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THREE PHOTON VIRTUAL ANNIHILATION CONTRIBUTIONS TO POSITRONIUM HYPERFINE STRUCTURE *)

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ABSTRACT

Three photon virtual annihilation contributions to the positronium hyperfine interval are calculated to order $\alpha^6 m$ ($\alpha^4 R_\infty$). The result is obtained in analytic form and leads to the small value of -0.91 MHz for the contribution to the ground state energy difference from this source.

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The positronium and muonium spectra and positronium decay currently constitute the most critical (and possibly the last) areas in which low energy QED needs to be tested. The particular interest of positronium bound states is that they provide a sensitive test of two body relativistic equations (the Bethe-Salpeter[1] equation, or related forms[2]) in a case where the interaction is entirely of electrodynamic origin. Agreement between theory[3] and experiment[4] at the level of first non-leading order ($\alpha^3 R_\infty$; $R_\infty = \frac{1}{2}m\alpha^2$) was demonstrated long ago for ground state hyperfine structure; agreement at the same level for selected fine structure transitions in the first excited states was shown to exist more recently[5,6]. The next order in the perturbation expansion ($\alpha^6 \ell n \alpha^{-1} m$ and $\alpha^6 m$) is of particular theoretical interest because it requires a more detailed treatment of the two-body relativistic equations and the surmounting of some difficulties in the infrared. On the experimental side, measurements of ground state hfs energies of an accuracy sufficient to test the theory at this level have been available for some time. (The most recent results are reported in Refs. 7 and 8.) The theoretical calculations, as is frequently the case in low energy QED, have lagged behind. Various contributions to order $\alpha^6 \ell n \alpha^{-1} m$ have been calculated and reconfirmed[2,9,10,11,12]. There is some indication that the numerical coefficient of the total contribution to this order is small[2]. The contrary is true of the one

known term[13] in order $\alpha^6 m$ (an extension of the results for hydrogen hfs) which has a large numerical coefficient. This suggests that $\alpha^6 m$ terms, though more difficult to obtain than $\alpha^6 \ell n \alpha^{-1} m$ terms, cannot be neglected relative to the latter, if a significant comparison between theory and experiment is to be made.

The present note is a report on the first stage of what is hoped will be a systematic calculation of positronium ground state hfs to order $m\alpha^6$. We have considered the combination of three photon virtual annihilation diagrams (shown in Fig. 1) to this energy difference. At this level, it is sufficient to consider the positron and electron to be free particles confined in a box of volume $|\phi(0)|^{-2}$, where $\phi(0)$ is the non-relativistic Schroedinger wave function at the origin of the relative coordinates. In order to provide a natural low-momentum cut-off, it is convenient to take the free particles off mass shell by letting

$$p_\mu = \bar{p}_\mu = \tfrac{1}{2}K_\mu, \quad K_\mu = (K_o; \vec{0}) \tag{1}$$

$$K_o \simeq (2-\varepsilon^2)m; \quad \varepsilon = \tfrac{1}{2}\alpha$$

The infrared divergent terms, though present at intermediate stages of the calculation, eventually cancel.

The six terms in the perturbation calculation arising from Figs. (1*a*)-(1*f*) can be reduced to two independent terms by a judicious use of the charge conjugation

matrix C. Insertion of CC^{-1} in a given electron line, together with a suitable relabeling of photon momenta, corresponds to a reflection of that line about a horizontal axis bisecting the line, with the photon vertices remaining attached. A reflection of both electron lines shows that Figs. (1*c*), (1*d*) on the one hand and (1*e*), (1*f*) on the other hand are topologically equivalent. A reflection of only one electron line serves to demonstrate the equivalence of Figs. (1*a*), (1*b*) and of Figs. (1*c*), (1*e*). The final contribution is thus twice that due to Fig. (1*a*) plus four times the one arising from Fig. (1*c*).

The calculation was carried to completion analytically. An essential feature of the analysis is the identification and isolation of singular parts which go as $ln^2\varepsilon$, $ln\varepsilon$ and $i\pi ln\varepsilon$. We take the limit $\varepsilon\to 0$ in the remaining parts. The contributions, ΔE_a, from the two terms related to Fig. (1*a*) are:

$$\Delta E_a = \frac{8}{\pi}\frac{\alpha^3}{m^2}|\phi(0)|^2\{-i\pi[\frac{1}{6}ln\varepsilon^2+(-\frac{1}{2}-\frac{1}{6}ln2+\frac{5\pi^2}{36})]$$
$$+ ln^2\varepsilon^2 + (\frac{2}{3}+\frac{1}{3}ln2)ln\varepsilon^2 + [-\frac{1}{6}ln^2 2 - (1+\frac{5\pi^2}{9})ln2$$
$$+ \frac{5}{4}\zeta(3) + \frac{3\pi^2}{8} + \frac{1}{12}]\}, \qquad (2)$$

where $\zeta(3)$ is the Riemann Zeta function of integer argument 3. The corresponding contributions from the four terms related to Fig. (1*c*) are

$$\Delta E_c = \frac{8}{\pi}\frac{\alpha^3}{m^2}|\phi(0)|^2\{-i\pi[-\frac{1}{6}ln\varepsilon^2+(-\frac{1}{2}+\frac{1}{6}ln2 - \frac{\pi^2}{36})]$$

$$- ln^2\varepsilon^2 - (\frac{2}{3}+\frac{1}{3}ln2)ln\varepsilon^2 + [\frac{1}{6}ln^2 2+(-1+\frac{19\pi^2}{36})ln2$$

$$- \frac{7}{8}\zeta(3) - \frac{23\pi^2}{36} + \frac{19}{6}]\}. \qquad (3)$$

The total contribution, the sum of (2) and (3), is

$$\Delta E^A_{3\gamma} = \alpha^2 R_\infty \frac{\alpha^2}{\pi^2}[-4ln2 - \frac{19\pi^2}{36} - \frac{\pi^2 ln2}{18} + \frac{3}{4}\zeta(3) + \frac{13}{2}$$

$$- \frac{2\pi i}{9}(\pi^2-9)], \qquad (4)$$

where we have used $|\phi(0)|^2 = (m\alpha)^3/8\pi$. As stated earlier, the logarithmic infrared singularities present in the intermediate result of Eqs. (2) and (3) have cancelled out in the final result, Eq. (4).

Of course, only the real part of Eq. (4) represents the contribution to the hfs energy. The imaginary part is related to the three photon annihilation lifetime of positronium through the expression

$$\text{Im}\ \Delta E^A_{3\gamma} = -\frac{1}{2}\Gamma_{3\gamma}. \qquad (5)$$

Eq. (5) agrees with the standard result[14] and serves as a partial check on our calculation. Evaluation of the real part of Eq. (4) gives

$$\Delta E^A_{3\gamma} = \frac{\alpha^4}{\pi^2} R_\infty\{-0.96\} = -0.91\ \text{MHz}. \qquad (6)$$

The result, Eq. (6) is too small by at least one order of magnitude to affect significantly the comparison with cur-

rent experimental results. This is because the "natural" expansion parameter (i.e., the one with coefficient unity) appears to be $\frac{\alpha}{\pi}$ rather than α for this term.

In conclusion, we may remark that it is not difficult to show that the contributions we have considered are independent of the choice of gauge, for the radiation gauge and for the wide class of covariant gauges which have a term $\xi k_\mu k_\nu / k^2$ (ξ arbitrary) in the photon propagator $D_{\mu\nu}(k)$.

Details of this work will be presented elsewhere.

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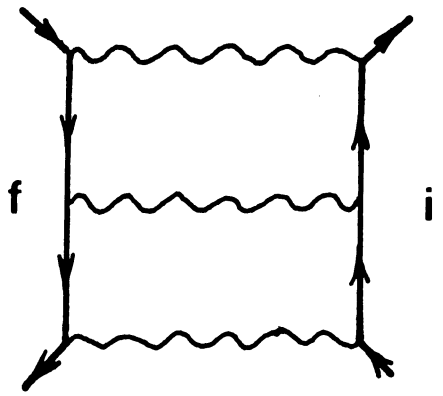
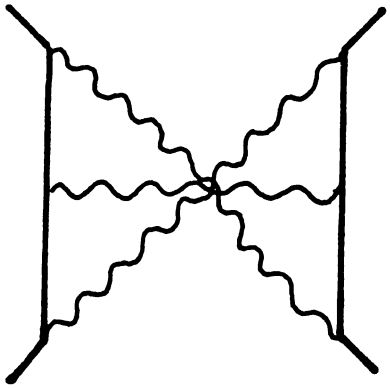
FIGURE CAPTION

These six diagrams represent the kernel for three photon virtual annihilation. The letters i and f denote initial and final states of the electron-positron system.



(a)



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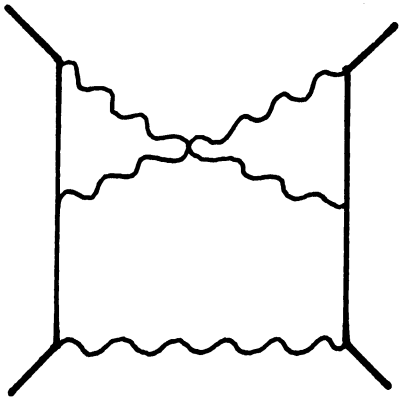

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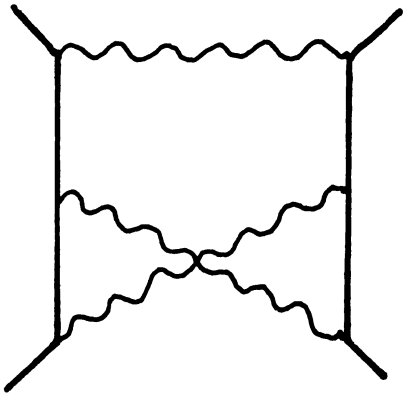

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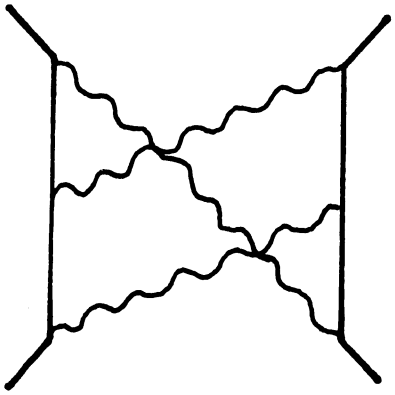

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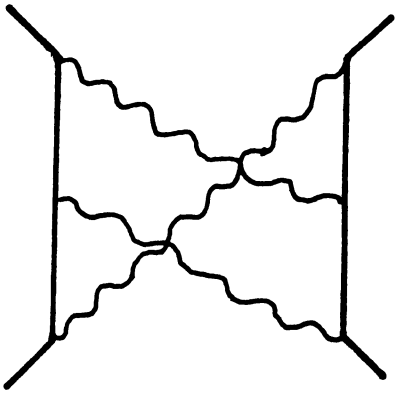

(f)