \Omega^n \rangle$ in which Ω^n has a particular structure, namely, it is the n-transitive closure of the tolerance relation associated with $\mathit{SForget}^1_{\mathcal{S}}(\mathcal{K})$.

Corollary 2. *Let $\langle \mathcal{U}, \Omega^n \rangle$ be the tolerance space with Ω^n as in Definition 5. Then $\mathit{Mod}(\mathit{SForget}^n_{\mathcal{S}}(\mathcal{K})) = \overline{\mathit{Mod}(\mathcal{K})}$.*

The semantic intuition of selective forgetting is that $\overline{\mathit{Mod}(\mathcal{K})}$ represents the weakest knowledge base resembling $\mathcal{K}$, with indiscernibility determined by a given set of atoms $\mathcal{S}$ and degree of transitivity n.

As in the case of forgetting, we obtain a syntactic characterization of selective forgetting when $\mathcal{K}$ is written as a single sentence τ:

Corollary 3. *Let $\tau \in \mathcal{L}$, $\mathcal{S} \subseteq \mathcal{P}$ and $0 \leq n \leq |\mathcal{S}|$. $\mathit{SForget}^n_{\mathcal{S}}(\tau)$ can be characterized as the singleton set $\mathit{SForget}^n_{\mathcal{S}}(\tau) := \left\{\bigvee_{\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}} \bigwedge \mathit{Forget}_{\mathcal{C}^n_{\mathcal{S}}}(\tau)\right\}$.*

With this result, together with Theorem 5 below, we obtain a method to compute selective forgetting via disjunctive normal form (DNF). Although computing the DNF of a sentence is itself computationally expensive, once preprocessing has been done, it provides an attractive alternative to computing forgetting directly.

Definition 6. *Let π be a term and let $\mathcal{S} \subseteq \mathcal{P}$. Then $\pi^-_{\mathcal{S}} := \bigwedge_{\ell \in \mathit{lit}(\pi), \mathit{atm}(\ell) \notin \mathcal{S}} \ell$.*

Let $\alpha[p/\bot]$ and $\alpha[p/\top]$ denote the sentences obtained from α by replacing the atom p with $\bot$ and $\top$, respectively.

Lemma 2 (Lang et al. [31]). *Let π be a consistent term and let $p \in \mathcal{P}$. Then $\pi^-_{\{p\}} \equiv \pi[p/\top] \vee \pi[p/\bot]$.*

Let $\mathit{DNF}(\tau)$ denote (some) DNF representation of τ as a set of terms. The proof of the theorem below then follows from iterative applications of Lemma 2.

Theorem 5 (Lang et al. [31]). $\mathit{Forget}_{\mathcal{S}}(\tau) \equiv \left\{\bigvee_{\pi \in \mathit{DNF}(\tau)} \pi^-_{\mathcal{S}}\right\}$.

Corollary 4. *Let $\tau \in \mathcal{L}$, $\mathcal{S} \subseteq \mathcal{P}$ and $0 \leq n \leq |\mathcal{S}|$. Then*

$$\mathit{SForget}^n_{\mathcal{S}}(\tau) \equiv \left\{\bigvee_{\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}} \bigvee_{\pi \in \mathit{DNF}(\tau)} \pi^-_{\mathcal{C}^n_{\mathcal{S}}}\right\}.$$

If τ has been pre-compiled as a disjunction of its prime implicants [37], the size of $\mathit{SForget}^n_{\mathcal{S}}(\tau)$ is at most $|\mathcal{S}|^n$ times the size of τ. Some post-processing may be required to remove redundant clauses to rewrite the result as a disjunction of prime implicants, but it is not hard to see that this does not affect the overall complexity of computation.

As an example, we consider the special case where $n = 1$. Given a knowledge base $\mathcal{K}$ and a set of atoms $\mathcal{S}$, our aim is to define $\mathcal{K}'$, first semantically and then syntactically, obtained from $\mathcal{K}$ by non-deterministically disregarding one atom from $\mathcal{S}$. The semantic characterization of $\mathcal{K}'$ follows directly from Definition 5. The tolerance space induced by $\mathcal{S}$ is $\langle \mathcal{U}, \Omega \rangle$, with Ω the set of all pairs of valuations that differ on at most one atom from $\mathcal{S}$. The upper approximation of $\mathit{Mod}(\mathcal{K})$ is $\mathit{Mod}(\mathcal{K}')$:

$$\overline{\mathit{Mod}(\mathcal{K})} = \mathit{Mod}(\mathcal{K}') = \bigcup \{\Omega(w) \mid \Omega(w) \cap \mathit{Mod}(\mathcal{K}) \neq \emptyset\}.$$

Finally, if $\mathcal{K} = \{\tau\}$, then $\mathcal{K}'$ may be characterized syntactically as follows:

$$\mathit{SForget}^1_{\mathcal{S}}(\tau) := \left\{\bigvee_{p \in \mathcal{S}} \tau[p/\bot] \vee \tau[p/\top]\right\}.$$

We illustrate these relationships in a simple concrete example:

Example 2. Let $\mathcal{P} = \mathcal{K} = \{p, q, r, s\}$ and $\mathcal{S} = \{p, q, r\}$. $\mathit{Forget}_{\mathcal{S}}(\mathcal{K}) = \mathit{SForget}^3_{\mathcal{S}}(\mathcal{K}) = \{s\}$, $\mathit{SForget}^2_{\mathcal{S}}(\mathcal{K}) = \{p \vee q \vee r, s\}$, $\mathit{SForget}^1_{\mathcal{S}}(\mathcal{K}) = \{(p \wedge q) \vee (p \wedge r) \vee (q \wedge r), s\}$, and $\mathit{SForget}^0_{\mathcal{S}}(\mathcal{K}) = \{p, q, r, s\}$. Clearly, selective forgetting does not eliminate the atoms that are forgotten syntactically from a knowledge base. Its effect is best thought of semantically: In our example, $\mathit{Mod}(\mathcal{K}) = \{1111\}$. To obtain $\mathit{Mod}(\mathit{SForget}^1_{\mathcal{S}}(\mathcal{K}))$, add all worlds that differ from 1111 in at most one atom, which must be from $\mathcal{S}$. This gives $\mathit{Mod}(\mathit{SForget}^1_{\mathcal{S}}(\mathcal{K})) = \{1111, 1101, 1011, 0111\}$. Similarly, to get $\mathit{Mod}(\mathit{SForget}^2_{\mathcal{S}}(\mathcal{K}))$, add all worlds that differ from 1111 in at most two atoms, which must be from $\mathcal{S}$, and to obtain $\mathit{Mod}(\mathit{Forget}_{\mathcal{S}}(\mathcal{K}))$, add all worlds that differ from 1111 only on $\mathcal{S}$.

5 A Neighbourhood Semantics for a Logic of Dilation

Both standard forgetting and selective forgetting are operators at the *meta*-level in the sense that we cannot refer to or explicitly use them in the logical language. In this section we turn our attention to internalizing the notion of dilation in the object level, which gives rise to a logic within which one can explicitly express and reason about dilation. (Our motivation is similar to that of internalizing belief revision [21] in modal logic [46] or the notions of typicality [5, 6, 20] and relative normality [2, 7, 9, 11, 12] in nonmonotonic reasoning.)

We have seen in the previous sections that dilation is carried out relative to a given set of atoms and that these can be considered either *collectively* or *selectively*. Hence, operators internalizing the notion of dilation should be parameterized by a given signature and in this way inform the operation. Moreover, they should also allow for more complex operations involving both collective and selective dilation, as in e.g. "dilate a theory by p and q or by r and s".

We first define the grammar of the dilation operators. Below, ε denotes the empty string and $p \in \mathcal{P}$. With $\cdot$ and $+$ we denote, respectively, collective and selective dilation as motivated above.

$$\mathrm{o} \ ::= \varepsilon \mid p \mid (\mathrm{o} \cdot \mathrm{o}) \mid (\mathrm{o} + \mathrm{o})$$

Assuming $\mathcal{P} = \{p, q, r\}$, examples of dilation operators generated by the above grammar are $(p \cdot (q + p))$ and $((p \cdot p) + (q \cdot (p + r)))$. (For the sake of readability, in what follows we shall assume that $\cdot$ has precedence over $+$ and therefore we shall omit some unnecessary parentheses. We shall also omit the outermost parentheses in operators.) We shall use $\mathrm{o}, \varrho, \sigma, \ldots$ to denote operators generated by the grammar. With $\mathcal{O}$ we denote the operator language generated as above. Given $p \in \mathcal{P}$ and $\mathrm{o} \in \mathcal{O}$, with $p \in \mathrm{o}$ we denote the fact that p appears in (is a symbol of) the operator o.

An *atomic* dilation operator in $\mathcal{O}$ is either an atom $p \in \mathcal{P}$ or it is ε. A *primitive collective* dilation operator is any operator of the form $\varrho_1 \cdot \ldots \cdot \varrho_k$, $k \geq 1$, where each ϱ_i is an atomic dilation operator. A dilation operator is in *dilation normal form* if it has the form $\sigma_1 + \ldots + \sigma_n$, $n \geq 1$, where each σ_i is a primitive collective dilation operator.

Given the set of all dilation operators $\mathcal{O}$ as defined above, we can extend our underlying propositional language in the following way:

$$\alpha \ ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha^{\mathrm{o}}$$

All the other Boolean truth-functional connectives ($\vee$, $\rightarrow$, $\leftrightarrow$, $\ldots$) are defined in terms of $\neg$ and $\wedge$ in the usual way. We use $\top$ as an abbreviation for $p \vee \neg p$ and $\bot$ as an abbreviation for $p \wedge \neg p$, for some $p \in \mathcal{P}$. With $\mathcal{L}^{\mathcal{O}}$ we denote the set of all sentences of our extended language.

A sentence of the form α^{o} is read "the dilation of α by o". The semantics of sentences of $\mathcal{L}^{\mathcal{O}}$ is in terms of *neighbourhoods*, which we define more precisely below.

Definition 7. *The neighbourhood space of $\mathcal{L}^{\mathcal{O}}$ is a mapping $\Omega : \mathcal{O} \longrightarrow 2^{\mathcal{U}\times\mathcal{U}}$ s.t.:*

- $\Omega_\varepsilon = \{(v, v) \mid v \in \mathcal{U}\}$*;*
- $\Omega_p = \{(v, w) \mid v(q) = w(q)$ *for all* $q \in \mathcal{P} \setminus \{p\}\}$*;*

– $\Omega_{\varrho\cdot\sigma} = (\Omega_\varrho \cup \Omega_\sigma)^*$, *the reflexive transitive closure of* $\Omega_\varrho \cup \Omega_\sigma$*;*
– $\Omega_{\varrho+\sigma} = \Omega_\varrho \cup \Omega_\sigma$.

Given $\mathrm{o} \in \mathcal{O}$, we abbreviate $\Omega(\mathrm{o})$ by Ω_o as in Definition 7. Also, given $w \in \mathcal{U}$, Ω_o defines (by a slight abuse of notation) a neighbourhood function $\Omega_\mathrm{o} : \mathcal{U} \longrightarrow 2^{\mathcal{U}}$, with $\Omega_\mathrm{o}(w)$ the image of w under Ω_o. The following properties of Ω are worthy of mention:

– $\Omega_{\varrho\cdot\sigma} = \Omega_{\sigma\cdot\varrho}$ and $\Omega_{\varrho+\sigma} = \Omega_{\sigma+\varrho}$ (Commutativity)
– $\Omega_{\mathrm{o}\cdot(\varrho\cdot\sigma)} = \Omega_{(\mathrm{o}\cdot\varrho)\cdot\sigma}$ and $\Omega_{\mathrm{o}+(\varrho+\sigma)} = \Omega_{(\mathrm{o}+\varrho)+\sigma}$ (Associativity)

It is not hard to see that distributivity of $\cdot$ over $+$ and of $+$ over $\cdot$ do not hold in general. However, we have the following additional properties:

Lemma 3 (Normal Form).

– *For every* $\varrho, \sigma \in \mathcal{O}$*, there exists some primitive collective dilation operator* o *such that* $\Omega_{\varrho\cdot\sigma} = \Omega_\mathrm{o}$*;*
– *For every* $\varrho \in \mathcal{O}$ *there exists* $\sigma \in \mathcal{O}$ *in dilation normal form such that* $\Omega_\varrho = \Omega_\sigma$*.*

Armed with the notion of neighbourhood functions we can give a precise and elegant semantics to the sentences of $\mathcal{L}^{\mathcal{O}}$:

Definition 8 (Satisfaction). *Given* $w \in \mathcal{U}$*:*

– $w \Vdash p$ *if and only if* $w(p) = 1$*;*
– $w \Vdash \neg\alpha$ *if and only if* $w \nVdash \alpha$*;*
– $w \Vdash \alpha \wedge \beta$ *if and only if* $w \Vdash \alpha$ *and* $w \Vdash \beta$*;*
– $w \Vdash \alpha^\mathrm{o}$ *if and only if, for some* $v \in \mathcal{U}$*,* $v \Vdash \alpha$ *and* $w \in \Omega_\mathrm{o}(v)$*.*

Given $\alpha \in \mathcal{L}^{\mathcal{O}}$, we say that α is *valid* if $Mod(\alpha) = \mathcal{U}$ and we denote it $\models_\mathrm{o} \alpha$. The next lemma shows that our neighbourhood semantics preserves the validity of all propositional tautologies.

Lemma 4. *Let* $\alpha \in \mathcal{L}$ *(i.e.,* α *is a propositional sentence). Then* $\models \alpha$ *iff* $\models_\mathrm{o} \alpha$*.*

From the perspective of knowledge representation and reasoning, it becomes important to address the question of what it means for a sentence α (or a theory $\mathcal{T}$) to *entail* a sentence β. For now we suffice with a standard (Tarskian) definition of logical consequence (at the end of the present section we shall define an alternative notion of entailment for $\mathcal{L}^{\mathcal{O}}$): given $\alpha, \beta \in \mathcal{L}^{\mathcal{O}}$, α entails β (denoted $\alpha \models_{\mathcal{L}^{\mathcal{O}}} \beta$) if and only if $Mod(\alpha) \subseteq Mod(\beta)$. (This notion of entailment can be extended to theories in the usual way.) We shall use $\alpha \equiv_{\mathcal{L}^{\mathcal{O}}} \beta$ as an abbreviation for $\alpha \models_{\mathcal{L}^{\mathcal{O}}} \beta$ and $\beta \models_{\mathcal{L}^{\mathcal{O}}} \alpha$. It then follows that, for every $\varrho, \sigma \in \mathcal{O}$, $(\alpha^\varrho)^\sigma \equiv_{\mathcal{L}^{\mathcal{O}}} \alpha^{\varrho\cdot\sigma}$.

The following result generalizes Theorem 4, and its proof is similar:

Theorem 6. *Let* $\alpha \in \mathcal{L}$ *and* $\mathrm{o} \in \mathcal{O}$ *be in dilation normal form, with* $o = \pi_1 + \ldots + \pi_n$*,* $n \geq 1$*. Then* $\alpha^\mathrm{o} \equiv_{\mathcal{L}^{\mathcal{O}}} \bigvee_{1\leq i\leq n} Forget_{\{p \mid p\in\pi_i\}}(\alpha)$*.*

Theorem 6 establishes that reasoning with $\mathcal{L}^{\mathcal{O}}$ can be reduced to reasoning in classical propositional logic together with the standard forgetting operator. From this it then becomes easy to analyze the complexity of reasoning within $\mathcal{L}^{\mathcal{O}}$.

Conversely, both standard and selective forgetting can be captured in the extended language $\mathcal{L}^{\mathcal{O}}$ presented above. For example, let $\mathcal{S} = \{p, q, r\}$. Then $\mathit{Forget}_{\mathcal{S}}(\tau) \equiv_{\mathcal{L}^{\mathcal{O}}} \tau^{pqr}$, $\mathit{SForget}^1_{\mathcal{S}}(\tau) \equiv_{\mathcal{L}^{\mathcal{O}}} \tau^{p+q+r}$ and $\mathit{SForget}^2_{\mathcal{S}}(\tau) \equiv_{\mathcal{L}^{\mathcal{O}}} \tau^{pq+pr+qr}$. However, the new language allows us to express dilations directly in the language, and it allows us to express more — the sentence τ^{p+qr} can, for example, not be expressed using the meta-language of selective forgetting.

Thus far, we have only considered the neighbourhoods defined in terms of collective and selective dilation. The grammar can in principle be extended to allow for the construction of *erosion*, *opening* and *closing* operators, to name a few [4].

We shall now turn our attention to an alternative notion of entailment for the extended language $\mathcal{L}^{\mathcal{O}}$. We start by observing that the dilation operation is a uniform weakening operator in the sense of Britz et al. [10]. That is, given $\alpha, \beta \in \mathcal{L}$ and $\mathrm{o} \in \mathcal{O}$, the following properties are satisfied:

- $\alpha \models_{\mathcal{L}^{\mathcal{O}}} \alpha^{\mathrm{o}}$ (Weakening)
- If $\alpha \models_{\mathcal{L}^{\mathcal{O}}} \beta$, then $\alpha^{\mathrm{o}} \models_{\mathcal{L}^{\mathcal{O}}} \beta^{\mathrm{o}}$ (Uniformity)

It then follows that, given $\mathrm{o} \in \mathcal{O}$, the operator $\mathit{Cn}_{\mathcal{L}^{\mathcal{O}}} : \mathcal{L}^{\mathcal{O}} \longrightarrow 2^{\mathcal{L}^{\mathcal{O}}}$, defined by $\mathit{Cn}_{\mathcal{L}^{\mathcal{O}}}(\alpha) \mapsto \{\beta \in \mathcal{L}^{\mathcal{O}} \mid \alpha \models_{\mathcal{L}^{\mathcal{O}}} \beta^{\mathrm{o}}\}$, is a parameterized *supra-classical* Tarskian consequence operator. We extend $\mathit{Cn}_{\mathcal{L}^{\mathcal{O}}}$ to a consequence operation on knowledge bases in the standard way.

Given $\mathrm{o} \in \mathcal{O}$, $\mathit{Cn}_{\mathcal{L}^{\mathcal{O}}}$ defines a tolerant entailment relation $\mid\!\sim$, allowing pairs $\mathcal{K} \mid\!\sim \alpha$ for which $\mathcal{K} \not\models \alpha$, and applicable in approximate reasoning where additional consequences that do not follow classically, but which resemble some classical consequences, are sought. (Note that $\mid\!\sim$ is a *monotonic* consequence relation.) Given a knowledge base $\mathcal{K}$ and $\alpha \in \mathcal{L}^{\mathcal{O}}$, entailment checking of $\mathcal{K} \mid\!\sim \alpha$ can then be reduced to standard forgetting, courtesy of Theorem 6 above.

6 Weak Selective Forgetting

We now turn to the syntactic characterization of *weak selective forgetting* in infinite theories. Of course, the finite characterization of selective forgetting then collapses. Nor does syntactic irrelevance suffice as vehicle for its representation, as in the case of forgetting [33].

The following alternative syntactic characterization of forgetting suggests a possible course of action. It holds for finite as well as infinite theories, and can therefore be used to obtain a syntactic representation of weak forgetting as proposed by Zhang and Zhou [52]:

Lemma 5. *Let $\mathcal{T} \subseteq \mathcal{L}$ and $p \in \mathcal{P}$. Let $\mathcal{T}_1 = \{\alpha \mid \mathcal{T} \models \alpha$ and p is irrelevant to $\alpha\}$ and $\mathcal{T}_2 = \{\alpha[p/\bot] \vee \beta[p/\top] \mid \alpha, \beta \in \mathcal{T}\}$. Then $\mathcal{T}_1 \equiv \mathcal{T}_2$.*

Proof. Routine, using compactness to deal with the infinite case. □

Definition 9. *Let $\mathcal{T} \subseteq \mathcal{L}$, and $\mathcal{S} \subseteq \mathcal{P}$ be a finite set of atoms. Define $\mathcal{T}_{\mathcal{S}}$ recursively as follows:*

(i) $\mathcal{T}_{\emptyset} = \mathcal{T}$;
(ii) $\mathcal{T}_{\mathcal{S}_0 \cup \{p\}} = \{\alpha[p/\bot] \vee \beta[p/\top] \mid \alpha, \beta \in \mathcal{T}_{\mathcal{S}_0}\}$, *for* $\mathcal{S}_0 \subset \mathcal{S}$.

Theorem 7. *Let $\mathcal{T} \subseteq \mathcal{L}$, $\mathcal{S} \subseteq \mathcal{P}$ a finite set of atoms, and $\mathcal{T}_{\mathcal{S}}$ as in Definition 9. If $\mathcal{T}$ is finite, then $\mathcal{T}_{\mathcal{S}} \equiv \mathit{Forget}_{\mathcal{S}}(\mathcal{T})$. If $\mathcal{T}$ is infinite, then*

$$\mathcal{T}_{\mathcal{S}} \equiv \{\alpha \mid \mathcal{T} \models \alpha \text{ and } \mathcal{S} \text{ is irrelevant to } \alpha\}.$$

Proof. From Lemma 5 and Theorem 2. □

The result above warrants us to call $\mathcal{T}_{\mathcal{S}}$ the result of weakly forgetting $\mathcal{S}$ in $\mathcal{T}$, or $\mathit{WForget}_{\mathcal{S}}(\mathcal{T})$ in the terminology of Zhang and Zhou [52]. We use this to define weak selective forgetting as a generalization of weak forgetting in the power set algebra associated with the propositional language $\mathcal{L}$. Brink [8] gives a general account of power structures in the context of logic.

Definition 10. *Let $f : A^n \longrightarrow A$ be an n-ary operation on the set A and let $X_1, \ldots, X_n \subseteq A$. Then $f^+ : (2^A)^n \longrightarrow 2^A$ is the power operation of f on 2^A defined by:*

$$f^+(X_1, \ldots, X_n) := \{y \in A \mid (\exists x_1 \in X_1) \ldots (\exists x_n \in X_n)[f(x_1, \ldots, x_n) = y]\}.$$

Definition 11. *Let $\mathcal{T} \subseteq \mathcal{L}$, $\mathcal{S} \subseteq \mathcal{P}$ a finite set of atoms and $0 \leq n \leq |\mathcal{S}|$. The weak selective forgetting of n atoms from $\mathcal{S}$ in $\mathcal{T}$ is the theory*

$$\mathit{WSForget}^n_{\mathcal{S}}(\mathcal{T}) := \textstyle\bigvee^+_{\mathcal{C}^n_{\mathcal{S}} \subseteq \mathcal{S}} \mathit{WForget}_{\mathcal{C}^n_{\mathcal{S}}}(\mathcal{T}),$$

where $\bigvee^+$ denotes the power operation of $\vee$.

In other words, if $\mathcal{T}_1, \ldots, \mathcal{T}_k$ is an enumeration of the theories $\mathit{WForget}_{\mathcal{C}^n_{\mathcal{S}}}(\mathcal{T})$ for all values of $\mathcal{C}^n_{\mathcal{S}}$, then $\mathit{WSForget}^n_{\mathcal{S}}(\mathcal{T}) = \{\bigvee_{1 \leq j \leq k} \alpha_j \mid \text{for each } j,\ \alpha_j \in \mathcal{T}_j\}$.

It is not difficult to see that, for Ω^n as in Definition 5,

$$\{w \mid \mathit{Mod}(\mathcal{T}) \cap \Omega^n(w) \neq \emptyset\} \subseteq \mathit{Mod}(\mathit{WSForget}^n_{\mathcal{S}}(\mathcal{T})).$$

Finally, our definition is further supported by the fact that, for knowledge bases, selective forgetting and weak selective forgetting produce the same results.

7 Concluding Remarks

The main contributions of the present paper can be summarized as follows: (*i*) defining a notion of selective indiscernibility, (*ii*) showing how to compute it via selective forgetting, and (*iii*) presenting a logic in which to express and reason with dilation at the object level and for which the reasoning problem can be reduced to entailment in classical propositional logic plus standard forgetting.

Syntactically, selective forgetting is measured in terms of the number and selection of atoms on which disagreement is allowed. It therefore provides a range of increasingly tolerant upper bounds to a given theory, which can be applied to approximate reasoning and nonmonotonic reasoning. To witness, we can define a belief contraction operator based on the following observation:

Lemma 6. *Let $\mathcal{K}$ be a knowledge base. For every non-tautological $\alpha \in \mathcal{L}$, if $\emptyset \neq \mathcal{S} \subseteq$ atm$!(\alpha)$, then there exists an n such that SForget$^n_{\mathcal{S}}(\mathcal{K}) \not\models \alpha$.*

Hence, given appropriate $\mathcal{S}$ and n, selective forgetting of atoms in $\mathcal{K}$ delivers a weaker knowledge base not entailing sentence α. The results in Section 5 together with Lemma 6 suggest we can also bring such a contraction operator into the object language.

The present paper also opens up a number of avenues for future research:

An investigation of the dual case of selective *remembering*, and its relationship with its semantic counterpart of lower approximations in a tolerance space, as well as the corresponding operators of *erosion*, remain to be done. Likewise, lifting the propositional results obtained here to knowledge forgetting in modal logics [15] and to concept and role forgetting in description logics [47] are worth investigating.

Another avenue for future exploration is the definition of different alternative notions of entailment for $\mathcal{L}^{\mathcal{O}}$ and their relationship with various forms of reasoning.

Finally, from a knowledge representation and reasoning perspective, when one deals with knowledge bases, issues related to modularization [14, 19, 22–24], knowledge base revision and update [21, 25, 26] as well as knowledge base maintenance and versioning [18, 27, 29] show up. These are tasks that also make sense in the setting studied in this paper. When moving beyond the classical case, though, such tasks have to be reassessed and specific methods and techniques redesigned. This constitutes a thread worthy of exploration.

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