A semantics for Rational Closure: Preliminary Results

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Abstract. We provide a semantical reconstruction of rational closure. We first consider rational closure as defined by Lehman and Magidor for propositional logic, and we provide a semantical characterization based on minimal models mechanism on rational models. Then, we extend the whole formalism and semantics to Description Logics focusing our attention to the standard $\mathcal{ALC}$: we first naturally adapt to Description Logics Lehman and Magidor's propositional rational closure, starting from an extension of $\mathcal{ALC}$ with a typicality operator $\mathbf{T}$ that selects the most typical instances of a concept C (hence $\mathbf{T}(C)$ stands for typical Cs). Then, we provide for $\mathcal{ALC}$ plus $\mathbf{T}$ a semantical characterization similar to the one for propositional logic. Last, we extend the notion of rational closure to the ABox.

1 Introduction

In [18] Kraus, Lehmann and Magidor (henceforth KLM) proposed a set of natural properties of non-monotonic reasoning. Plausible inferences are represented by non-monotonic conditionals of the form $A \mid\sim B$, to be read as "typically or normally A entails B": for instance $monday \mid\sim go\_work$ can be used to represent that "normally if it is Monday I go to work". Conditional entailment is non-monotonic since from $A \mid\sim B$ one cannot derive $A \wedge C \mid\sim B$, in our example from $monday \mid\sim go\_work$ one cannot monotonically derive $monday \wedge ill \mid\sim go\_work$ ("normally if it is Monday, even if I am ill I go to work"). KLM organized the core properties of non-monotonic reasoning into a hierarchy of systems, from the weakest to the strongest: cumulative logic **C**, loop-cumulative logic **CL**, preferential logic **P**. Preferential logic has been strengthened into rational logic **R** in [20]. In this work, we restrict our attention to the rational logic **R** on which rational closure is built.

KLM system **R** formalizes desired properties of non-monotonic inference but it is too weak to perform useful non-monotonic inferences. We have just seen that by the non-monotonicity of $\mid\sim$, $A \mid\sim B$ does not entail $A \wedge C \mid\sim B$, and this is a wanted property of $\mid\sim$. However, there are cases in which, in the absence of information to the contrary, we want to be able to tentatively infer that also $A \wedge C \mid\sim B$, with the possibility of withdrawing the inference in case we discovered that it is inconsistent. For instance, we might want to infer that $A \wedge C \mid\sim B$ when C is irrelevant with respect to the property B: we might want to tentatively infer from $monday \mid\sim go\_work$ that $monday \wedge shines \mid\sim go\_work$ ("normally if it is Monday, even if the sun shines I go to work"), with the possibility of withdrawing the conclusion if we discovered that indeed the sun shining prevents from going to work. **R** cannot handle irrelevant information in conditionals, and the inferences just exemplified are not supported.

Partially motivated by this weakness, Lehmann and Magidor have proposed a true non-monotonic mechanism on the top of **R**. Rational closure on the one hand preserves the properties of **R**, on the other hand it allows to perform some truthful non-monotonic inferences, like the one just mentioned ($monday \wedge shines \mid\sim go\_work$). In [20] the authors give a syntactic procedure to calculate the set of conditionals entailed by the rational closure as well as a quite complex semantic construction. It is worth noticing that a strongly related construction has been proposed by Pearl [22] with his notion of 1-entailment, motivated by a probabilistic interpretation of conditionals.

The first problem we tackle in this work is that of giving a purely semantic characterization of the syntactic notion of rational closure. Our semantic characterization has as its main ingredient the modal semantics of logic **R**, over which we build a minimal models' mechanism, based on the minimization of the *rank* of worlds. Intuitively, we prefer the models that minimize the rank of domain elements: the lower the rank of a world, the more normal (or less exceptional) is the world and our minimization corresponds intuitively to the idea of minimizing less-plausible worlds (or maximizing most plausible ones). We show that a semantic reconstruction of rational closure can be given in terms of a specific case of a general semantic framework for non-monotonic reasoning.

In the second part of the paper we consider Description Logics (DLs for short). A large amount of discussion has recently been done in order to extend the basic formalism of DLs with non-monotonic reasoning features [1, 2, 4, 6, 7, 14, 19, 17, 3, 21]; the purpose of these extensions is that of allowing reasoning about prototypical properties of individuals or classes of individuals. In spite of the load of work in this direction, finding a solution to the problem of extending DLs for reasoning about prototypical properties seems far from being solved. The best known semantics for non-monotonic reasoning have been used to the purpose, from default logic [1], to circumscription [2], from Lifschitz's non-monotonic logic MKNF [6, 21] to KLM logics. Concerning KLM logics, in [10] a preferential extension of $\mathcal{ALC}$ is defined, based on the logic **P**, and in [14] a minimal model semantics for this logic is proposed; in [3], a defeasible description logic based on the logic **R** is introduced and, in [4], a notion of rational closure is defined for $\mathcal{ALC}$ through an algorithmic construction similar to the one introduced by Freund for the propositional calculus. Although [4] provides axiomatic properties of this notion of rational closure, it does not provide a semantics for it.
We here extend to $\mathcal{ALC}$ the definition of rational closure by Lehmann and Magidor [20] and define a minimal model semantics for rational closure in $\mathcal{ALC}$ by adapting the semantics introduced in the propositional case. We start from the extension of the description logic $\mathcal{ALC}$ with a typicality operator $\mathbf{T}$, first proposed in [10], that allows to directly express typical properties such as $\mathbf{T}(\mathit{HeartPosition}) \sqsubseteq \mathit{Left}$, $\mathbf{T}(\mathit{Bird}) \sqsubseteq \mathit{Fly}$, and $\mathbf{T}(\mathit{Penguin}) \sqsubseteq \neg \mathit{Fly}$, whose intuitive meaning is that normally, the heart is positioned in the left-hand side of the chest, that typical birds fly, whereas penguins do not. In this paper, the $\mathbf{T}$ operator is intended to enjoy the well-established properties of rational logic **R**. Even if $\mathbf{T}$ is a non-monotonic operator (so that for instance $\mathbf{T}(\mathit{HeartPosition}) \sqsubseteq \mathit{Left}$ does not entail that $\mathbf{T}(\mathit{HeartPosition} \sqcap \mathit{SitusInversus}) \sqsubseteq \mathit{Left}$) the logic itself is monotonic. Indeed, in this logic it is not possible to monotonically infer from $\mathbf{T}(\mathit{Bird}) \sqsubseteq \mathit{Fly}$, in the absence of information to the contrary, that also $\mathbf{T}(\mathit{Bird} \sqcap \mathit{Black}) \sqsubseteq \mathit{Fly}$. Nor it can non-monotonically be inferred from $\mathit{Bird}(\mathit{tweety})$, in the absence of information to the contrary, that $\mathbf{T}(\mathit{Bird})(\mathit{tweety})$ and that $\mathit{Fly}(\mathit{tweety})$. Non-monotonicity is achieved, from a semantic point of view, by defining, on the top of $\mathcal{ALC}$ with typicality, a minimal

model semantics which is similar to the one in [14], with the difference that the notion of minimality is based on the minimization of the ranks of the worlds, rather than on the minimization of specific formulas, as in [14]. This semantics provides a characterization to the rational closure construction for $\mathcal{ALC}$, which assigns a *rank* (a level of exceptionality) to every concept; this rank is used to evaluate defeasible inclusions of the form $\mathbf{T}(C) \sqsubseteq D$: the inclusion is supported by the rational closure whenever the rank of C is strictly smaller than the one of $C \sqcap \neg D$.

Last, we tackle the problem of extending rational closure to ABox reasoning: in order to ascribe defeasible properties to individuals we maximize their typicality. This is done by minimizing their ranks (that is, their level of exceptionality). Because of the interaction between individuals (due to roles) it is not possible to separately assign a unique minimal rank to each individual and alternative minimal ranks must be considered. We end up with a kind of *skeptical* inference with respect to the ABox.

The rational closure construction that we propose has not just a theoretical interest and a simple minimal model semantics, we show that it is also *feasible*. Its complexity is EXPTIME in the size of the knowledge base (and the query), the same complexity as the underlying logic $\mathcal{ALC}$. In this respect it is less complex than other approaches to non-monotonic reasoning in DLs [14, 2] and comparable with the approaches in [4, 21], and thus a good candidate to define effective non-monotonic extensions of DLs.

2 Propositional rational closure: a semantic characterization

2.1 KLM rational system R

The language of logic **R** consists just of conditional assertions $A \mid\!\sim B$. Here we consider a richer language which also allows boolean combinations of assertions. Our language $\mathcal{L}$ is defined from a set of propositional variables ATM, the boolean connectives and the conditional operator $\mid\!\sim$. We assume that the set ATM is finite. We use $A, B, C, \ldots$ to denote propositional formulas (that do not contain conditional formulas), whereas $F, G, \ldots$ are used to denote all formulas (including conditionals). The formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\!\sim B \in \mathcal{L}$; if F is a boolean combination of formulas of $\mathcal{L}$, $F \in \mathcal{L}$. A knowledge base K is any set of formulas: in this work we restrict our attention to finite knowledge bases.

Here is the axiomatization of logic **R** [11]. We use $\vdash_{PC}$ (resp. $\models_{PC}$) to denote provability (resp. validity) in the propositional calculus:

- All axioms and rules of propositional logic
- $A \mid\!\sim A$ (REF)
- if $\vdash_{PC} A \leftrightarrow B$ then $(A \mid\!\sim C) \rightarrow (B \mid\!\sim C)$, (LLE)
- if $\vdash_{PC} A \rightarrow B$ then $(C \mid\!\sim A) \rightarrow (C \mid\!\sim B)$ (RW)
- $((A \mid\!\sim B) \wedge (A \mid\!\sim C)) \rightarrow (A \wedge B \mid\!\sim C)$ (CM)
- $((A \mid\!\sim B) \wedge (A \mid\!\sim C)) \rightarrow (A \mid\!\sim B \wedge C)$ (AND)
- $((A \mid\!\sim C) \wedge (B \mid\!\sim C)) \rightarrow (A \vee B \mid\!\sim C)$ (OR)
- $((A \mid\!\sim B) \wedge \neg(A \mid\!\sim \neg C)) \rightarrow ((A \wedge C) \mid\!\sim B)$ (RM)

The axiom (CM) is called cumulative monotony and it is characteristic of all KLM logics, axiom (RM) is called rational monotony and it characterizes the logic of rational entailment **R** (it is what distinguishes rational from the weaker preferential entailment). **R** seems to capture the core properties of non-monotonic reasoning, as shown by Friedman and Halpern these properties are quite ubiquitous being characterized by different semantics (all of them being instances of so-called *plausibility structures* [8]).

The logic **R** enjoys a simple modal semantics, actually it turns out that it is the flat fragment (i.e. without nested conditionals) of the well-known conditional logic **VC**. The modal semantics is defined by considering a set of worlds $\mathcal{W}$ equipped by an accessibility (or preference) relation $<$. Intuitively the meaning of $x < y$ is that x is more normal/less exceptional than y. We say that a conditional $A \mathrel{|\!\sim} B$ is true in a model if B holds in all most normal worlds where A is true, i.e. in all $<$-minimal worlds satisfying A.

Definition 1. *A* rational *model is a triple $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ where: • $\mathcal{W}$ is a non-empty set of worlds; • $<$ is an irreflexive, transitive relation on $\mathcal{W}$ satisfying modularity: for all x, y, z, if $x < y$ then either $x < z$ or $z < y$. $<$ further satisfies the Smoothness condition defined below; • V is a function $V : \mathcal{W} \longmapsto 2^{ATM}$, which assigns to every world w the set of atoms holding in that world. If F is a boolean combination of formulas, its truth conditions ($\mathcal{M}, w \models F$) are defined as for propositional logic. Let A be a propositional formula; we define $Min^{\mathcal{M}}_{<}(A) = \{w \in \mathcal{W} \mid \mathcal{M}, w \models A$ and $\forall w'$, $w' < w$ implies $\mathcal{M}, w' \not\models A\}$. Hence $\mathcal{M}, w \models A \mathrel{|\!\sim} B$ if for all w', if $w' \in Min^{\mathcal{M}}_{<}(A)$ then $\mathcal{M}, w' \models B$.*
We define the Smoothness condition*: if $\mathcal{M}, w \models A$, then $w \in Min^{\mathcal{M}}_{<}(A)$ or there is $w' \in Min^{\mathcal{M}}_{<}(A)$ s.t. $w' < w$. Validity and satisfiability of a formula are defined as usual. Given a set of formulas K of $\mathcal{L}$ and a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, we say that $\mathcal{M}$ is a model of K, written $\mathcal{M} \models K$, if for every $F \in K$ and every $w \in \mathcal{W}$, $\mathcal{M}, w \models F$. K rationally entails a formula F ($K \models F$) if F is valid in all rational models of K.*

Since in this work we limit our attention to a language containing finitely many atoms, and to finite knowledge bases, we can restrict our attention to finite models, as the logic enjoys the finite model property (observe that in this case the smoothness condition is ensured trivially by the irreflexivity of the $<$). It is easy to see from Definition 1 that the truth condition of $A \mathrel{|\!\sim} B$ is "global" in a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$: given a world w, we have that $\mathcal{M}, w \models A \mathrel{|\!\sim} B$ if, for all w', if $w' \in Min^{\mathcal{M}}_{<}(A)$ then $\mathcal{M}, w' \models B$. It immediately follows that $A \mathrel{|\!\sim} B$ holds in w if and only if $A \mathrel{|\!\sim} B$ is valid in a model, i.e. it holds that $\mathcal{M}, w' \models A \mathrel{|\!\sim} B$, for all w' in $\mathcal{W}$; for this reason we will often write $\mathcal{M} \models A \mathrel{|\!\sim} B$. Moreover, when the reference to the model $\mathcal{M}$ is unambiguous, we will simply write $Min_{<}(A)$ instead of $Min^{\mathcal{M}}_{<}(A)$.

Rational models can be equivalently defined by postulating the existence of a rank function $k : \mathcal{W} \to \mathbb{N}$, and then letting $x < y$ iff $k(x) < k(y)$. For this reason rational models are also called "ranked models".

Definition 2 (Rank of a world). *Given a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, the rank $k_{\mathcal{M}}$ of a world $w \in \mathcal{W}$, written $k_{\mathcal{M}}(w)$, is the length of the longest chain $w_0 < \cdots < w$ from w to a minimal w_0 (i.e. there is no w' such that $w' < w_0$).*

Definition 3 (Rank of a formula). *The rank $k_{\mathcal{M}}(F)$ of a formula F in a model $\mathcal{M}$ is $i = min\{k_{\mathcal{M}}(w) : \mathcal{M}, w \models F\}$. If there is no $w : \mathcal{M}, w \models F$, F has no rank in $\mathcal{M}$.*

Proposition 1. *For any $\mathcal{M} = \langle \mathcal{W}, V, < \rangle$ and any $w \in \mathcal{W}$, we have $\mathcal{M} \models A \mathrel{|\!\sim} B$ iff $k_{\mathcal{M}}(A \land B) < k_{\mathcal{M}}(A \land \neg B)$ or A has no rank in $\mathcal{M}$.*

2.2 Lehmann and Magidor's definition of rational closure

As already mentioned, although the operator $\mid\!\sim$ is *non-monotonic*, the notion of logical entailment just defined is itself *monotonic*. In order to strengthen **R** and to obtain non-monotonic entailment, Lehmann and Magidor in [20] propose the well-known mechanism of rational closure. Since in rational closure no boolean combinations of conditionals are allowed, in the following, the knowledge base K is just a finite set of positive conditional assertions of the form $A \mid\!\sim B$.

Definition 4 (Exceptionality of propositional formulas and conditional formulas). *Let K be a knowledge base (i.e. a finite set of positive conditional assertions) and A a propositional formula. A is said to be* exceptional *for K if and only if $K \models \top \mid\!\sim \neg A$. A conditional formula $A \mid\!\sim B$ is exceptional for K if its antecedent A is exceptional for K. The set of conditional formulas which are exceptional for K will be denoted as $E(K)$.*

It is possible to define a non increasing sequence of subsets of K, $C_0 \supseteq C_1, \dots$ by letting $C_0 = K$ and, for $i > 0$, $C_i = E(C_{i-1})$. Observe that, being K finite, there is a $n \geq 0$ such that for all $m > n, C_m = C_n$ or $C_m = \emptyset$.

Definition 5 (Rank of a formula). *Let K be a knowledge base and let A be a propositional formula. A has* rank i *(for K) if and only if i is the least natural number for which A is not exceptional for C_i. If A is exceptional for all C_i then A has no rank.*

Definition 5 above allows to define the rational closure of a knowledge base K.

Definition 6 (Rational closure $\bar{\mathbf{K}}$ of K). *Let K be a conditional knowledge base. The rational closure $\bar{K}$ of K is the set of all $A \mid\!\sim B$ such that either (1) the rank of A is strictly less than the rank of $A \wedge \neg B$ (this includes the case A has a rank and $A \wedge \neg B$ has none), or (2) A has no rank.*

This mechanism, which is now well-established, allows to overcome some weaknesses of **R** . First of all it is closed under rational monotonicity (RM): if $(A \mid\!\sim B) \in \bar{\mathrm{K}}$ and $(A \mid\!\sim \neg C) \notin \bar{\mathrm{K}}$ then $(A \wedge C) \mid\!\sim B \in \bar{\mathrm{K}}$. Furthermore, rational closure supports some of the wanted inferences that **R** does not support. For instance rational closure allows to deal with irrelevance: from $monday \mid\!\sim go\_work$, it does support the non-monotonic conclusion that $monday \ \wedge \ shines \mid\!\sim go\_work$.

2.3 A semantical characterization of rational closure

We provide a semantical reconstruction of rational closure in terms of a minimal models' mechanism, thus providing an instantiation of the following general recipe for non-monotonic reasoning:
(i) fix an underlying modal semantics for conditionals (here we concentrate on **R** but another possible choice could have been the weaker **P** as in [12]),
(ii) obtain non-monotonic inference by restricting semantic consequence to a class of *minimal* models. These minimal models should be chosen on the basis of semantic considerations, independent from the *language* and from the *set of conditionals* (knowledge base) whose non-monotonic consequences we want to determine.
In the next proposition we will use $\mathcal{M}_i$ defined as follows. Let $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ be any rational model of K. Let $\mathcal{M}_0 = \mathcal{M}$ and, for all i, let $\mathcal{M}_i = \langle \mathcal{W}_i, <_i, V_i \rangle$ be the

rational model obtained from $\mathcal{M}$ by removing all the worlds w with $k_{\mathcal{M}}(w) < i$, i.e., $\mathcal{W}_i = \{w \in \mathcal{W} : k_{\mathcal{M}}(w) \geq i\}$.

Proposition 2. *Let $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ be any rational model of K. For any propositional formula A, if rank(A) $\geq i$, then 1) $k_{\mathcal{M}}(A) \geq i$, and 2) if $A \mathrel{|\!\sim} B$ is entailed by C_i, then $\mathcal{M}_i$ satisfies $A \mathrel{|\!\sim} B$.*

The semantics we propose is a *fixed interpretations minimal semantics*, for short $FIMS$. In some respects our approach is similar in spirit to minimal models approaches to non-monotonic reasoning, such as circumscription[4].

Definition 7 ($FIMS$)**.** *Given $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to the fixed interpretations minimal semantics, and we write $\mathcal{M} <_{FIMS} \mathcal{M}'$, if $\mathcal{W} = \mathcal{W}'$, $V = V'$, and for all x, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $x' : k_{\mathcal{M}}(x') < k_{\mathcal{M}'}(x')$. We say that $\mathcal{M}$ is minimal w.r.t. $<_{FIMS}$ in case there is no $\mathcal{M}'$ such that $\mathcal{M}' <_{FIMS} \mathcal{M}$. We say that K minimally entails a formula F w.r.t. FIMS, and we write $K \models_{FIMS} F$, if F is valid in all models of K which are minimal w.r.t. $<_{FIMS}$.*

Can we capture rational closure within the semantics of Definition 7 above? We are soon forced to recognize that this is not the case. For instance, consider the following:

Example 1. Let $K = \{penguin \mathrel{|\!\sim} bird, penguin \mathrel{|\!\sim} \neg fly, bird \mathrel{|\!\sim} fly\}$. We derive that $K \not\models_{FIMS} penguin \wedge black \mathrel{|\!\sim} \neg fly$. Indeed in $FIMS$ there can be a model $\mathcal{M}$ in which $\mathcal{W} = \{x, y, z\}$, $V(x) = \{penguin, bird, fly, black\}$, $V(y) = \{penguin, bird\}$, $V(z) = \{bird, fly\}$, and $z < y < x$. $\mathcal{M}$ is a model of K, and it is minimal with respect to $FIMS$ (indeed once fixed $V(x), V(y), V(z)$ as above, it is not possible to lower the rank of x nor of y nor of z unless we falsify K). Furthermore, in $\mathcal{M}$, x is a typical world in which "it flies" and "it is black" hold (since there is no other world satisfying the same propositions which is preferred to it). Therefore, $K \not\models_{FIMS} penguin \wedge black \mathrel{|\!\sim} \neg fly$.

We have that $\{penguin \mathrel{|\!\sim} bird, penguin \mathrel{|\!\sim} \neg fly, bird \mathrel{|\!\sim} fly\} \not\models_{FIMS} penguin \wedge black \mathrel{|\!\sim} \neg fly$. On the contrary, it can be verified that $penguin \wedge black \mathrel{|\!\sim} \neg fly$ is in the rational closure of $\{penguin \mathrel{|\!\sim} bird, penguin \mathrel{|\!\sim} \neg fly, bird \mathrel{|\!\sim} fly\}$. Therefore, $FIMS$ as it is does not allow us to define a semantics corresponding to rational closure. Things change if we consider $FIMS$ applied to models that contain *all possible valuations compatible (see Definition 8 below) with a given knowledge base* K. We call these models *canonical models*.

Example 2. Consider Example 1 above. If we restrict our attention to models that also contain a w with $V(w) = \{penguin, bird, black\}$ which satisfies "it is a penguin", "it is black" and "it does not fly" in which w is a typical world satisfying "it is a penguin", we are able to conclude that typically it holds that if it is a penguin and it is black then it does not fly, as in rational closure. Indeed, in all minimal models of K that also contain w with $V(w) = \{penguin, bird, black\}$, it holds that $penguin \wedge black \mathrel{|\!\sim} \neg fly$.

[4] As for circumscription, there are mainly two ways of comparing models with the same domain: by keeping the valuation function fixed (only comparing $\mathcal{M}$ and $\mathcal{M}'$ if V and V' in the two models respectively coincide); or by also comparing $\mathcal{M}$ and $\mathcal{M}'$ in case $V \neq V'$. In this work we consider the latter alternative.

We are led to the conjecture that $FIMS$ restricted to canonical models could be the right semantics for rational closure. Canonical models are defined w.r.t. the language $\mathcal{L}$. A truth assignment $v : ATM \longrightarrow \{true, false\}$ is *compatible* with K, if there is no formula $A \in \mathcal{L}$ such that $v(A) = true$ and $K \models A \mathrel{|\!\sim} \bot$.

Definition 8. *A model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ satisfying a knowledge base K is said to be* canonical *if it contains (at least) a world associated to each truth assignment compatible with K, that is to say: if v is compatible with K, then there exists a world w in $\mathcal{W}$, such that for all propositional formulas B $\mathcal{M}, w \models B$ iff $v(B) = true$.*

It can be shown that for any knowledge base a minimal canonical model exists: this is any canonical model in which every possible world w has the rank associated to the conjunction of all atoms and negated atoms in $\mathcal{L}$ that it satisfies. We can also prove that the canonical models that are minimal with respect to $FIMS$ are an adequate semantic counterpart of rational closure.

Theorem 1. *Let K be a knowledge base and $\mathcal{M}$ be a canonical model of K minimal w.r.t. $<_{FIMS}$. We show that, for all conditionals $A \mathrel{|\!\sim} B$, $\mathcal{M} \models A \mathrel{|\!\sim} B$ if and only if $A \mathrel{|\!\sim} B \in \overline{K}$, where $\overline{K}$ is the rational closure of K.*

3 Rational closure in Description Logics

As mentioned, the interest towards non-monotonic reasoning in DLs has grown in the last years. In this section, we extend to $\mathcal{ALC}$ the notion of rational closure proposed by Lehmann and Magidor [20], recalled in Section 2.2, and we define a semantic characterization of this notion of rational closure by introducing a minimal model semantics for $\mathcal{ALC}$ with defeasible inclusions. This semantics is a direct generalization of the minimal (canonical) model semantics introduced in Section 2.3

To express defeasible inclusions, $\mathcal{ALC}$ is extended with a typicality operator $\mathbf{T}$, following the approach in [10, 14]. Differently from [14], here we consider special kinds of preferential models, namely, rational models, to define the semantics of the $\mathbf{T}$ operator, and we use a different notion of preference between models, namely, the preference relation $<_{FIMS}$, introduced in Section 2.3. Given the typicality operator, the defeasible assertion $\mathbf{T}(C) \sqsubseteq D$ (all the typical C's are D's) plays the role of the conditional assertion $C \mathrel{|\!\sim} D$ in **R**.

3.1 The logic $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$

Similarly to rational closure which is a non-monotonic mechanism built over **R**, our application of rational closure to DLs is done in two steps. First, similarly to what done in [10], we extend the standard $\mathcal{ALC}$ by a typicality operator $\mathbf{T}$ that allows to single out the typical instances of a concept $\mathbf{T}$. Since we are dealing here with rational closure (that builds over **R**), we attribute to $\mathbf{T}$ properties related to **R**. The resulting logic is called $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$. As a second step, we build over $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ a rational closure mechanism.

Our starting point is therefore the extension of logic $\mathcal{ALC}$ with a typicality operator $\mathbf{T}$. The intuitive idea is to extend the standard $\mathcal{ALC}$ allowing concepts of the form $\mathbf{T}(C)$ whose intuitive meaning is that $\mathbf{T}(C)$ selects the *typical* instances of a concept C. We can therefore distinguish between the properties that hold for all instances of concept C ($C \sqsubseteq D$), and those that only hold for the typical such instances ($\mathbf{T}(C) \sqsubseteq D$).

Definition 9. *We consider an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individual constants $\mathcal{O}$. Given $A \in \mathcal{C}$ and $R \in \mathcal{R}$, we define $C_R := A \mid \top \mid \bot \mid \neg C_R \mid C_R \sqcap C_R \mid C_R \sqcup C_R \mid \forall R.C_R \mid \exists R.C_R$, and $C_L := C_R \mid \mathbf{T}(C_R)$. A KB is a pair (TBox, ABox). TBox contains a finite set of concept inclusions $C_L \sqsubseteq C_R$. ABox contains assertions of the form $C_L(a)$ and $R(a, b)$, where $a, b \in \mathcal{O}$.*

The **T** operator satisfies a set of postulates that are essentially a reformulation of rational logic **R**: in this respect, the **T**-assertion $\mathbf{T}(C) \sqsubseteq D$ is equivalent to the conditional assertion $C \mid\sim D$ in **R**.

A first semantic characterization of **T** can be given by means of a set of postulates that are essentially a restatement of axioms and rules of non-monotonic entailment in rational logic **R**. Given a domain Δ and a valuation function I one can define the function $f_{\mathbf{T}}(S)$ that selects the *typical* instances of S, and in case $S = C^I$ for a concept C, it selects the typical instances of C. In this semantics, $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$, and $f_{\mathbf{T}}$ has the following intuitive properties for all subsets S of Δ:

$(f_{\mathbf{T}} - 1)\ f_{\mathbf{T}}(S) \subseteq S$ $\quad$ $(f_{\mathbf{T}} - 2)$ if $S \neq \emptyset$, then also $f_{\mathbf{T}}(S) \neq \emptyset$

$(f_{\mathbf{T}} - 3)$ if $f_{\mathbf{T}}(S) \subseteq R$, then $f_{\mathbf{T}}(S) = f_{\mathbf{T}}(S \cap R)$ $\quad$ $(f_{\mathbf{T}} - 4)\ f_{\mathbf{T}}(\bigcup S_i) \subseteq \bigcup f_{\mathbf{T}}(S_i)$

$(f_{\mathbf{T}} - 5)\ \bigcap f_{\mathbf{T}}(S_i) \subseteq f_{\mathbf{T}}(\bigcup S_i)$ $\quad$ $(f_{\mathbf{T}} - \mathbf{R})$ if $f_{\mathbf{T}}(S) \cap R \neq \emptyset$, then $f_{\mathbf{T}}(S \cap R) \subseteq f_{\mathbf{T}}(S)$

$(f_{\mathbf{T}} - 1)$ enforces that typical elements of S belong to S. $(f_{\mathbf{T}} - 2)$ enforces that if there are elements in S, then there are also *typical* such elements. $(f_{\mathbf{T}} - 3)$ expresses a weak form of monotonicity, namely *cautious monotonicity*. The next properties constraint the behavior of $f_{\mathbf{T}}$ wrt $\cap$ and $\cup$ in such a way that they do not entail monotonicity. Last, $(f_{\mathbf{T}} - \mathbf{R})$ corresponds to rational monotonicity, and forces again a form of monotonicity: if there is a typical S having the property R, then all typical S and Rs inherit the properties of typical Ss.

The semantics of $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ can be equivalently formulated in terms of rational models: models of $\mathcal{ALC}$ are equipped by a *preference relation* $<$ on the domain, whose intuitive meaning is to compare the "typicality" of domain elements, that is to say $x < y$ means that x is more typical than y. Typical members of a concept C, that is members of $\mathbf{T}(C)$, are the members x of C that are minimal with respect to this preference relation (s.t. there is no other member of C more typical than x). This semantics with one single preference relation $<$ is the one that, as we will show, corresponds to rational closure[5].

Definition 10 (Semantics of $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$). *A model $\mathcal{M}$ of $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ is any structure $\langle \Delta, <, I \rangle$ where: Δ is the domain; $<$ is an irreflexive, transitive and modular relation over Δ ($<$ is modular if, for all $x, y, z \in \Delta$, if $x < y$ then either $x < z$ or $z < y$); I is the extension function that maps each concept C to $C^I \subseteq \Delta$, and each role R to $R^I \subseteq \Delta^I \times \Delta^I$. For concepts of $\mathcal{ALC}$, C^I is defined in the usual way. For the $\mathbf{T}$ operator, we have $(\mathbf{T}(C))^I = Min_<(C^I)$, where $Min_<(S) = \{u : u \in S$ and $\nexists z \in S$*

[5] One may think of considering a sharper semantics with several preference relations. We aim to explore this possibility in future works, for the moment, we just notice that (i) the definition of such a semantics is not straightforward (what does differentiate one preference relation from another? What are the dependencies between the different preference relations? Has the typicality operator to be made parametric?) (ii) it cannot be expected that the resulting semantics, being stronger than the one just proposed, can correspond to rational closure below.

s.t. $z < u\}$. *Furthermore,* $<$ *satisfies the* Smoothness Condition*, i.e., for all concepts* C*,* C^I *is* smooth*. For* $S \subseteq \Delta$*, we say that* S *is* smooth *iff for all* $x \in S$*, either* $x \in Min_<(S)$ *or* $\exists y \in Min_<(S)$ *such that* $y < x$*,*

Theorem 2. *[Theorem* 1 *in [9]] A KB=(TBox,ABox) is satisfiable in a model described in Definition 10 iff it is satisfiable in a model* $\langle \Delta, I, f_{\mathbf{T}} \rangle$ *where* $f_{\mathbf{T}}$ *satisfies* $(f_{\mathbf{T}} - 1) - (f_{\mathbf{T}} - 5)$ *and* $(f_{\mathbf{T}} - \mathbf{R})$*, and* $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$*.*

In the following, we will refer to the definition of the semantics given in Definition 10.

Definition 11 (Model satisfying a Knowledge Base). *Given a model* $\mathcal{M}$*,* I *is extended to assign a distinct element* a^I *of* Δ *to each individual constant* a *of* $\mathcal{O}$ *(i.e. we assume the* unique name assumption*).*
We say that: a model $\mathcal{M}$ *satisfies an inclusion* $C \sqsubseteq D$ *if it holds* $C^I \subseteq D^I$*;* $\mathcal{M}$ *satisfies an assertion* $C(a)$ *if* $a^I \in C^I$*; and* $\mathcal{M}$ *satisfies an assertion* $R(a, b)$ *if* $(a^I, b^I) \in R^I$*.*
We say that: $\mathcal{M}$ satisfies a knowledge base K=(TBox,ABox)*, if it satisfies both its TBox and its ABox, where:* $\mathcal{M}$ *satisfies TBox if* $\mathcal{M}$ *satisfies all inclusions in TBox and* $\mathcal{M}$ *satisfies ABox if* $\mathcal{M}$ *satisfies all assertions in ABox.*

From now on, in this section, we restrict our attention to $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ and to finite models. Given a knowledge base K and an inclusion $C_L \sqsubseteq C_R$, we say that the inclusion is derivable from K (we write $K \models_{\mathcal{ALC}^{\mathbf{R}}\mathbf{T}} C_L \sqsubseteq C_R$) if $C_L^I \subseteq C_R^I$ holds in all models $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K.

Definition 12 (Rank of a domain element). *The rank* $k_{\mathcal{M}}$ *of a domain element* x *in a model* $\mathcal{M}$ *is the length of the longest chain* $x_0 < \cdots < x$ *from* x *to a minimal* x_0 *(s.t. for no* x'*,* $x' < x_0$*).*

Finite $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ models can be equivalently defined by postulating the existence of a function $k : \Delta \to \mathbb{N}$, and then letting $x < y$ iff $k(x) < k(y)$.

Definition 13 (Rank of a concept). *Given a model* $\mathcal{M} = \langle \Delta, <, I \rangle$*, the* rank $k_{\mathcal{M}}(C_R)$ of a concept C_R in the model $\mathcal{M}$ *is* $i = min\{k_{\mathcal{M}}(x) : x \in C_R^I\}$*. If* $C_R^I = \emptyset$*, then* C_R *has no rank and we write* $k_{\mathcal{M}}(C_R) = \infty$*.*

Proposition 3. *For any* $\mathcal{M} = \langle \Delta, <, I \rangle$*, we have that* $\mathcal{M}$ *satisfies* $\mathbf{T}(C) \sqsubseteq D$ *iff* $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$*.*

As already mentioned, although the typicality operator $\mathbf{T}$ itself is non-monotonic (i.e. $\mathbf{T}(C) \sqsubseteq D$ does not imply $\mathbf{T}(C \sqcap E) \sqsubseteq D$), the logics $\mathcal{ALC}+\mathbf{T}$ and $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ are monotonic: what is inferred from K can still be inferred from any K' with $K \subseteq K'$. This is a clear limitation in DLs. As a consequence of non-monotonicity in $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ one cannot deal with irrelevance for instance. So one cannot derive from $K = \{Penguin \sqsubseteq Bird$, $\mathbf{T}(Bird) \sqsubseteq Fly$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$ that $K \models_{min} \mathbf{T}(Penguin \sqcap Black) \sqsubseteq \neg Fly$, even if the property of being black is irrelevant with respect to flying. In the same way if we added to K the information that jim is a bird ($Bird(jim)$), in $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ one cannot non-monotonically derive that it is a typical bird and therefore flies ($\mathbf{T}(Bird)(jim)$ and $Fly(jim)$). We investigate the possibility of overcoming this weakness by extending to $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$ the notion of rational closure. We first consider the rational closure of the TBox alone. Next we will consider rational closure that also takes into account the ABox.

3.2 Rational Closure of the TBox in $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$

Let us first define the notion of *query*. Intuitively, a query is either an inclusion relation or an assertion of the ABox; we want to check whether it is entailed from a given KB.

Definition 14 (Query). *A* query *F is either an assertion $C_L(a)$ or an inclusion relation $C_L \sqsubseteq C_R$. Given a model $\mathcal{M} = \langle \Delta, <, I\rangle$, a query F holds in $\mathcal{M}$ if $\mathcal{M}$ satisfies F.*

Definition 15. *Let T_B be a TBox and C a concept. C is said to be* exceptional *for T_B iff $T_B \models_{\mathcal{ALC}^{\mathbf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg C$. A $\mathbf{T}$-inclusion $\mathbf{T}(C) \sqsubseteq D$ is exceptional for T_B if C is exceptional for T_B. The set of $\mathbf{T}$-inclusions of T_B which are exceptional in T_B will be denoted as $\mathcal{E}(T_B)$.*

Given a DL knowledge base K=(TBox,ABox), it is possible to define a sequence of non-increasing subsets of TBox $E_0 \supseteq E_1, \ldots$ by letting $E_0 =$ TBox and, for $i > 0$, $E_i = \mathcal{E}(E_{i-1}) \cup \{C \sqsubseteq D \in$ TBox s.t. $\mathbf{T}$ does not occurr in $C\}$. Observe that, being K finite, there is an $n \geq 0$ such that for all $m > n, E_m = E_n$ or $E_m = \emptyset$. Observe also that the definition of the E_i's is the same as the definition of the C_i's in Lehmann and Magidor's definition of rational closure in Section 2.2, except for the fact that here, at each step, we also add all the strict inclusions.

Definition 16. *A concept C has* rank *i (denoted by $rank(C) = i$) for K=(TBox,ABox), iff i is the least natural number for which C is not exceptional for E_i. If C is exceptional for all E_i then $rank(C) = \infty$, and we say that C has no rank.*

As for propositional logic, the notion of rank of a formula allows to define the rational closure of the TBox of a knowledge base K.

Definition 17 (Rational closure of TBox). *Let K=(TBox,ABox) be a DL knowledge base. We define, $\overline{TBox}$, the* rational closure of TBox, *as*

$$\overline{TBox} = \{\mathbf{T}(C) \sqsubseteq D \mid \textit{either } rank(C) < rank(C \sqcap \neg D) \textit{ or } rank(C) = \infty\} \cup \{C \sqsubseteq D \mid K \models_{\mathcal{ALC}} C \sqsubseteq D\}$$

It can be easily seen that the rational closure of TBox is a non-monotonic strengthening of $\mathcal{ALC}^{\mathbf{R}}\mathbf{T}$. For instance it allows to deal with irrelevance. If TBox = $\{Penguin \sqsubseteq Bird$, $\mathbf{T}(Bird) \sqsubseteq Fly$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$, then it can be verified that $\mathbf{T}(Bird \sqcap Black) \sqsubseteq Fly \in \overline{TBox}$. This is a non-monotonic inference that does no longer follow if we knew that indeed black birds are non typical birds that do not fly: in this case from TBox'= TBox $\cup\{\mathbf{T}(Bird \sqcap Black) \sqsubseteq \neg Fly\}$ (in this case $\mathbf{T}(Bird \sqcap Black) \sqsubseteq Fly \notin \overline{TBox'}$). Similarly, as for the propositional case, rational closure is closed under rational monotonicity: from $\mathbf{T}(Bird) \sqsubseteq Fly \in \overline{TBox}$ and $\mathbf{T}(Bird) \sqsubseteq \neg LivesEurope \notin \overline{TBox}$ it follows that $\mathbf{T}(Bird \sqcap LivesEurope) \sqsubseteq Fly \in \overline{TBox}$.

As for the propositional case, in order to semantically characterize the rational closure, we first restrict our attention to minimal rational models that minimize *the rank of domain elements*. Informally, given two models of K, one in which a given domain element x has rank 2 (because for instance $z < y < x$) , and another in which it has rank 1 (because only $y < x$), we would prefer the latter, as in this model the element x is "more normal" than in the former.

From now on, we restrict our attention to *canonical minimal models*. First, we define a set of concepts $\mathcal{S}$ closed under negation and subconcepts. We assume that all the concepts in K and in the query F belong to $\mathcal{S}$. In order to define canonical models, we consider all the sets of concepts $\{C_1, C_2, \dots, C_n\} \subseteq \mathcal{S}$ that are *consistent with* K, i.e., s.t. $K \not\models_{\mathcal{ALC}} C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$.

Definition 18 (Canonical model w.r.t. $\mathcal{S}$). *Given K=(TBox,ABox) and a query F, a model $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K is* canonical w.r.t. $\mathcal{S}$ *if it contains at least a domain element $x \in \Delta$ s.t. $x \in (C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n)^I$, for each set of concepts $\{C_1, C_2, \dots, C_n\} \subseteq \mathcal{S}$ that are consistent with K.*

Definition 19 (Minimal canonical models (w.r.t. $\mathcal{S}$)). *Consider two models $\mathcal{M} = \langle \Delta, <, I \rangle$ and $\mathcal{M}' = \langle \Delta', <', I' \rangle$, canonical w.r.t. $\mathcal{S}$. We say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ ($\mathcal{M} < \mathcal{M}'$) if $\Delta = \Delta'$, and for all $x \in \Delta$, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$. Given a knowledge base K, we say that $\mathcal{M}$* is a minimal canonical model of K *if it is a canonical model satisfying K and there is no canonical model $\mathcal{M}'$ satisfying K such that $\mathcal{M}' < \mathcal{M}$.*

The following results hold (more details and proofs can be found in [15, 16]):

Theorem 3. *For any K there exists a minimal canonical model w.r.t. TBox.*

Theorem 4. *Let K=(TBox,ABox) be a knowledge base and $C \sqsubseteq D$ a query. We have that $C \sqsubseteq D \in \overline{TBox}$ if and only if $C \sqsubseteq D$ holds in all minimal canonical models of K with respect to $\mathcal{S}$.*

Theorem 5 (Complexity of rational closure over the TBox). *Given a knowledge base K =(TBox,ABox), the problem of deciding whether $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$ is in* EXPTIME.

3.3 Rational Closure Over the ABox

In this section we extend the notion of rational closure defined in the previous section in order to take into account the individual constants in the ABox. We address this question by first considering the semantic aspect, in order to treat individuals explicitly mentioned in the ABox in a uniform way with respect to the other domain elements: as for all the domain elements we would like to attribute to each individual constant named in the ABox the lowest possible rank. So we further refine Definition 19 of minimal canonical models with respect to TBox by taking into account the interpretation of individual constants of the ABox: given two minimal canonical models $\mathcal{M}$ and $\mathcal{M}'$, we prefer $\mathcal{M}$ to $\mathcal{M}'$ if there is an individual constant b occurring in ABox such that $k_{\mathcal{M}}(b^I) < k_{\mathcal{M}}(b^{I'})$ (whereas $k_{\mathcal{M}}(a^I) \leq k_{\mathcal{M}}(a^{I'})$ for all other individual constants occurring in ABox).

Definition 20 (Minimal canonical model of K minimally satisfying ABox). *Given K=(TBox,ABox), let $\mathcal{M} = \langle \Delta, <, I \rangle$ and $\mathcal{M}' = \langle \Delta', <', I' \rangle$ be two canonical models of K which are minimal w.r.t. Definition 19. We say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to ABox ($\mathcal{M} <_{ABox} \mathcal{M}'$) if for all individual constants a occurring in ABox, $k_{\mathcal{M}}(a^I) \leq k_{\mathcal{M}}(a^{I'})$ and there is at least one individual constant b occurring in ABox such that $k_{\mathcal{M}}(b^I) < k_{\mathcal{M}}(b^{I'})$.*

Theorem 6. *For any $K = (TBox, ABox)$ there exists a minimal canonical model of K minimally satisfying ABox.*

In order to see the power of the above semantic notion, consider the standard birds and penguins example.

Example 3. Suppose we have a knowledge base K where TBox = $\{\mathbf{T}(Bird) \sqsubseteq Fly$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly, Penguin \sqsubseteq Bird\}$, and ABox = $\{Penguin(pio), Bird(tweety)\}$. Knowing that tweety is a bird and pio is a penguin, we would like to be able to assume, in the absence of other information, that tweety is a typical bird, whereas pio is a typical penguin, and therefore tweety flies whereas pio does not. Consider any minimal canonical model $\mathcal{M}$ of K. Being canonical, $\mathcal{M}$ will contain, among other elements:
- $x \in (Bird)^I$, $x \in (Fly)^I$, $x \in (\neg Penguin)^I$, $k_{\mathcal{M}}(x) = 0$;
- $y \in (Bird)^I$, $y \in (\neg Fly)^I$, $y \in (\neg Penguin)^I$, $k_{\mathcal{M}}(y) = 1$;
- $z \in (Penguin)^I$, $z \in (Bird)^I$, $z \in (\neg Fly)^I$, $k_{\mathcal{M}}(z) = 1$;
- $w \in (Penguin)^I$, $w \in (Bird)^I$, $w \in (Fly)^I$, $k_{\mathcal{M}}(w) = 2$;

Notice that in the definition of minimal canonical model there is no constraint on the interpretation of the ABox constants tweety and pio. As far as Definition 19 is concerned for instance tweety can be mapped onto x ($(tweety)^I = x$) or onto y ($(tweety)^I = y$): the minimality of $\mathcal{M}$ with respect to Definition 19 is not affected by this choice. However in the first case it would hold that tweety is a typical bird, in the second tweety is not a typical bird. We want to prefer the first case, and this is what derives from Definition 20: if in $\mathcal{M}$ $tweety^I = x$ whereas in $\mathcal{M}_1$ (which for the rest is identical to $\mathcal{M}$) it holds that $tweety^I = y$, then $\mathcal{M}$ is preferred to $\mathcal{M}_1$. The same for pio. As a result in all models of K minimal with respect to both TBox and ABox (Definition 20), it holds what we wanted: that tweety is a typical bird ($T(Bird)(tweety)$), and therefore it flies, whereas pio is a typical penguin ($T(Penguin)(pio)$), and therefore it does not fly.

We conclude this section by providing an algorithmic construction for the rational closure of ABox, whose idea is that of considering all the possible minimal consistent assignments of ranks to the individuals explicitly named in the ABox. Each assignment adds some properties to named individuals which can be used to infer new conclusions. We adopt a skeptical view of considering only those conclusions which hold for all assignments. The equivalence with the semantics shows that the minimal entailment captures a skeptical approach when reasoning about the ABox.

More formally, in order to calculate the rational closure of ABox ($\overline{ABox}$) for all individual constants of the ABox we find out what is the lowest possible rank they can have in minimal canonical models w.r.t. Definition 19, with the idea that an individual constant a_i can have a given rank ($k_j(a_i)$) just in case it is compatible with all the inclusions of the TBox whose antecedent A's rank is $\geq k_j(a_i)$ (the inclusions whose antecedent A's rank is $< k_j(a_i)$ do not matter. The minimal possible rank assignment k_j for all a_i is computed in the algorithm below: μ_i^j computes all the concepts that a_i would need to satisfy in case it had the rank attributed by k_j ($k_j(a_i)$). The algorithm verifies whether μ_i^j is compatible with ($\overline{TBox}$, ABox) and whether it is minimal. Notice that in this phase all constants are considered simultaneously (indeed the possible ranks of different individual constants depend on each other). For this reason μ^j takes into account the ranks attributed to all individual constants, being the union of all μ_i^j for all a_i, and the consistency of this union with ($\overline{TBox}$, ABox) is verified (instead of the consistency of all separate μ_i^j). Once computed the minimal rank assignments these are used to define $\overline{ABox}$) as the set of all assertions derivable in $\mathcal{ALC}$ from ABox $\cup \mu^j$ for all minimal consistent rank assignments k_j.

Definition 21 **($\overline{ABox}$: rational closure of ABox).** *Let $a_1, \ldots, a_m$ be the individuals explicitly named in the ABox. Let $k_1, k_2, \ldots, k_h$ be all the possible rank assignments (ranging from 1 to n) to the individuals occurring in ABox.*

- *Given a rank assignment k_j we define:*
  - *for each a_i: $\mu_i^j = \{(\neg C \sqcup D)(a_i)$ s.t. $C, D \in \mathcal{S}$, $\mathbf{T}(C) \sqsubseteq D$ in $\overline{TBox}$, and $k_j(a_i) \leq rank(C)\} \cup \{(\neg C \sqcup D)(a_i)$ s.t. $C \sqsubseteq D$ in TBox $\}$;*
  - *let $\mu^j = \mu_1^j \cup \cdots \cup \mu_m^j$ for all $\mu_1^j \ldots \mu_m^j$ just calculated for all $a_1, \ldots, a_m$ in the ABox*
- *k_j is* minimal and consistent *with ($\overline{TBox}$, ABox) if:*
  - *ABox $\cup \mu^j$ is consistent in $\mathcal{ALC}$;*
  - *there is no k_i consistent wih ($\overline{TBox}$, ABox) s.t. for all a_i, $k_i(a_i) \leq k_j(a_i)$ and for some b, $k_i(b) < k_j(b)$.*
- *The rational closure of ABox ($\overline{ABox}$) is the set of all assertions derivable in $\mathcal{ALC}$ from ABox $\cup \mu^j$ for all minimal consistent rank assignments k_j, i.e:*

$$\overline{ABox} = \bigcap_{k_j \textit{minimal consistent}} \{C(a) : \textit{ABox} \cup \mu^j \models_{\mathcal{ALC}} C(a)\}$$

The following theorems hold (again, see [15, 16] for details and proofs):

Theorem 7 **(Soundness and Completeness of $\overline{ABox}$).** *Given K=(TBox, ABox), for all individual constant a in ABox, we have that $C(a) \in \overline{ABox}$ if and only if $C(a)$ holds in all minimal canonical models of K minimally satisfying ABox.*

Theorem 8 **(Complexity of rational closure over the ABox).** *Given a knowledge base K =(TBox,ABox), an individual constant a and a concept C, the problem of deciding whether $C(a) \in \overline{ABox}$ is* EXPTIME*-complete.*

4 Related work

In [14] non-monotonic extensions of DLs based on the **T** operator have been proposed. In these extensions, the semantics of **T** is based on preferential logic **P**. Non-monotonic inference is obtained by restricting entailment to *minimal models*, where minimal models are those that minimize the truth of formulas of a special kind. In this work, we have presented an alternative approach. First, the semantics underlying the **T** operator is **R** . Moreover and more importantly, we have adopted a minimal model semantics, where, as a difference with [14], the notion of minimal model is completely independent from the language and is determined only by the relational structure of models.

Casini and Straccia [4] study the application of rational closure to DLs. They extend to $\mathcal{ALC}$ the algorithmic construction proposed by Freund for capturing the rational closure in the propositional calculus. While in the propositional calculus this construction is proved to be equivalent with the notion of rational closure in [20], the equivalence is not known to hold for the case of $\mathcal{ALC}$. While Casini and Straccia prove axiomatic properties of their notion of rational closure, here we focus on an extension of Lehmann and Magidor definition of rational closure for $\mathcal{ALC}$ and we define a semantics for it. [4] also keeps the ABox into account, and defines closure operations over individuals. It introduces a consequence relation $\Vdash$ among a knowledge base K and assertions, under the requirement that the TBox is unfoldable and the ABox is closed under completion rules, such as, for instance, that if $a : \exists R.C \in$ ABox, then both aRb and $b : C$ (for some individual constant b) must belong to the ABox too. Under such restrictions they are able to define a procedure to compute the rational closure of the ABox assuming that the individuals explicitly named are linearly ordered, and different orders determine different sets of consequences. The authors show that, for each order s, the consequence relation $\Vdash_s$ is rational and can be computed in PSPACE. In a subsequent work [5], the authors introduce an approach based on the combination of rational closure and *Defeasible Inheritance Networks* (INs).

5 Conclusions

In the first part of the paper we have provided a semantic reconstruction of the well known rational closure, in detail a minimal model semantics based on the idea that preferred rational models are those ones in which the height of the worlds is minimized. Adding suitable possibility assumptions to a knowledge base, such a minimal model semantics corresponds to rational closure.

The correspondence between the proposed minimal model semantics and rational closure suggests the possibility of defining variants of rational closure by varying the ingredients underlying our approach, namely: (i) the properties of the preference relation $<$: for instance just preorder, or multi-linear or weakly-connected; (ii) the comparison relation on models: based for instance on the rank of the worlds or on the inclusion between the relations $<$, or on negated boxed formulas satisfied by a world, as in the logic $\mathbf{P}_{min}$ [12]. The systems obtained by various combinations of these ingredients are largely unexplored and may give rise to useful non-monotonic logics.

In the second part of the paper we have defined a rational closure construction for the Description Logic $\mathcal{ALC}$ extended with a typicality operator and provided a minimal model semantics for it, based on the idea of minimizing the rank of objects in the domain, that is their level of "untypicality". This semantics corresponds to a natural extension to DLs of Lehmann and Magidor's notion of rational closure. We have also extended the notion of rational closure to the ABox, by providing an algorithm for computing it that is sound and complete with respect to the minimal model semantics. Last, we have shown an EXPTIME upper bound for the algorithm.

In future work, concerning Description Logics, we will consider further ingredients in the recipe for non-monotonic DLs. First, we aim to study stronger versions of rational closure that allow to overcome the weaknesses of the basic one, for instance the fact that we cannot reason separately on the inheritance of different properties. Furthermore, non-monotonic extensions of *low complexity* DLs based on the $\mathbf{T}$ operator have been recently provided [13]. In future works, we aim to study the application of the proposed semantics to DLs of the $\mathcal{EL}$ and DL-Lite families, in order to define a rational closure for low complexity DLs.

References

1. F. Baader and B. Hollunder. Priorities on defaults with prerequisites, and their application in treating specificity in terminological default logic. *J. Autom. Reasoning*, 15(1):41–68, 1995.
2. Piero A. Bonatti, Carsten Lutz, and Frank Wolter. The Complexity of Circumscription in DLs. *Journal of Artificial Intelligence Research (JAIR)*, 35:717–773, 2009.
3. Katarina Britz, Johannes Heidema, and Thomas Meyer. Semantic preferential subsumption. In G. Brewka and J. Lang, editors, *KR 2008*, pages 476–484, 2008. AAAI Press.
4. G. Casini and U. Straccia. Rational Closure for Defeasible Description Logics. In T. Janhunen and I. Niemelä, editors, *Proc. of JELIA 2010*, LNAI 6341, pages 77–90, 2010. Springer.
5. Giovanni Casini and Umberto Straccia. Defeasible Inheritance-Based Description Logics. In *Proc of IJCAI 2011*, pages 813–818, 2011. Morgan Kaufmann.
6. F. M. Donini, D. Nardi, and R. Rosati. Description logics of minimal knowledge and negation as failure. *ACM Transactions on Computational Logic (ToCL)*, 3(2):177–225, 2002.
7. T. Eiter, T. Lukasiewicz, R. Schindlauer, and H. Tompits. Combining Answer Set Programming with Description Logics for the Semantic Web. In *KR 2004*, pages 141–151, 2004.
8. N. Friedman and J. Y. Halpern. Plausibility measures and default reasoning. *Journal of the ACM*, 48(4):648–685, 2001.
9. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Preferential vs Rational Description Logics: which one for Reasoning About Typicality? *ECAI 2010*, pp. 1073 - 1074, IOS Press.
10. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. ALC+T: a preferential extension of Description Logics. *Fundamenta Informaticae*, 96:1–32, 2009.
11. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Analytic Tableaux Calculi for KLM Logics of Nonmonotonic Reasoning. *ACM Trans. on Comput. Logics (TOCL)*, 10(3), 2009.
12. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. A nonmonotonic extension of KLM preferential logic P. In *LPAR 2010*, LNCS 6397, pages 317–332, 2010. Springer-Verlag.
13. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Reasoning about typicality in low complexity DLs: the logics $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and *DL-Lite*$_c\mathbf{T}_{min}$. In *Proc. of IJCAI 2011*, pages 894–899, 2011. Morgan Kaufmann.
14. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. A NonMonotonic Description Logic for Reasoning About Typicality. *Artificial Intelligence*, pages 165–202, 2012.
15. Laura Giordano, Valentina Gliozzi, Nicola Olivetti, and Gian Luca Pozzato. Preliminary result on the definition of a minimal model semantics for Rational Closure . Technical report, Dipartimento di Informatica, Università degli Studi di Torino, 2013.
16. Laura Giordano, Valentina Gliozzi, Nicola Olivetti, and Gian Luca Pozzato. Minimal model semantics and rational closure in description logics. In *Informal Proc. of DL2013)*, CEUR 1014, pages 168–180, 2013.
17. P. Ke and U. Sattler. Next Steps for Description Logics of Minimal Knowledge and Negation as Failure. In *Proc. of DL2008*, CEUR 353, 2008. CEUR-WS.org.
18. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44(1-2):167–207, 1990.
19. Adila Alfa Krisnadhi, Kunal Sengupta, and Pascal Hitzler. Local closed world semantics: Keep it simple, stupid! In *Proc. of DL2011*, CEUR 745, 2011.
20. Daniel Lehmann and Menachem Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55(1):1–60, 1992.
21. Boris Motik and Riccardo Rosati. Reconciling Description Logics and rules. *Journal of the ACM*, 57(5), 2010.
22. J. Pearl. System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *TARK*, pages 121–135,1990. Morgan Kaufmann.