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RATIONALIZABILITY AND CORRELATED EQUILIBRIA

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We discuss the unity between the two standard approaches to noncooperative solution concepts for games. The decision-theoretic approach starts from the assumption that the rationality of the players is common knowledge. This leads to the notion of correlated rationalizability. It is shown that correlated rationalizability is equivalent to a posteriori equilibrium—a refinement of subjective correlated equilibrium. Hence a decision-theoretic justification for the equilibrium approach to game theory is provided. An analogous equivalence result is proved between independent rationalizability, which is the appropriate concept if each player believes that the others act independently, and conditionally independent a posteriori equilibrium. A characterization of Nash equilibrium is also provided.

KEYWORDS: Rationalizability, correlated equilibrium, subjective and common priors, independence, Nash equilibrium.

1. INTRODUCTION

THE FUNDAMENTAL SOLUTION CONCEPT for noncooperative games is that of a Nash equilibrium (Nash (1951)). Many justifications for Nash equilibrium have been provided in the literature. Probably the most common view of Nash equilibrium is as a self-enforcing agreement. A game is envisaged as being preceded by a more or less explicit period of communication between the players. It is argued that if the players agree on a certain profile of strategies, then these must constitute a Nash equilibrium. Otherwise some player will have an incentive to deviate from the agreement. Aumann (1974) proposed the ideas of objective and subjective correlated equilibrium as extensions of Nash equilibrium to allow for correlation between the players' randomizations and for subjectivity in the players' probability assessments.

The Nash equilibrium solution concept has been criticized from two opposing directions. On the one hand, the literature on refinements of Nash equilibrium (Selten (1965, 1975), Myerson (1978), Kreps and Wilson (1982), Kohlberg and Mertens (1986) and others) starts from the contention that not every Nash equilibrium can be viewed as a plausible agreed-upon way to play the game. On the other hand, Bernheim (1984) and Pearce (1984) have argued that Nash equilibrium is too restrictive in that it rules out behavior that does not contradict the rationality of the players. Bernheim and Pearce propose instead the concept of rationalizability as the logical consequence of assuming that the structure of the game and the rationality of the players (and nothing more) is common knowledge.

This paper starts with the solution concept of rationalizability, since this is what is implied by the basic decision-theoretic analysis of a game. However, it is shown that rationalizability is more closely related to an equilibrium approach

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than one might at first think. The main result we prove in this paper is an equivalence between rationalizability and a posteriori equilibrium—a refinement of subjective correlated equilibrium. So in fact a certain kind of equilibrium arises from assuming no more than common knowledge of rationality of the players in a game. This paper therefore provides a formal decision-theoretic justification for using equilibrium concepts in game theory.

The solution concepts of rationalizability and a posteriori equilibrium will now be briefly described. Call a strategy of player i justifiable if it is optimal given some belief (probability measure) over the possible strategies of player j. (For simplicity suppose that there are only two players.) Define a justifiable strategy of j similarly. A strategy of i is rationalizable if it is justifiable using a belief which assigns positive probability only to strategies of j which are justifiable, if these latter strategies are justified using beliefs which assign positive probability only to justifiable strategies of i, and so on. In this way the notion of rationalizability captures the idea that a player should only choose a strategy which respects common knowledge of rationality. Tan and Werlang (1984) and Bernheim (1985) provide formal proofs of the equivalence between rationalizability and common knowledge of rationality.

Aumann (1974) introduced various notions of objective and subjective correlated equilibrium, including the notion of a posteriori equilibrium which refines subjective correlated equilibrium in a way we now discuss. Objective and subjective correlated equilibrium differ in that the first requires the players' priors to be the same while the second allows them to be different. In both cases the equilibrium requirement is that each player i's strategy should be ex ante optimal, that is, should maximize i's expected utility before any private information is observed. Of course this requirement is equivalent to having i's strategy maximize conditional expected utility on every information cell which is assigned positive prior probability. A possible strengthening of the definition of equilibrium is to require optimality even on null information cells. In the case of objective equilibrium this strengthening makes no difference, but it is significant for subjective equilibrium (see the example in Figure 1 of Section 2). A posteriori equilibrium is exactly this strengthening of subjective correlated equilibrium.

The equivalence result in this paper comes in two parts depending on whether one starts with "correlated" or "independent" rationalizability. The difference is that the second requires a player to believe that the other players choose their strategies independently, while the first does not. (Of course, the two versions of rationalizability coincide for two-person games.) Independent rationalizability is the concept originally defined by Bernheim (1984) and Pearce (1984). It is appropriate if one thinks of the players in a "laboratory" situation: any correlating devices are explicitly modelled, the players are placed in separate rooms, and then informed of the game they are to play. Correlated rationalizability seems more appropriate when the players are able to coordinate their actions via a large collection of correlating devices (such as sunspots) which are not explicitly modelled in the game but are taken into account by allowing for correlated beliefs.

Our starting point is that rationalizability is the solution concept implied by common knowledge of rationality of the players in a game. It is then shown that there is an equivalence between rationalizability and a posteriori equilibrium. Of course, most applications of game theory in economics assume that the players have a common prior, that is, most applications use either the Nash or objective correlated equilibrium concepts. Section 4 of the paper discusses characterizations of these solution concepts.

In a related paper, Aumann (1987) adopts a somewhat different notion of Bayesian rationality from that in this paper. Bayesian rationality is formalized using a standard model of differential information with the additional feature that the state space includes the actions of the players. Under an assumption of common knowledge of rationality together with an assumption of common priors (the Common Prior Assumption) one is again led to objective correlated equilibrium. For the details of this characterization and a discussion of the Common Prior Assumption the reader should consult Aumann (1987). Alternative characterizations of objective solution concepts can also be found in Tan and Werlang (1984) and Bernheim (1985).

The organization of the rest of the paper is as follows. Section 2 provides formal definitions of correlated rationalizability and a posteriori equilibrium, and proves the equivalence result between these two concepts. In Section 3 an analogous equivalence result is proved between independent rationalizability and conditionally independent a posteriori equilibrium. Section 4 discusses characterizations of objective correlated equilibrium and Nash equilibrium.

2. CORRELATED RATIONALIZABILITY AND A POSTERIORI EQUILIBRIA

This section starts by defining the sets of correlated rationalizable strategies and payoffs in a game. The approach is based on that in Pearce (1984). However, unlike Pearce's paper, players are not allowed to select mixed strategies—allowing them to do so would not expand the set of rationalizable payoffs. Also, a player's beliefs over the actions of the other players may be correlated (cf. Pearce (1984, p. 1035)). The next section examines the case in which these beliefs are independent.

Consider an n-person game $\Gamma = \langle A^1, \ldots, A^n; u^1, \ldots, u^n \rangle$ where for each $i = 1, \ldots, n$, A^i is a finite set of pure strategies (henceforth actions) of player i and $u^i : \prod_{j=1}^{n} A^j \to \mathscr{R}$ is i's payoff function. For any finite set Y, let $\Delta(Y)$ denote the set of probability measures on Y. Given sets $Y^1, \ldots, Y^n$, Y^{-i} denotes the set $Y^1 \times \cdots \times Y^{i-1} \times Y^{i+1} \times \cdots \times Y^n$, and $y^{-i} = (y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^n)$ is a typical element of Y^{-i}.

DEFINITION 2.1: A subset $B^1 \times \cdots \times B^n$ of $A^1 \times \cdots \times A^n$ is a *best reply set* if for every i and each $a^i \in B^i$ there is a $\sigma \in \Delta(B^{-i})$ to which a^i is a best reply.

The set of correlated rationalizable actions $R^1 \times \cdots \times R^n$ is the (finite) component-by-component union $(\bigcup_\alpha B^1_\alpha) \times \cdots \times (\bigcup_\alpha B^n_\alpha)$ of all best reply sets $B^1_\alpha \times \cdots \times B^n_\alpha$. It is easy to check that $R^1 \times \cdots \times R^n$ is itself a best reply set. This fact will be used below. There are two equivalent definitions of the set $R^1 \times \cdots \times$

R^n. One is in terms of the systems of justifiable actions discussed in the Introduction. The other is in terms of iterated deletion of strongly dominated actions. (Proofs of the equivalence of the three definitions are easily adapted from arguments in Bernheim (1984) and Pearce (1984).) i's maximal expected payoff against a $\sigma \in \Delta(R^{-i})$ is a correlated rationalizable payoff to i. Let Π^i denote the set of all possible correlated rationalizable payoffs to i.

We now want to define an a posteriori equilibrium (Aumann (1974, Section 8)) of the game Γ. First, the definition of a subjective correlated equilibrium of Γ is reviewed, and then an a posteriori equilibrium is defined as a special kind of subjective correlated equilibrium. To define a subjective correlated equilibrium of Γ, one must add to the basic description of the game a finite space Ω. The finiteness of Ω entails no loss of generality. Each player i has a prior P^i—a probability measure on Ω—and a partition $\mathscr{H}^i$ of Ω. A strategy of player i is an $\mathscr{H}^i$-measurable map $f^i : \Omega \to A^i$. An n-tuple of strategies $(f^1, \ldots, f^n)$ is a subjective correlated equilibrium if for every i

$$\sum_{\omega \in \Omega} P^i(\{\omega\}) u^i[f^i(\omega), f^{-i}(\omega)] \geq \sum_{\omega \in \Omega} P^i(\{\omega\}) u^i[\tilde{f}^i(\omega), f^{-i}(\omega)]$$

for every strategy $\tilde{f}^i$ of i. The definition of subjective correlated equilibrium is more general than that of objective correlated equilibrium in that it allows the players' priors P^i to be different. If the P^i's are required to be equal then one gets objective correlated equilibrium.

In a subjective correlated equilibrium the players' strategies are only required to be ex ante optimal. In an a posteriori equilibrium the players' strategies must be optimal even after they have learned their private information. The following simple example motivates this distinction. Refer to Figure 1. The set Ω consists of two points ω_1, ω_2. Row is informed of the true state, Column has no private information. Row assigns (prior) probability 1 to ω_1, Column assigns probability $\frac{1}{2}$ to ω_1, $\frac{1}{2}$ to ω_2. The following strategies form a subjective correlated equilibrium: Row plays U if informed that ω_1 happens, D if ω_2 happens; Column plays L. Notice that this equilibrium relies on Row playing a strongly dominated action

	L	R
U	3, 0	1, 2
D	0, 4	0, 1

FIGURE 1.

if ω_2 happens. As in the literature on refinements of Nash equilibrium, it seems natural to rule out such situations by requiring optimal behavior even on null events—in this case after a move by Nature which is assigned prior probability zero. The definition of an a posteriori equilibrium (Definition 2.2) is designed to deal with this issue. The unique a posteriori equilibrium of the game in Figure 1 has Row playing U and hence Column playing R for sure.

To define an a posteriori equilibrium formally, start again with the game Γ. As for a subjective correlated equilibrium, add to Γ a finite space Ω and for each player i a probability measure P^i on Ω and a partition $\mathcal{H}^i$ of Ω. Furthermore, in order to deal with the kind of difficulty raised in the example above, the players' posterior beliefs at every $\omega \in \Omega$ must be specified. So for each player i choose a version of conditional probability which is regular and proper. That is, for *every* $H^i \in \mathcal{H}^i$, $P^i(\cdot \mid H^i)$ is required to be a probability measure on Ω and to satisfy $P^i(H^i \mid H^i) = 1$. (This last requirement is properness in the sense of Blackwell and Dubins (1975).) Of course if $P^i(H^i) > 0$, then by Bayes' rule $P^i(\cdot \mid H^i)$ automatically satisfies both requirements, but the point is that $P^i(\cdot \mid H^i)$ is required to satisfy them even if $P^i(H^i) = 0$. For each i, let $\mathcal{H}^i(\omega)$ denote the cell of i's partition that contains ω.

DEFINITION 2.2: An n-tuple of strategies $(f^1, \ldots, f^n)$ is an *a posteriori equilibrium of Γ* if for each i

$$\forall \omega \in \Omega \qquad \sum_{\omega' \in \Omega} P^i[\{\omega'\} \mid \mathcal{H}^i(\omega)] u^i[f^i(\omega), f^{-i}(\omega')]$$
$$\geqslant \sum_{\omega' \in \Omega} P^i[\{\omega'\} \mid \mathcal{H}^i(\omega)] u^i[a^i, f^{-i}(\omega')] \qquad \forall a^i \in A^i.$$

Notice that by a change of variables, i's optimality condition can be rewritten as:

$$\forall \omega \in \Omega \qquad \sum_{a^{-i} \in A^{-i}} P^i[\{\omega' : f^{-i}(\omega') = a^{-i}\} \mid \mathcal{H}^i(\omega)] u^i[f^i(\omega), a^{-i}]$$
$$\geqslant \sum_{a^{-i} \in A^{-i}} P^i[\{\omega' : f^{-i}(\omega') = a^{-i}\} \mid \mathcal{H}^i(\omega)] u^i(a^i, a^{-i}) \quad \forall a^i \in A^i.$$

From the point of view of the players there are two stages to the game: the ex ante and the interim stages corresponding to before and after they receive their private information. It will be helpful to distinguish between a player's payoffs at these two stages.

DEFINITION 2.3: Given an a posteriori equilibrium $(f^1, \ldots, f^n)$ of Γ, *i's interim payoff* at ω is

$$\sum_{a^{-i} \in A^{-i}} P^i[\{\omega' : f^{-i}(\omega') = a^{-i}\} \mid \mathcal{H}^i(\omega)] u^i[f^i(\omega), a^{-i}];$$

i's ex ante payoff is

$$\sum_{\omega\in\Omega} P^i(\{\omega\}) \sum_{a^{-i}\in A^{-i}} P^i[\{\omega': f^{-i}(\omega') = a^{-i}\} \mid \mathscr{H}^i(\omega)] u^i[f^i(\omega), a^{-i}].$$

The basic equivalence result in this section (Proposition 2.1) is between correlated rationalizable payoffs and interim payoffs from a posteriori equilibria. The idea behind rationalizability is that (according to Bayesian decision theory) player i has a certain given belief over the actions of the other players, and this determines i's (maximal) expected payoff. On the other hand, at the ex ante stage in an a posteriori equilibrium i does not yet know what his/her belief over the other players' actions will be. This belief will be equal to i's conditional probability which is determined by i's information, i.e. it is i's belief at the interim stage. This is why the basic equivalence result is stated in terms of interim payoffs. In fact, because of convexity of the set of correlated rationalizable payoffs to i (Lemma 2.1) one can also prove an equivalence between correlated rationalizable payoffs and ex ante payoffs from a posteriori equilibria—see Proposition 2.2.

PROPOSITION 2.1: *$(\pi^1, \cdots, \pi^n) \in \Pi^1 \times \cdots \times \Pi^n$ if and only if there is an a posteriori equilibrium of Γ in which $(\pi^1, \ldots, \pi^n)$ is a vector of interim payoffs.*

PROOF: *Only if.* Given a vector $(\pi^1, \ldots, \pi^n) \in \Pi^1 \times \cdots \times \Pi^n$, we have to show that there is an a posteriori equilibrium of Γ in which $(\pi^1, \ldots, \pi^n)$ is a vector of interim payoffs. To see this, consider a mediator (cf. Myerson (1985)) who randomly selects a joint action $(a^1, \ldots, a^n) \in R^1 \times \cdots \times R^n$ and recommends to each player i to play a^i. Since π^i is a correlated rationalizable payoff to i, there is an $\tilde{a}^i \in R^i$ and a $\tilde{\sigma} \in \Delta(R^{-i})$ such that $\tilde{a}^i$ is a best reply to $\tilde{\sigma}$ and π^i is i's expected payoff from playing $\tilde{a}^i$ against $\tilde{\sigma}$. If i is recommended to play $\tilde{a}^i$ then the conditional probability with which i believes the mediator chooses actions in R^{-i} is $\tilde{\sigma}$. For any other $a^i \in R^i$ choose a $\sigma \in \Delta(R^{-i})$ to which a^i is a best reply. If i is recommended to play a^i, then the conditional probability with which i believes the mediator chooses actions in R^{-i} is σ. With these conditional probabilities i will be willing to follow the mediator's recommendations, and when informed of $\tilde{a}^i$, i's conditional expected payoff from this a posteriori equilibrium is π^i.

If. We have to show that a vector of interim payoffs from an a posteriori equilibrium $(f^1, \ldots, f^n)$ of Γ is an element of $\Pi^1 \times \cdots \times \Pi^n$. For each i let $A_+^i = \{a^i \in A^i: a^i = f^i(\omega)$ for some $\omega \in \Omega\}$. The set $A_+^1 \times \cdots \times A_+^n$ is a best reply set. To prove this, it has to be shown that for every i and each $a^i \in A_+^i$ there is a $\sigma \in \Delta(A_+^{-i})$ to which a^i is a best reply. Given an $a^i \in A_+^i$ choose an ω such that $f^i(\omega) = a^i$. Since $(f^1, \ldots, f^n)$ is an a posteriori equilibrium, and only strategies $a^{-i} \in A_+^{-i}$ "enter" into the equilibrium, i's optimality condition at ω can be written as

$$\sum_{a^{-i}\in A_+^{-i}} P^i[\{\omega': f^{-i}(\omega') = a^{-i}\} \mid \mathscr{H}^i(\omega)] u^i(a^i, a^{-i})$$
$$\geqslant \sum_{a^{-i}\in A_+^{-i}} P^i[\{\omega': f^{-i}(\omega') = a^{-i}\} \mid \mathscr{H}^i(\omega)] u^i(\tilde{a}^i, a^{-i}) \qquad \forall \tilde{a}^i \in A^i.$$

This says that a^i is a best reply to the strategy σ which assigns probability $P^i[\{\omega' : f^{-i}(\omega') = a^{-i}\} | \mathscr{H}^i(\omega)]$ to a^{-i} for $a^{-i} \in A_+^{-i}$. It follows that i's conditional expected payoff on $\mathscr{H}^i(\omega)$ is a correlated rationalizable payoff to i. *Q.E.D.*

Proposition 2.1 also implies that for each player i the set of actions in A^i which are played in some a posteriori equilibrium of Γ is equal to R^i, i.e. the set of correlated rationalizable actions of i. Hence the equivalence result in Proposition 2.1 could be stated in terms of actions as well as payoffs. The same remark applies to Propositions 3.1 and 4.1 later in the paper.

Suppose that in the first half of the proof of Proposition 2.1, player i assigns (prior) probability 1 to being recommended to play $\tilde{a}^i$. Then i's ex ante payoff is also π^i. It was shown in the second half of the proof that i's conditional expected payoff on $\mathscr{H}^i(\omega)$ is a correlated rationalizable payoff to i. So i's ex ante payoff from the a posteriori equilibrium $(f^1, \ldots, f^n)$ is a convex combination of correlated rationalizable payoffs to i. Lemma 2.1 below says that the set of correlated rationalizable payoffs to i is convex. Putting these observations together shows that the term "interim" in Proposition 2.1 can be replaced with "ex ante."

PROPOSITION 2.2: *The set of ex ante payoff vectors from the a posteriori equilibria of Γ is equal to $\Pi^1 \times \cdots \times \Pi^n$.*

As argued above, Proposition 2.2 will be implied by the following Lemma.

LEMMA 2.1: *Π^i is convex for every i.*

PROOF: For any $\sigma \in \Delta(R^{-i})$, let $v^i(a^i, \sigma)$ be i's expected payoff from playing a^i against σ. Then $\Pi^i = \{\max_{a^i \in A^i} v^i(a^i, \sigma) : \sigma \in \Delta(R^{-i})\}$. But $\{\max_{a^i \in A^i} v^i(a^i, \sigma) : \sigma \in \Delta(R^{-i})\}$ is the image of the continuous map $\sigma \mapsto \max_{a^i \in A^i} v^i(a^i, \sigma)$, and is therefore a closed interval since the domain $\Delta(R^{-i})$ is compact and connected. *Q.E.D.*

3. INDEPENDENT RATIONALIZABILITY AND CONDITIONALLY INDEPENDENT A POSTERIORI EQUILIBRIA

The previous section established an equivalence between correlated rationalizability and a posteriori equilibrium. In this section analogous results are obtained starting from independent rather than correlated rationalizability. Independent rationalizability is the concept originally defined by Bernheim (1984) and Pearce (1984). The set of independent rationalizable payoffs is the subset of the set of correlated rationalizable payoffs obtained by restricting each player's beliefs over the actions of the other players to be independent. Clearly these sets of payoffs are the same in two-person games. To see that the set of independent rationalizable payoffs is a strict subset of the set of correlated rationalizable payoffs in games with three or more players, consider the example in Figure 2. Player 1 chooses the row, 2 the column, 3 the matrix. 0.7 is a correlated rationalizable payoff to

A

	L	R
U	1, 1, 1	1, 0, 1
D	0, 1, 0	0, 0, 0

B

	L	R
U	2, 2, .7	0, 0, 0
D	0, 0, 0	2, 2, .7

C

	L	R
U	1, 1, 0	1, 0, 0
D	0, 1, 1	0, 0, 1

FIGURE 2.

3 as follows. Player 3 believes 1, 2 play (U, L) with probability $\frac{1}{2}$, (D, R) with probability $\frac{1}{2}$ (to which B is the best reply). Player 1 believes 2 plays L with probability $\frac{1}{2}$, R with probability $\frac{1}{2}$, and 3 plays B (to which U and D are best replies). Player 2 believes 1 plays U with probability $\frac{1}{2}$, D with probability $\frac{1}{2}$, and 3 plays B (to which L and R are best replies). On the other hand, 1 is the unique independent rationalizable payoff to 3. To see this, first note that B is not a best reply to any pair of mixed strategies of 1, 2. Hence 1, 2 must assign probability 0 to 3 playing B. But then U, L, strongly dominate D, R for 1, 2 respectively.

DEFINITION 3.1: A subset $\hat{B}^1 \times \cdots \times \hat{B}^n$ of $A^1 \times \cdots \times A^n$ is an *independent best reply set* if for every i and each $a^i \in \hat{B}^i$ there is a $\sigma^{-i} \in \prod_{j \neq i} \Delta(\hat{B}^j)$ to which a^i is a best reply.

The set of independent rationalizable actions $\hat{R}^1 \times \cdots \times \hat{R}^n$ is the (finite) component-by-component union $(\bigcup_\alpha \hat{B}^1_\alpha) \times \cdots \times (\bigcup_\alpha \hat{B}^n_\alpha)$ of all independent best reply sets $\hat{B}^1_\alpha \times \cdots \times \hat{B}^n_\alpha$. Player i's maximal expected payoff against a $\sigma^{-i} \in \prod_{j \neq i} \Delta(\hat{R}^j)$ is then an independent rationalizable payoff to i. Let $\hat{\Pi}^i$ denote the set of all possible independent rationalizable payoffs to i.

The results of the previous section would suggest an equivalence between independent rationalizable payoffs and interim payoffs from "mixed" a posteriori equilibria (and if the set of independent rationalizable payoffs is convex, that the equivalence holds for ex ante payoffs as well). This intuition is correct; however "mixed" should not be taken to mean independence of the players' partitions of Ω in terms of their priors (which is the usual approach). What is needed is a form of conditional independence, which is not implied by a definition in terms of priors.

DEFINITION 3.2: $\mathscr{H}^1, \ldots, \mathscr{H}^n$ are *P^i-prior independent* if $P^i(\bigcap_{j=1}^n H^j) = \prod_{j=1}^n P^i(H^j)$ for every $H^j \in \mathscr{H}^j, j = 1, \ldots, n$. $\mathscr{H}^1, \ldots, \mathscr{H}^{i-1}, \mathscr{H}^{i+1}, \ldots, \mathscr{H}^n$ are *P^i-conditionally independent* given $\mathscr{H}^i$ if for every $H^i \in \mathscr{H}^i$, $P^i(\bigcap_{j \neq i} H^j \mid H^i) = \prod_{j \neq i} P^i(H^j \mid H^i)$ for every $H^j \in \mathscr{H}^j$, $j \neq i$.

Prior independence is the standard definition of independent σ-fields (Chung (1974, p. 61)). It is also the notion of independence used in Aumann (1974) to define mixed strategies. Our definition of conditional independence is a strengthening of the standard definition of conditionally independent σ-fields (Chung (1974, p. 306)) from an almost everywhere to an everywhere requirement. Conditional independence says that *whatever* information i receives, i believes that the other players choose their actions independently. Prior independence implies that if $P^i(H^i) > 0$, then $P^i(\bigcap_{j\neq i} H^j \mid H^i) = \prod_{j\neq i} P^i(H^j)$. So prior independence only implies that i believes with P^i-probability 1 that the others play independently. Prior independence says nothing about $P^i(\bigcap_{j\neq i} H^j \mid H^i)$ if H^i is P^i-null, so prior independence does not imply conditional independence. Nor does conditional independence imply prior independence. (Let $\Omega = \{\omega_1, \omega_2\}$ and suppose all the players have the finest partition. If $P^i(\{\omega_1\}) = \frac{1}{2}$ then conditional independence is satisfied but prior independence is not.)

DEFINITION 3.3: A *conditionally independent a posteriori equilibrium of* Γ is an a posteriori equilibrium of Γ in which for every i, $\mathcal{H}^1, \ldots, \mathcal{H}^{i-1}, \mathcal{H}^{i+1}, \ldots, \mathcal{H}^n$ are P^i-conditionally independent given $\mathcal{H}^i$.

PROPOSITION 3.1: *The sets of interim and ex ante payoff vectors from the conditionally independent a posteriori equilibria of* Γ *are both equal to* $\hat{\Pi}^1 \times \cdots \times \hat{\Pi}^n$.

PROOF: *Only if.* Given a vector $(\pi^1, \ldots, \pi^n) \in \hat{\Pi}^1 \times \cdots \times \hat{\Pi}^n$, we have to show that there is a conditionally independent a posteriori equilibrium of Γ in which $(\pi^1, \ldots, \pi^n)$ is a vector of interim and ex ante payoffs. The proof is like the first half of the proof of Proposition 2.1. A mediator randomly selects a joint action $(a^1, \ldots, a^n) \in \hat{R}^1 \times \cdots \times \hat{R}^n$ and recommends to each player i to play a^i. Since π^i is an independent rationalizable payoff to i, there is an $\tilde{a}^i \in \hat{R}^i$ and a $\tilde{\sigma}^{-i} \in \prod_{j\neq i} \Delta(\hat{R}^j)$ such that $\tilde{a}^i$ is a best reply to $\tilde{\sigma}^{-i}$ and π^i is i's expected payoff from playing $\tilde{a}^i$ against $\tilde{\sigma}^{-i}$. If i is recommended to play $\tilde{a}^i$ then the conditional probability with which i believes the mediator chooses actions in $\hat{R}^{-i}$ is $\tilde{\sigma}^{-i}$. Notice that $\tilde{\sigma}^{-i}$ is a product measure on $\hat{R}^{-i}$. Continuing in this way, after any recommendation i's conditional probability on $\hat{R}^{-i}$ is a product measure. So the a posteriori equilibrium which is constructed is conditionally independent, and π^i is an interim payoff to i. By letting i assign (prior) probability 1 to the mediator recommending $\tilde{a}^i$, this is also the ex ante payoff to i.

If. We have to show that a vector of interim or ex ante payoffs from a conditionally independent a posteriori equilibrium $(f^1, \ldots, f^n)$ of Γ is an element of $\hat{\Pi}^1 \times \cdots \times \hat{\Pi}^n$. The proof is essentially the same as the second half of the proof of Proposition 2.1. For each i let $A^i_+ = \{a^i \in A^i : a^i = f^i(\omega) \text{ for some } \omega \in \Omega\}$. The set $A^1_+ \times \cdots \times A^n_+$ is an independent best reply set. This follows from the same argument as before, noting that

$$P^i[\{\omega' : f^{-i}(\omega') = a^{-i}\} \mid \mathcal{H}^i(\omega)] = \prod_{j\neq i} P^i[\{\omega' : f^j(\omega') = a^j\} \mid \mathcal{H}^i(\omega)]$$

because of conditional independence. It follows that i's conditional expected payoff on $\mathscr{H}^i(\omega)$ is an independent rationalizable payoff to i. Player i's ex ante payoff will then be a convex combination of independent rationalizable payoffs, and a trivial modification of Lemma 2.1 shows that each $\hat{\Pi}^i$ is convex. *Q.E.D.*

To prove an equivalence with independent rationalizability it was necessary to use conditionally independent a posteriori equilibrium. Recall that conditional independence does not in general imply prior independence. Nevertheless, when considering conditionally independent a posteriori equilibria, prior independence can be assumed without loss of generality. More precisely, the sets of interim and ex ante payoffs from the conditionally independent a posteriori equilibria which also satisfy prior independence are again both equal to $\hat{\Pi}^1 \times \cdots \times \hat{\Pi}^n$. To see this, first note that by definition these sets must be contained in $\hat{\Pi}^1 \times \cdots \times \hat{\Pi}^n$. Second note that the a posteriori equilibrium constructed in the first half of the proof of Proposition 3.1 satisfies prior independence (when each player i assigns probability 1 to the mediator recommending $\tilde{a}^i$).

4. OBJECTIVE SOLUTION CONCEPTS

The starting point of this paper is that rationalizability (either correlated or independent) is the solution concept implied by common knowledge of rationality of the players in a game. The previous two sections established equivalences between rationalizability (correlated and independent) and equilibrium concepts (a posteriori and conditionally independent a posteriori). It follows that common knowledge of rationality alone implies equilibrium behavior on the part of the players. Notice that the players may have subjective, i.e. different, priors. However, as discussed in the Introduction, it is usual in applications to assume that the players have the same prior. If an assumption of common priors is adopted then, as Aumann (1987) has shown, one is led to objective correlated equilibrium rather than a posteriori equilibrium. In order to go further and characterize Nash equilibrium, additional assumptions must be made.

One way to characterize Nash equilibrium is to adopt, in addition to a common prior, the assumption of prior independence (Definition 3.2). In the alternative characterization provided below the assumption of common priors is weakened to the requirement that any two players share the same beliefs about a third player's choice of action. This requirement is met by assuming "concordant" priors. Technically this assumption differs only slightly from common priors, in that under concordant priors player i's belief over events in $\mathscr{H}^i$ need not be the same as the (common) beliefs of the other players. However, this is perhaps more natural, since i's beliefs over events in $\mathscr{H}^i$ have no decision theoretic significance for the play of the game.

DEFINITION 4.1: $P^1, \ldots, P^n$ are *concordant* if for each i and every $j, k \neq i$, $P^j(H^i) = P^k(H^i)$ for every $H^i \in \mathscr{H}^i$.

To compensate for weakening the assumption of common priors, prior independence must be strengthened to hold "everywhere." Recall that prior independence says that with probability 1: (i) i's beliefs over the other players' choice of action is a product measure; and (ii) i will not update his/her prior. Conditional independence is designed to strengthen (i) to an everywhere condition. It remains to strengthen (ii) to an everywhere condition. This is achieved by assuming "informational independence." (We are grateful to a referee for suggesting the following definition.) Notice that both concordant priors and conditional independence are automatically satisfied for two-person games (whereas the assumption that players beliefs do not vary with their private information *is* needed in two-person games).

DEFINITION 4.2: $\mathscr{H}^1, \ldots, \mathscr{H}^{i-1}, \mathscr{H}^{i+1}, \ldots, \mathscr{H}^n$ are *P^i-informationally independent* of $\mathscr{H}^i$ if for every H^i and $\tilde{H}^i \in \mathscr{H}^i$, $P^i(\bigcap_{j\neq i} H^j \mid H^i) = P^i(\bigcap_{j\neq i} H^j \mid \tilde{H}^i)$ for every $H^j \in \mathscr{H}^j, j \neq i$.

PROPOSITION 4.1: *Consider the a posteriori equilibria of Γ which have concordant priors and in which for every i, $\mathscr{H}^1, \ldots, \mathscr{H}^{i-1}, \mathscr{H}^{i+1}, \ldots, \mathscr{H}^n$ are P^i-conditionally independent given $\mathscr{H}^i$ and P^i-informationally independent of $\mathscr{H}^i$. The sets of interim and ex ante payoffs from these equilibria are both equal to the set of expected payoff vectors from the Nash equilibria of Γ.*

PROOF: Consider an a posteriori equilibrium $(f^1, \ldots, f^n)$ of Γ and for each i let $A^i_+ = \{a^i \in A^i : a^i = f^i(\omega) \text{ for some } \omega \in \Omega\}$. i's conditional expected payoff on $H^i \in \mathscr{H}^i$ from playing a^i is

$$\sum_{a^{-i} \in A^{-i}_+} P^i[\{\omega : f^{-i}(\omega) = a^{-i}\} \mid H^i] u^i(a^i, a^{-i})$$

which is equal to

$$\sum_{a^{-i} \in A^{-i}_+} \prod_{j \neq i} P^i[\{\omega : f^j(\omega) = a^j\} \mid H^i] u^i(a^i, a^{-i})$$

by conditional independence, which in turn equals

$$\sum_{a^{-i} \in A^{-i}_+} \prod_{j \neq i} P^i[\{\omega : f^j(\omega) = a^j\}] u^i(a^i, a^{-i})$$

by informational independence. Write $P^i[\{\omega : f^j(\omega) = a^j\}] = \sigma^j(a^j)$ and let $\sigma^j \in \Delta(A^j_+)$ be the mixed strategy which assigns probability $\sigma^j(a^j)$ to each $a^j \in A^j_+$. Note that σ^j does not depend on i by the assumption of concordant priors. In other words, i's conditional expected payoff from playing a^i is the expected payoff from playing a^i against the $(n-1)$-tuple of mixed strategies σ^{-i}. Let $BR(\sigma^{-i})$ denote the set of i's best replies to σ^{-i}. Then $A^i_+ \subset BR(\sigma^{-i})$. So there are sets $A^1_+ \subset A^1, \ldots, A^n_+ \subset A^n$ and mixed strategies $\sigma^1 \in \Delta(A^1_+), \ldots, \sigma^n \in \Delta(A^n_+)$ such that $A^1_+ \subset BR(\sigma^{-1}), \ldots, A^n_+ \subset BR(\sigma^{-n})$. That is, $(\sigma^1, \ldots, \sigma^n)$ is a Nash equilibrium. So i's conditional expected payoff on any H^i—and hence i's ex ante payoff—is equal to i's expected payoff from a Nash equilibrium. The converse direction is immediate. *Q.E.D.*

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