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# CONSTRAINTS ON REGGEON AMPLITUDES FROM ANALYTICITY AND PLANAR UNITARITY*)

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ABSTRACT

Using planar dual amplitudes as a guide, we discuss some features of Reggeon amplitudes which are relevant in the context of the Topological Expansion. We look into the analytic properties and, in particular, discuss the validity of Finite-Mass-Sum-Rules for Reggeon-Reggeon scattering. We investigate the form taken by planar unitarity when a multiperipheral assumption is added. The integral equations obtained are not of the standard Chew-Goldberger-Low type. We find that pure pole-type solutions (i.e. without Regge cuts) to planar unitarity are possible in a way consistent with the symmetry and factorization properties of Reggeon-Reggeon amplitudes. The appearance of "good" FMSR in the unitarity integrals follows from a careful treatment of phase space -- all possible configurations are counted uniquely -- and is crucial in achieving the cut cancellation. Throughout the paper we emphasize various subtle points that have been overlooked in the literature.

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## 1. INTRODUCTION

A recent approach to hadronic physics, known as the Topological Expansion (TE) or dual unitarization[1)] has received much attention in the last few years. This approach is conceptually located between a fundamental approach to hadron physics in which everything can be calculated from a given input (such as a Lagrangian or a good dual model) and a purely phenomenological approach (such as Regge-Mueller phenomenology of inclusive reactions[2)]).

As discussed recently by one of us[3)], the TE has a natural place in a future "theory" of strong interactions which starts from the Lagrangian of Quantum-Chromodynamics (QCD), proceeds through dual models[4)] and ends up with Reggeon-Field-Theory (RFT)[5)] through successive use of non-perturbative expansions of the "large N" variety[1,6)].

Despite much beautiful progress in quantum field theory, the possibility of computing the spectrum and scattering amplitudes of QCD seems rather remote. Hence, at this stage, the TE can be used as a surrogate of a more basic approach. The TE allows us to relate several aspects of hadronic physics and to introduce a small dimensionless parameter into strong interaction theory. In this way, several interesting results have been obtained for low energies (hadron spectroscopy)[7)], present accelerator energies[8)] and super asymptotic-energies[9)].

One of the most important and most difficult parts of the TE programme is computing the zero-order term of the expansion -- the planar S-matrix. Attempts in this direction have so far involved abstracting some properties of planar dual models and then utilizing them together with some phenomenological input in planar unitarization. Bootstrap constraints[10)] have emerged when a multiperipheral picture is assumed for the planar production amplitudes which saturate unitarity at large s and small t. Planar dual models with almost linear trajectories seem to be close to fulfilling these constraints, provided the intercept and coupling of the leading planar Regge pole (the $\rho$-f system) are appropriately chosen[11)]. The values obtained do provide a more or less correct normalization of hadronic amplitudes.

In this paper we maintain the attitude of looking to the dual model as a guide. In particular, we study properties of Reggeon amplitudes which are relevant in the TE programme.

In Section 2 we examine the analytic properties of Reggeon amplitudes in Reggeon-particle (Rp) and Reggeon-Reggeon (RR) scattering. For the former, analytic properties are rather simple and Finite-Mass-Sum-Rules (FMSR) are straightforward[12]; however, for the Reggeon-Reggeon amplitude the situation is more involved. Turning to the dual model example, we find that certain asymmetric FMSR are "good", that is, they are free of fixed pole contributions.

In Section 3 we derive the integral equations for Reggeon-particle and Reggeon-Reggeon scattering that follow from planar unitarity combined with standard multiperipheral assumptions. A no-double-counting condition (NDC) gives non-trivial limits of integration. Consequently, the equations are not simply diagonalized by a Mellin transformation. Instead, the careful treatment of kinematics in the unitarity equations yields precisely the "good" FMSR of Section 2. These equations are easily solved by pure-pole solutions and are consistent with the symmetry and factorization properties of Reggeon amplitudes. No contradictions arise in several self-consistency checks. The final outcome is a bootstrap condition on the triple Regge coupling and the trajectory identical to the one already analysed by several authors[10,11,13-15]. Just how cut-cancellation works term-by-term in the energy plane is spelled out. The dangers of over-simplifying the planar bootstrap in rapidity variables is emphasized.

Section 4 contains a summary of our conclusions. Some typical technical calculations are given in the Appendices.

## 2. ANALYTIC PROPERTIES OF PLANAR REGGEON AMPLITUDES

### 2.1 Definition of Reggeon amplitudes

Complications with Reggeon amplitudes compared to particle amplitudes arise already at the level of definition. For spinless particle-particle scattering, the invariant amplitude $A_{ab\to a'b'}(s,t)$ has two basic properties in the planar limit:

i) Normal analytic structure in s for fixed, negative t;

ii) Regge-pole dominance with an asymptotic behaviour

$$A_{ab\to a'b'}(s,t) \xrightarrow[s\to\infty]{} \gamma_{aa'}(t)\,\gamma_{bb'}(t)\,\Gamma(-\alpha(t))\left(-\frac{s}{s_0}\right)^{\alpha(t)} .$$

The power of s is independent of the external particles and their masses.

The complications of Reggeon amplitudes follow from the impossibility of defining them so that properties (i) and (ii) are obeyed simultaneously. Consider first Reggeon-particle scattering. The amplitude is extracted from the appropriate limit of a six-point function $A_{abc\to a'b'c'}$. Referring to Fig. 1 for notation, if we take $s/M^2$, $s'/M^2$ large then we expect

$$A_{abc\to a'b'c'} \longrightarrow \gamma_{bc}(t_1^-)\gamma_{b'c'}(t_1^+)\Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+)) \cdot \\ (-\tfrac{s}{s_0})^{\alpha(t_1^-)}(-\tfrac{s}{s_0})^{\alpha(t_1^+)} A_{R_1 a\to R_1 a'}(M^2, t, t_1^{\pm}) \, . \tag{2.1}$$

Here $A_{R_1 a\to R_1 'a'}$ defines the Reggeon-particle (Rp) amplitude. Starting from Eq. (2.1) one can argue, using field theoretic or dual examples that this amplitude has only normal threshold singularities in $M^2$. However, the large $M^2$ behaviour is given by

$$\frac{1}{2i}\,\mathrm{Disc}_{M^2} A_{R_1 a\to R_1 a'} \xrightarrow[M^2\to\infty]{} \pi \frac{\gamma_{aa'}(t)\, g_{R_1 R_1'}(t, t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \left(\frac{M^2}{s_0}\right)^{\alpha(t)-\alpha(t_1^-)-\alpha(t_1^+)} \tag{2.2}$$

where $g_{R_1 R_1'}(t, t_1^{\pm})$ is the triple Regge vertex. Thus a t-channel Regge-pole results in a large $M^2$ behaviour with dependence on $t_1^{\pm}$. A more natural definition of a Reggeon-particle amplitude is perhaps

$$A_{abc\to a'b'c'} \longrightarrow \gamma_{bc}(t_1^-)\gamma_{b'c'}(t_1^+)\Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+)) \cdot \\ (-\tfrac{s}{M^2})^{\alpha(t_1^-)}(-\tfrac{s}{M^2})^{\alpha(t_1^+)} A'_{R_1 a\to R_1 a'}(M^2, t, t_1^{\pm}) \, . \tag{2.3}$$

Obviously the asymptotic behaviour for $A'_{R_1 a\to R_1 'a'}$ is simple:

$$\frac{1}{2i}\,\mathrm{Disc}_{M^2} A'_{R_1 a\to R_1 a'} \xrightarrow[M^2\to\infty]{} \pi \frac{\gamma_{aa'}(t)\, g_{R_1 R_1}(t, t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \left(\frac{M^2}{s_0}\right)^{\alpha(t)} \, . \tag{2.4}$$

On the other hand, if A does not contain kinematical singularities, then A' does (due to the factor $(M^2)^{\alpha(t_1^-)+\alpha(t_1^+)}$). Hence, the application of FMSR to A' is delicate, while for A it is straightforward[12]. Our work hinges a lot upon the correct analytic structure of Reggeon amplitudes and upon the validity of certain FMSR. Therefore, we must be careful in the definition of Reggeon amplitudes to be used and also in the study of their analytic properties.

The Reggeon-Reggeon (RR) amplitude is extracted from the eight-point function $A_{abcd\to a'b'c'd'}$ depicted in Fig. 2. Going to the limit $s$, $s_1$, $s_2$, $s/s_1$ and $s/s_2$ large, we write (our definition differs from that of Kwiecinski[16] by some unimportant factors):

$$A_{abcd\to a'b'c'd'} \cong \gamma_{bd}(t_1^-)\gamma_{b'd'}(t_1^+)\gamma_{ac}(t_2^-)\gamma_{a'c'}(t_2^+)\cdot \\ \cdot \Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+))\Gamma(-\alpha(t_2^-))\Gamma(-\alpha(t_2^+)) \\ \cdot (-s_1/s_0)^{\alpha(t_1^-)}(-s_1'/s_0)^{\alpha(t_1^+)}(-s_2/s_0)^{\alpha(t_2^-)}(-s_2'/s_0)^{\alpha(t_2^+)} \\ \cdot A_{R_1R_2}(M^2, s_1s_2/s, t, t_1^\pm, t_2^\pm) \quad , \tag{2.5}$$

where

$$s_1s_2/s \underset{s,s_1,s_2 \text{ large}}{\cong} M^2+(\vec{p}_c^{\,\perp}+\vec{p}_d^{\,\perp})^2 \equiv M_\perp^2 \quad . \tag{2.6}$$

Introducing

$$A'_{RR} = \left(\frac{M^2}{s_0}\right)^{\alpha(t_1^-)+\alpha(t_1^+)+\alpha(t_2^-)+\alpha(t_2^+)} A_{RR} \quad , \tag{2.7}$$

we expect $A'_{RR} \to (M^2)^{\alpha(t)}$. But $A'_{RR}$ should have a rather complicated singularity structure. It is not completely trivial to define a RR amplitude which has just the correct threshold singularities in $M^2$. To find out how to use FMSR for RR scattering we resort to the explicit example provided by the dual model.

## 2.2 Reggeon amplitudes in the dual model

In the planar dual model the Rp amplitude has the form[17]:

$$A_{R_1a\to R_1a'} = g^2 B_4(-\alpha(t)+\alpha(t_1^-)+\alpha(t_1^+), -\alpha(M^2)) \quad . \tag{2.8}$$

As for the RR amplitude, in Appendix A we study the dual eight-point function corresponding to Fig. 2. By a straightforward use of Beta-transform techniques[17] we find:

$$B_8 \cong g^4 (-\alpha(s_1))^{\alpha(t_1^-)} (-\alpha(s_1'))^{\alpha(t_1^+)} (-\alpha(s_2))^{\alpha(t_2^-)} (-\alpha(s_2'))^{\alpha(t_2^+)}$$

$$\cdot \Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+))\Gamma(-\alpha(t_2^-))\Gamma(-\alpha(t_2^+))$$

$$\cdot A_{R_1R_2}(M^2, M_\perp^2 = s_1s_2/s\, , t, t_1^\pm, t_2^\pm) \quad . \tag{2.9}$$

Here

$$A_{R_1R_2} = g^2 \sum_{k=0}^{\infty} \sum_{\ell,\ell'=0}^{k} C_k^{\ell\ell'} \frac{\Gamma(-\alpha(t_1^-)+\ell)}{\Gamma(-\alpha(t_1^-))} \frac{\Gamma(-\alpha(t_1^+)+\ell')}{\Gamma(-\alpha(t_1^+))}$$

$$\cdot \frac{\Gamma(-\alpha(t_2^-)+\ell')}{\Gamma(-\alpha(t_2^-))} \frac{\Gamma(-\alpha(t_2^+)+\ell)}{\Gamma(-\alpha(t_2^+))} \cdot \frac{1}{\Gamma(-\alpha_{c,1}-1+\ell+\ell')\Gamma(-\alpha_{c,2}-1+\ell+\ell')}$$

$$\cdot \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \Gamma(-\beta+\ell+\ell')\Gamma(\beta-\alpha_{c,1}-1)\Gamma(\beta-\alpha_{c,2}-1)(-1)^\beta (M_\perp^2)^{-\beta} \tag{2.10}$$

$$\cdot B_4(-\beta-\alpha(t)+\alpha_{c,1}+\alpha_{c,2}+2+k, -\alpha(M^2)) \quad ,$$

where

$$\alpha_{c,i} = \alpha(t_i^-) + \alpha(t_i^+) - 1 \quad . \tag{2.11}$$

Notice that in $A_{R_1R_2}$ of Eq. (2.10) the dynamical $M^2$ singularities appear explicitly as poles in $B_4$. Hence, the analytic structure of $A_{RR}$ is relatively simple.

For large $M^2$ we have $M_\perp^2 \cong M^2$ and

$$B_4(-\beta-\alpha(t)+\alpha_{c,1}+\alpha_{c,2}+2+k, -\alpha(M^2)) \xrightarrow[M^2\to\infty]{}$$

$$\Gamma(-\beta-\alpha(t)+\alpha_{c,1}+\alpha_{c,2}+2+k)(-\alpha' M^2)^{\beta+\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2-k} \quad . \tag{2.12}$$

The leading term of $A_{R_1R_2}$ is (k = 0)

$$A_{R_1R_2} \xrightarrow[M^2\to\infty]{} g^2 (-\alpha' M^2)^{\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2} \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} \cdot$$

$$\Gamma(-\beta) \frac{\Gamma(\beta-\alpha_{c,1}-1)}{\Gamma(-\alpha_{c,1}-1)} \frac{\Gamma(\beta-\alpha_{c,2}-1)}{\Gamma(-\alpha_{c,2}-1)} \Gamma(-\beta-\alpha(t)+\alpha_{c,1}+\alpha_{c,2}+2) \tag{2.13}$$

$$= g^2 \frac{(-\alpha' M^2)^{\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2}}{\Gamma(-\alpha(t))} \Gamma(-\alpha(t)+\alpha_{c,1}+1)\Gamma(-\alpha(t)+\alpha_{c,2}+1) .$$

A similar result holds for the Mueller discontinuity:

$$\frac{1}{2i} \mathrm{Disc}_{M^2} A_{R_1R_2} \cong$$

$$\pi g^2 \frac{(\alpha' M^2)^{\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2}}{\Gamma(\alpha(t)+1)} \cdot \frac{(\Gamma(\alpha(t)+1))^2}{\Gamma(\alpha(t)-\alpha_{c,1})\Gamma(\alpha(t)-\alpha_{c,2})} . \tag{2.14}$$

Using the high $M^2$ limit of Eq. (2.8) and the standard form for $A_{ab\to a'b'}(M^2,t)$ we find the desired factorization property*)

$$\frac{\mathrm{Disc}_{M^2} A_{R(t_1^-)R(t_2^-)\to R(t_1^+)R(t_2^+)}}{\mathrm{Disc}_{M^2} A_{ab\to a'b'}} =$$

$$\frac{\mathrm{Disc}_{M^2} A_{R(t_1^-)a\to R(t_1^+)a'}}{\mathrm{Disc}_{M^2} A_{ab\to a'b'}} \times \frac{\mathrm{Disc}_{M^2} A_{R(t_2^-)b\to R(t_2^+)b'}}{\mathrm{Disc}_{M^2} A_{ab\to a'b'}} . \tag{2.15}$$

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*) Notice that Eq. (2.14) gives a singularity in $\mathrm{Disc}_{M^2}\, A_{R_1R_2}$ (and in $\mathrm{Disc}_{M^2}\, B_8$) at $\alpha(t) = -1, -2 \cdots$ . $B_8$ itself has no singularity at these points. The presence of these poles comes unavoidably from factorization [Eq. (2.15)] and from the non-vanishing of $\mathrm{DISC}_{M^2}\, B_6$ at $\alpha(t) = -1, -2 \ldots$ . The way out of this is probably the introduction of the new "$\beta$-trajectory" of Hoyer, Tornqvist and Webber[18]), which collides with $\alpha$ at these points. Elimination of these singularities is also welcome for building the Pomeron in the TE.

Note that in the asymptotic limit $A_{RR}$ depends on the external Reggeon masses ($A_{R_1R_2} \sim (M^2)^{\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2}$). If, as in Eq. (2.7) we define the more natural $A'_{RR}$ it will have a more complicated singularity structure in $M^2$.

In Table 1 we summarize our definitions. The correspondence with the dual model is:

$$\gamma(t) \to g \quad , \quad g(t,t_1^\pm) \longrightarrow g \frac{\Gamma(\alpha(t)+1)}{\Gamma(\alpha(t)-\alpha_{c,1})} .$$

## 2.3 FMSR and Mellin transforms of Reggeon amplitudes

For the planar Rp amplitude $A_{Ra}$ one expects the following FMSR to hold for sufficiently large $\bar{s}$ [12]

$$\int_0^{\bar{s}} dM^2 \, (M^2)^n \frac{1}{2i} \, Disc_{M^2} A_{Ra}(M^2,t,t_1^\pm) \cong \pi \sum_{\alpha(t)} \frac{(\bar{s})^{\alpha(t)-\alpha_{c,1}+n}}{\alpha(t)-\alpha_{c,1}+n} \; \frac{g(t,t_1^\pm)\,\gamma_{aa'}(t)}{\Gamma(\alpha(t)+1)} , \tag{2.16}$$

which is trivially consistent with the dual model amplitude (2.8).

Notice that $A_{Ra}$ obeys a FMSR of a simple form. The "natural" entity $A'_{Ra}$ [see Eq. (2.3)] does not satisfy an integer moment FMSR. Defining a Mellin transform over $A_{Ra}$

$$\tilde{A}_{Ra}(J,t,t_1^\pm) \equiv \int_0^\infty dM^2 \, (M^2)^{-J-1} A_{Ra}(M^2,t,t_1^\pm) , \tag{2.17}$$

we note that FMSR (2.16) hinges upon the absence of fixed poles in $\tilde{A}_{Ra}(J,t,t_1^\pm)$. Namely, the latter vanishes at $J = -1, -2, -3, \ldots$ .

Turning to RR scattering one wonders whether any simple FMSR holds. The extra complication here is that the variables s, $M^2$, $s_1$ and $s_2$ are not independent [see Eq. (2.6)]. Kwiecinski[16] has proposed the following decomposition of $A_{RR}$, based on a hybrid Feynman diagram model:

$$A_{RR} = (-M_\perp^2)^{-\alpha_{c,1}-1} F^{(1)}(M^2,M_\perp^2,t,t_1^\pm,t_2^\pm) + (-M_\perp^2)^{-\alpha_{c,2}-1} F^{(2)}(M^2,M_\perp^2,t,t_1^\pm,t_2^\pm) . \tag{2.18}$$

It is $F^{(i)}$ ($i = 1,2$) that are presumed to satisfy integer moment FMSR. The dual model example does not contradict this proposition. However, we shall now show that there are particular, asymmetric moments of the full $A_{RR}$ for which naive FMSR may hold.

In calculating a FMSR for $A_{RR}$ one has to specify which variables in the eight-point function of Fig. 2 are kept fixed while $M^2$ is integrated over*). We consider here two cases that are relevant to the unitarity equation:

i) $s$ and $s_1$ kept fixed, and $s_2 = M_\perp^2(s/s_1)$ (or the equivalent case with $s_1 \leftrightarrow s_2$);

ii) $s/s_1$ and $s/s_2$ kept fixed, and $s_1 = M_\perp^2(s/s_2)$, $s_2 = M_\perp^2(s/s_1)$, $s = M_\perp^2(s/s_1)(s/s_2)$.

In the first case consider the general Mellin transform ($i = 1,2$):

$$\tilde{A}_8^{as}(J,s,s_1,t,t_i^\pm) \equiv \int_0^\infty dM^2\,(M^2)^{-J-1}\,\frac{1}{2i}\,\mathrm{Disc}_{M^2} A_8(s,s_1,s_2 = M_\perp^2(s/s_1),M^2,t,t_i^\pm) \quad (2.19)$$

From Table 1 we see that, for the RR amplitude, Eq. (2.19) corresponds to a Mellin transform over an asymmetric entity:

$$\tilde{A}^{as}_{R_1R_2^*}(J,t,t_i^\pm) \equiv \int_0^\infty dM^2\,(M^2)^{-J-1}\,(M_\perp^2)^{\alpha_{c,2}+1}\,\frac{1}{2i} \cdot \mathrm{Disc}_{M^2} A_{R_1R_2}(M^2,M_\perp^2,t,t_i^\pm) \,, \quad (2.20)$$

where the asterisk over $R_2$ indicates which way the asymmetry goes.

In Appendix A we show in the dual model example that $\tilde{A}^{as}_{R_1R_2^*}(J,t,t_i^\pm)$ vanishes at $J = -1, -2, -3, \ldots$ ; that is, no fixed poles occur at these points. Abstracting this property from the dual model, we deduce that the RR amplitude satisfies the following asymmetric FMSR:

$$\int_0^{\bar{s}} dM^2\,(M^2)^n\,(M_\perp^2)^{\alpha_{c,2}+1} \cdot \frac{1}{2i}\,\mathrm{Disc}_{M^2} A_{R_1R_2}(M^2,M_\perp^2,t,t_i^\pm) \cong \pi\,\frac{g(t,t_1^\pm)\,g(t,t_2^\pm)}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-\alpha_{c,1}+n}}{\alpha(t)-\alpha_{c,1}+n} \qquad (n = 0,1,2,\cdots) \quad (2.21)$$

*) It is also crucial to take into account only the "physical" singularities in $M^2$, i.e. those related to states in the missing-mass channel.

Here for sufficiently large $\bar{s}$ only the leading pole contribution has been kept, and terms of order $(P_\perp^2/\bar{s})$ - neglected ($\bar{s} >> P_\perp^2$).

Notice that the asymmetric transform (2.20) has zeros also at $J = \alpha_{c,2} - \alpha_{c,1} - 1 - m$ ( $m = 0,1,2, \ldots$) for $P_\perp^2 = 0$ ($M_\perp^2 = M^2$). For $P_\perp \neq 0$ they disappear (unlike the zeros at $J = -1, -2, -3, \ldots$). This is seen in the term by term expansion in powers of $P_\perp^2$ (see Eq. (A.7) and discussion following it). We do not know whether the re-summation of the series re-establishes these zeros.

For the <u>second (symmetric) case</u> the appropriate Mellin transform is

$$\tilde{A}^{sy}_{\gamma}(J, s/s_1, s/s_2, t, t_i^{\pm}) \equiv \int_0^\infty dM^2 (M^2)^{-J-1} \frac{1}{2i} \mathrm{Disc}_{M^2} A_\gamma(s, s/s_1, s/s_2, M^2, t, t_i^{\pm}) \quad , \tag{2.22}$$

or, equivalently (see Table 1)

$$\tilde{A}^{sy}_{R_1R_2}(J,t,t_i^{\pm}) \equiv \int_0^\infty dM^2 (M^2)^{-J-1} (M_\perp^2)^{\alpha_{c,1}+\alpha_{c,2}+2} \frac{1}{2i} \mathrm{Disc}_{M^2} A_{R_1R_2}(M^2, M_\perp^2, t, t_i^{\pm}) . \tag{2.23}$$

Our study of the dual example in Appendix A shows that for $P_\perp = 0$ this transform has zeros at both $J = \alpha_{c,1} - n$ and $J = \alpha_{c,2} - m$ ($n,m = 0,1,2, \ldots$). Both sets of zeros seem to disappear when $P_\perp^2 \neq 0$, at least term by term in a power expansion in $P_\perp^2$. However, we cannot rule out the possibility of these zeros reappearing after the re-summation of the $P_\perp^2$ expansion.

Notice that at $P_\perp = 0$ both $\tilde{A}^{sy}_{R_1R_2}$ and $\tilde{A}^{as}_{R_1R_2^*}$ have two sets of zeros in J. In fact, in this limit the two transforms are related by a simple shift in J:

$$(J+1)_{asym.} \longleftrightarrow (J - \alpha_{c,1})_{sym.} \tag{2.24}$$

In order to preserve this simple relationship and thus overcome the effect of transverse momenta on the fixed pole contributions in (2.20) and (2.23) one may define a modified transform with $(M^2)^{-J-1}$ replaced by $(M_\perp^2)^{-J-1}$:

$$\tilde{A}^{\perp}_{R_1R_2}(J,t,t_i^{\pm}) \equiv \int_0^\infty dM^2 (M_\perp^2)^{-J-1} (M_\perp^2)^{\alpha_{c,1}+\alpha_{c,2}+2} \frac{1}{2i} \mathrm{Disc}_{M^2} A_{R_1R_2}(M^2, M_\perp^2, t, t_i^{\pm}) . \tag{2.25}$$

This symmetric transform does, indeed, vanish at both $J - \alpha_{c,1} = -n$ and $J - \alpha_{c,2} = -m$ ($n,m = 0,1,2, \ldots$), for $P_\perp \neq 0$ as well. Moreover, the analogous asymmetric transform is obtained by the simple shift of Eq. (2.24), and therefore has zeros at $J = -1 - n$ and $J = \alpha_{c,2} - \alpha_{c,1} - m$ for $P_\perp \neq 0$ as well. This interesting property of $\tilde{A}^\perp$ will not be studied further in this paper since such an object does not seem to enter in the unitarity equations. On the other hand, unitarity plus multiperipheral dynamics will lead us automatically to FMSR of the type (2.21) which will be used repeatedly in Section 3.

## 3. CONSTRAINTS ON REGGEON AMPLITUDES FROM PLANAR UNITARITY

### 3.1 Multiperipheral cluster production model

Planar unitarity provides a set of non-linear constraints on multi-particle amplitudes. In order to derive integral equations from these constraints, some multiperipheral assumptions have to be imposed on the production amplitudes (which are relevant for saturation of the unitarity sum at high energy and small momentum transfers).

In this context, a model with both theoretical and phenomenological appeal, is the multiperipheral cluster model. In this model, the extreme assumption of multiperipheral production of stable particles is replaced by the weaker assumption that some relatively long-lived resonant states (clusters) are first produced multiperipherally, then decay independently into the final state. In the planar dual theory the identification of clusters with low-energy resonances is almost automatic and restricts considerably the concept of clusters. Some theoretical justification for the validity of a cluster picture within QCD has recently been discussed by one of us[3]. The phenomenological validity of the picture has been discussed by several authors[7,8].

However, when one tries to classify an event in terms of clusters, a delicate problem arises of avoiding double-or-under-counting of final state configurations. For instance, suppose that a set of particles which are adjacent in the planar diagram forms a cluster if the invariant mass of the set obeys $s_i < \bar{s}$. Then two adjacent clusters of masses $s_1, s_2$ should satisfy not only $s_1, s_2 < \bar{s}$ but also (at least) the constraint $s_{12} > \bar{s}$, where $s_{12}$ is the invariant mass squared of the system 1 + 2. Furthermore, this system may be sometimes split into two adjacent sets in other ways, 1' + 2', such that $s_{1'}, s_{2'} < \bar{s}$. Should one count such subdivisions separately or not? This type of question has been dealt with in the literature. Essentially, two possible attitudes can be adopted:

i) Clusters are physical. In this case, suppose that below $\bar{s}$ the system is dominated by sufficiently narrow resonances, and above $\bar{s}$ is described by some rather smooth Regge exchange amplitude. The two events 1 + 2 and 1' + 2' mentioned above can be safely counted separately. The symmetric no-double-counting condition (NDC)[7,8,15,19,20)]

$$s_i < \bar{s} \quad , \quad s_{i,i+1} > \bar{s} \qquad\qquad (\text{all } i) \qquad\qquad , \tag{3.1}$$

should be appropriate to avoid either double counting or under counting of events. This is, of course, an approximate statement which may never be exact (strictly speaking, this NDC double-counts some special configurations). In this case, $\bar{s}$ has a physical meaning, output parameters will depend on it, and should be chosen in some optimal way (similar to what is done with FESR). These clusters are of the type discussed in Ref. 3).

ii) Clusters are mathematical. From this point of view $\bar{s}$ is a mathematical parameter; a device used in order to group final state particles into bins over which averaging out of individual particle properties is achieved. Double counting is avoided by the asymmetric NDC of Finkelstein and Koplik[20)] and Freeman and Zarmi[15)]. The condition states that a cluster (or bin) is a set of particles such that i) its invariant $(\text{mass})^2$ is less than $\bar{s}$, and ii) the addition of the next particle adjacent to this set on the left (or, alternatively, on the right) results in a new set whose invariant $(\text{mass})^2$ is larger than $\bar{s}$. Thus in the notation of Fig. 3, for a given particle configuration divided into n clusters (bins) we have:

$$s_i < \bar{s} \quad , \quad s_i' \geq \bar{s} \qquad i = 1,\dots,n-1 \qquad\qquad \tag{3.2}$$
$$s_n \leq \bar{s} \qquad\qquad .$$

Thus, every cluster is defined in a manner which correlates it to the gap lying between it and the next cluster, except for the last ("left over") cluster. In this scheme, clusters have nothing to do with resonances, or better, nothing to do with narrow resonances. If one wants to derive integral equations with this classification of the intermediate state, one has to use Regge exchange between adjacent clusters even when the gap separating them is small, or even if, e.g., the last particle of one cluster resonates with the first particle of the next one. This model is therefore expected to be good only when there are essentially no physical clusters (large width limit, see Ref. 3)). Note, however, that counting of events is precise, and results should not depend on the bin cut-off $\bar{s}$.

In this paper, our attention is focused on delicate cancellations and on qualitative properties, and therefore the second point of view is adopted. Let us stress, however, that for practical applications, the model with physical clusters is more appealing. The extension of our conclusions to the case of physical clusters would seem very desirable. We shall comment again on this case at the end of this paper and from now on we shall work with the asymmetric NDC of Refs. 15) and 20).

3.2 Integral equations for Reggeon-particle amplitudes

The equations that can be derived from planar unitarity and multiperipheral cluster model constrained by the asymmetric NDC (3.2) are represented in Fig. 4, where a box represents a finite mass object and a blob - an unrestricted mass object. The two equations stem from the two ways of counting, or classifying, the intermediate state (starting cluster assignment from either end of the chain of Fig. 3). The second equation (Fig. 4b) is not an integral equation. It represents a self-consistency check on the RR amplitude.

a) Equation of Fig. 4a in the energy plane

Consider the equation represented by Fig. 4a. At the level of six-point functions it has the form (without specifying it, all equations from now on deal only with the imaginary parts of amplitudes):

$$A_6(s,s_{23},t,t_1^\pm) = A_6^{(1)}(s,s_{23},t,t_1^\pm) + T_6(s,s_{23},t,t_1^\pm) \quad , \tag{3.3}$$

where $A_6^{(1)}$ is the low-energy, one-cluster amplitude, playing the role of the inhomogeneous term, and $T_6$ is the homogeneous term. In the appropriate kinematical limit ($s/s_{23} \gg 1$, $t_1^\pm$ fixed) we can extract Rp amplitudes as in Eq. (21) and write:

$$\begin{aligned} A_{Ra}(s_{23},t,t_1^\pm) &= A_{Ra}^{(1)}(s_{23},t,t_1^\pm) + T_{Ra}(s_{23},t,t_1^\pm) \quad , \\ A_{Ra}^{(1)} &= \theta(\bar{s}-s_{23})A_{Ra} \quad . \end{aligned} \tag{3.4}$$

From factorization and by keeping the leading poles in the $t_2^\pm$ channels, one gets for $T_6$ (we have confirmed this explicitly in the dual model):

$$T_6 = \theta(s_{23}-\bar{s}) \int_0^{\bar{s}} \frac{ds_2}{s_{23}} \int_0^{s_{23}/\bar{s}} ds_3 \int d\phi_2 \, \gamma_{bc}(t_1^-)\gamma_{b'c'}(t_1^+)\Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+))$$

$$\cdot \Gamma(-\alpha(t_2^-))\Gamma(-\alpha(t_2^+)) \cos\pi(\alpha(t_2^-)-\alpha(t_2^+)) \, (-s_{12}^{\prime})^{\alpha(t_1^-)}(-s_{12}^{\prime})^{\alpha(t_1^+)}$$

$$\cdot (s_{23})^{\alpha(t_2^-)+\alpha(t_2^+)} A_{R_1R_2}(s_2, \tfrac{s_{12}s_{23}}{s}, t, t_1^{\pm}, t_2^{\pm}) \, A_{R_2a}(s_3, t, t_2^{\pm}) \quad , \tag{3.5}$$

where the loop integral is converted (using strong damping of transverse momenta) into an integral over $t_2^{\pm}$ denoted by $d\phi_2$ ($d\phi_2 \equiv N/16\pi^4 \times dt_2^- dt_2^+ \, \theta(-\lambda)(-\lambda)^{-\frac{1}{2}}$, with $\lambda$ the usual triangular function) and over $s_2, s_3$. The limits of integration over $s_2$ and $s_3$ are determined by the asymmetric NDC (3.2) which implies here: $s_2 < \bar{s}$, $s_{23}/s_3 > \bar{s}$ (always in units of $s_0 = 1/\alpha' \cong 1$). In performing the integrations, $s_{12}$ is not to be regarded as an independent variable. Rather, one substitutes

$$s_{12} = s/s_{23} \left( \frac{s_{12}\cdot s_{23}}{s} \right) \cong s/s_{23} \cdot s_2^{\perp} \quad . \tag{3.6}$$

Eq. (3.5) now becomes:

$$T_6 = \theta(s_{23}-\bar{s}) \gamma_{bc}(t_1^-)\gamma_{b'c'}(t_1^+)\Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+))$$

$$\cdot(-s/s_{23})^{\alpha(t_1^-)}(-s/s_{23})^{\alpha(t_1^+)} \int d\phi_2 \, \Gamma(-\alpha(t_2^-))\Gamma(-\alpha(t_2^+)) \cos\pi(\alpha(t_2^-)-\alpha(t_2^+))$$

$$\cdot(s_{23})^{\alpha_{c,2}} \int_0^{\bar{s}} ds_2 \int_0^{s_{23}/\bar{s}} ds_3 \, (s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1R_2}(s_2, s_2^{\perp}, t, t_1^{\pm}, t_2^{\pm}) \, A_{R_2a}(s_3, t, t_2^{\pm}) \quad . \tag{3.7}$$

Here $\alpha_{c,i}$ are defined by Eq. (2.11). Using the definition (2.1) of Rp amplitudes we get

$$A_{R_1a} - A_{R_1a}^{(1)} = T_{R_1a} = \theta(s_{23}-\bar{s})(s_{23})^{-\alpha_{c,1}-1} \int d\phi_2 \, \eta_2 \, (s_{23})^{\alpha_{c,2}}$$

$$\cdot \int_0^{\bar{s}} ds_2 \int_0^{s_{23}/\bar{s}} ds_3 \, (s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1R_2}(s_2, s_2^{\perp}, t, t_i^{\pm}) \, A_{R_2a}(s_3, t, t_2^{\pm}) \quad (i=1,2) \quad . \tag{3.8}$$

Here,

$$\eta_2 \equiv \Gamma(-\alpha(t_2^-))\,\Gamma(-\alpha(t_2^+))\cos\pi\,(\alpha(t_2^-)-\alpha(t_2^+)) \quad . \tag{3.9}$$

The $s_3$ integral can be done by use of FMSR (2.16):

$$\int_0^{s_{23}/\bar{s}} ds_3\, A_{R_2 a}(s_3, t, t_2^{\pm}) = \pi\,\frac{\gamma_{aa'}(t)\,g(t,t_2^{\pm})}{\Gamma(\alpha(t)+1)}\;\frac{(s_{23}/\bar{s})^{\alpha(t)-\alpha_{c,2}}}{\alpha(t)-\alpha_{c,2}} \quad . \tag{3.10}$$

Moreover, the integral over $s_2$ is precisely of the type free of fixed pole contributions. Using FMSR (2.21) for n = 0 we get

$$\int_0^{\bar{s}} ds_2\,(s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1 R_2}(s_2, s_2^{\perp}, t, t_i^{\pm}) \cong \pi\,\frac{g(t,t_1^{\pm})\,g(t,t_2^{\pm})}{\Gamma(\alpha(t)+1)}\cdot\frac{(\bar{s})^{\alpha(t)-\alpha_{c,2}}}{\alpha(t)-\alpha_{c,2}} \quad . \tag{3.11}$$

Inserting Eqs. (3.10)-(3.11) into Eq. (3.8) we find

$$T_{R,a} = \theta(s_{23}-\bar{s})\,(s_{23})^{\alpha(t)-\alpha_{c,1}-1}\,\frac{g(t,t_1^{\pm})\,\gamma_{aa'}(t)}{\Gamma(\alpha(t)+1)}\;\cdot\;\frac{\pi^2}{\Gamma(\alpha(t)+1)}\int d\phi_2\,\eta_2\,\frac{g^2(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})^2} \quad . \tag{3.12}$$

On the other hand, from Eq. (2.2) we expect

$$A_{R,a} - A^{(1)}_{R,a} = \theta(s_{23}-\bar{s})\,(s_{23})^{\alpha(t)-\alpha_{c,1}-1}\,\pi\,\frac{g(t,t_1^{\pm})\,\gamma_{aa'}(t)}{\Gamma(\alpha(t)+1)} \quad . \tag{3.13}$$

Thus, complete agreement with a pole-type solution is possible provided the bootstrap condition

$$I(\alpha,t) \equiv \frac{\pi}{\Gamma(\alpha(t)+1)}\int d\phi_2\,\eta_2\,\frac{g^2(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})^2} = 1 \tag{3.14}$$

is satisfied. This is precisely the condition derived in Ref. 10) and studied in Ref. 11).

Comparing our Eq. (3.14) with Eq. (4) of Ref. 11) we find, using our definition of $d\phi_2$ and the dual expression for g (see end of Section 2.2 with the appropriate Neveu-Schwarz shifts explained in Ref. 11)), that they agree up to a factor of $\pi$. This factor was overlooked in Ref. 11) and should be introduced on the left-hand side of Eq. (4). One should also multiply $d\phi_2$, in Eq.(4) of Ref. 11), by a factor of 2 (planar unitarity has non-identical particle phase space) but this is compensated by another factor of two which one gains by adding cyclic and anticyclic permutations in the duality diagram. The final correct result of the bootstrap calculation of Ref. 11) can thus be found to be:

$$\sigma_{\pi^+\bar{\pi}^-} - \sigma_{\bar{\pi}^+\bar{\pi}^+} \cong \frac{100}{N}\, mb\, (\alpha' s)^{\alpha(0)-1} \quad ,$$

which is probably a little too large. Using the arguments of Ref. 3), one finds that this is actually an upper limit to the cross-section. The factor by which it has to be reduced (because of clustering) is indeed about 1.5. On the other hand the fact that we obtain approximately the right normalization of the cross-section is already quite remarkable.

While the last calculation shows that a Regge-pole is consistent with the integral equation, it does not demonstrate how Regge cuts are cancelled. In a series expansion of the integral equation, the n-cluster contribution will obviously have cuts in J.

b) <u>Equation of Fig. 4a in the J-plane</u>

We now turn to the J-plane in order to see how cut-cancellation takes place. Consider the Mellin transform of $T_{Ra}$. Using Eq. (3.8) we find

$$\begin{aligned}\tilde{T}_{R_1 a}(J,t,t_1^{\pm}) &\equiv \int_0^{\infty} ds_{23}\,(s_{23})^{-J-1}\, T_{R_1 a}(s_{23},t,t_1^{\pm}) = \\ &\int d\phi_2 \eta_2 \int_{\bar{s}}^{\infty} \frac{ds_{23}}{s_{23}}\,(s_{23})^{-J-1+\alpha_{c,2}-\alpha_{c,1}} \\ &\cdot \int_0^{\bar{s}} ds_2 \int_0^{s_{23}/\bar{s}} ds_3\,(s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1 R_2}(s_2,s_2^{\perp},t,t_i^{\pm})\, A_{R_2 a}(s_3,t,t_2^{\pm}) \;.\end{aligned} \qquad (3.15)$$

Changing variables $s_2$, $s_3$, $s_{23} \to s_2$, $s_3$, $x = s_{23}/s_2 s_3$, we obtain

$$\tilde{T}_{R_1 a} = \int d\phi_2 \eta_2 \int_0^{\bar{s}} ds_2 \, (s_2^{\perp})^{\alpha_{c,1}+1} (s_2)^{-J-1+\alpha_{c,2}-\alpha_{c,1}} A_{R_1 R_2}(s_2, s_2^{\perp}, t, t_i^{\pm})$$
$$\cdot \int_{\bar{s}/s_2}^{\infty} dx \, x^{-J-2+\alpha_{c,2}-\alpha_{c,1}} \int_{\bar{s}/s_2 x}^{\infty} ds_3 \, (s_3)^{-J-1+\alpha_{c,2}-\alpha_{c,1}} A_{R_2 a}(s_3, t, t_2^{\pm}) \quad . \qquad (3.16)$$

The lower limits of integration over $s_3$ and $x$ result from the NDC (3.2). Due to the non-zero lower limit of integration over $s_3$, the expression for $\tilde{T}_{R_1 a}$ does not immediately contain the Mellin transform of $A_{R_2 a}$. Writing $\tilde{T}_{R_1 a}$ naively (as is commonly done) as a product of Mellin transforms leads to inconsistencies and makes cut-cancellation in the J-plane impossible. One way to proceed is to introduce the inverse Mellin transform:

$$A_{R_2 a}(s_3, t, t_2^{\pm}) \equiv \int_{c-i\infty}^{c+i\infty} \frac{dJ'}{2\pi i} (s_3)^{J'} \tilde{A}_{R_2 a}(J', t, t_2^{\pm}) \quad . \qquad (3.17)$$

Performing now the $s_3$ and $x$ integrations we find:

$$\tilde{T}_{R_1 a}(J, t, t_1^{\pm}) = \int d\phi_2 \eta_2 \int_0^{\bar{s}} ds_2 \, (s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1 R_2}(s_2, s_2^{\perp}, t, t_i^{\pm})$$
$$\cdot \int \frac{dJ'}{2\pi i} \frac{(\bar{s})^{-J-1+\alpha_{c,2}-\alpha_{c,1}}}{(J'+1)(J-J'+\alpha_{c,1}-\alpha_{c,2})} \tilde{A}_{R_2 a}(J', t, t_2^{\pm}) \quad . \qquad (3.18)$$

Due to the special form of NDC we use, the cut-off Mellin transform over $A_{R_1 R_2}$ is converted into a "good" FMSR of the type (2.21). Hence, defining $J = j - \alpha_{c,1} - 1$, $J' = j' - \alpha_{c,2} - 1$ we have

$$\tilde{T}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm}) = (\bar{s})^{\alpha(t)-j} g(t, t_1^{\pm}) \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2$$
$$\cdot \int \frac{dj'}{2\pi i (j'-\alpha_{c,2})(j-j')} \frac{g(t, t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})} \tilde{A}_{R_2 a}(j'-\alpha_{c,2}-1, t, t_2^{\pm}) \quad . \qquad (3.19)$$

We thus see that, in general, Eq. (3.19) is not a standard integral equation of the type discussed by Chew, Goldberger and Low[21] as it is not diagonal in j. The j' integration contour is to the left of the pole at j' = j and to the right of all other singularities. In order to avoid a cut in j, $\tilde{A}_{R_2 a}(j' - \alpha_{c,2} - 1, t, t_2^{\pm})$ must vanish at $j' = \alpha_{c,2}$ (J' = -1), which is exactly what one expects for planar Reggeon-particle amplitudes (see discussion following Eq. (2.17)). Writing

$$\tilde{A}_{R_2 a}(j' - \alpha_{c,2} - 1, t, t_2^{\pm}) = \pi \frac{\gamma_{aa'}(t)\, g(t, t_2^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{1}{j' - \alpha(t)} \cdot \frac{j' - \alpha_{c,2}}{\alpha(t) - \alpha_{c,2}}\, h(j', t, t_2^{\pm}), \tag{3.20}$$

we get by closing the j' integration contour to the left

$$\tilde{T}_{R_1 a}(j - \alpha_{c,1} - 1, t, t_1^{\pm}) = \pi \frac{\gamma_{aa'}(t)\, g(t, t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-j}}{j - \alpha(t)}\, I(\alpha, t), \tag{3.21}$$

with $I(\alpha,t)$ given by Eq. (3.14). Notice that h is a smooth function of j' satisfying $h\left[j' = \alpha(t), t, t_1^{\pm}\right] \equiv 1$ and that only the residue of the pole at $j' = \alpha(t)$ of $\tilde{A}_{R_2 a}(j' - \alpha_{c,2} - 1, t, t_2^{\pm})$ is picked up in Eq. (3.19). Using the bootstrap condition (3.14) one finds

$$\tilde{T}_{R_1 a}(j - \alpha_{c,1} - 1, t, t_1^{\pm}) = \pi \frac{\gamma_{aa'}(t)\, g(t, t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-j}}{j - \alpha(t)}. \tag{3.22}$$

This is exactly the form expected for $\tilde{T}_{R_1 a}$, since by Eq. (3.4)

$$\begin{aligned} \tilde{T}_{R_1 a}(j - \alpha_{c,1} - 1, t, t_1^{\pm}) &= \tilde{A}_{R_1 a}(j - \alpha_{c,1} - 1, t, t_1^{\pm}) - \tilde{A}^{(1)}_{R_1 a}(j - \alpha_{c,1} - 1, t, t_1^{\pm}) \\ &= \int_{\bar{s}}^{\infty} ds\, A_{R_1 a}(s, t, t_1^{\pm})\, s^{-j+\alpha(t)} = \int_{\bar{s}}^{\infty} ds\, \pi \frac{\gamma_{aa'}(t)\, g(t, t_1^{\pm})}{\Gamma(\alpha(t)+1)}\, s^{\alpha(t)-j-1} \\ &= \pi \frac{\gamma_{aa'}(t)\, g(t, t_1^{\pm})}{\Gamma(\alpha(t)+1)} \frac{(\bar{s})^{\alpha(t)-j}}{j - \alpha(t)}. \end{aligned} \tag{3.23}$$

Finally, we get the expected form of $\tilde{A}^{(1)}_{R_1 a}$:

$$\tilde{A}^{(1)}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) = \tilde{A}_{R_1 a} - \tilde{T}_{R_1 a} =$$

$$\pi \frac{\gamma_{aa'}(t)\, g(t,t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{1}{j-\alpha(t)} \left[\frac{j-\alpha_{c,1}}{\alpha(t)-\alpha_{c,1}} h(j,t,t_1^{\pm}) - (\bar{s})^{\alpha(t)-j}\right]. \tag{3.24}$$

At this point one can regard $\tilde{A}_{Ra}$ of Eq. (3.20) as the solution to the integral equation (3.19) with the inhomogeneous term given by (3.24). This amounts to writing the solution as:

$$\tilde{A}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) = \tilde{A}^{(1)}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) + \pi \frac{\gamma_{aa'}(t)\, g(t,t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-j}}{j-\alpha(t)} \cdot \frac{\dfrac{\pi}{\Gamma(\alpha(t)+1)} \displaystyle\int d\phi_2 \eta_2 \frac{g^2(t,t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})} \left[\frac{j-\alpha_{c,2}}{\alpha(t)-\alpha_{c,2}} - (\bar{s})^{\alpha(t)-j}\right]}{1-(\bar{s})^{\alpha(t)-j} \dfrac{\pi}{\Gamma(\alpha(t)+1)} \displaystyle\int d\phi_2 \eta_2 \frac{g^2(t,t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})}}. \tag{3.25}$$

The second term in Eq. (3.25) reduces to the explicit cut-free form (3.22) of $\tilde{T}_{R_1 a}$ provided the bootstrap condition (3.14) is satisfied. The series expansion of the fraction in Eq. (3.25) gives the cluster expansion of the cross-section, and shows that the n-cluster term has explicit cuts in the j-plane. Notice that all the complications in the j-plane structure of $\tilde{A}_{Ra}$ are buried in the inhomogeneous term $\tilde{A}^{(1)}_{Ra}$. The homogeneous term does not exhibit the detailed form of the full amplitude (it is independent of $h(j,t,t_2^{\pm})$ since only the value of h at $j = \alpha$, i.e., unity, appears in $\tilde{T}_{R_2 a}$). This is achieved because of the correct manner in which the j' integration contour has been closed to the left. In general, one does not know whether closing the contour to the right is allowed. Let us see what happens if this is nevertheless done. The result is that in Eq. (3.19) the j' integration picks up only the pole at j' = j to give:

$$\tilde{A}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) = \tilde{A}^{(1)}_{R_1 a} + (\bar{s})^{\alpha(t)-j} g(t,t_1^{\pm}) \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \frac{g(t,t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})} \tilde{A}_{R_2 a}(j-\alpha_{c,2}-1,t,t_2^{\pm}) \tag{3.26}$$

.

This diagonal form has been used, for instance, in Ref. 13) for $\bar{s} = s_0 = 1/\alpha' \cong 1$. However, in general, only the more general form (3.19) is valid. To illustrate this point, the reader may easily convince himself that Eq. (3.26) leads to a bootstrap condition that reads:

$$\frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \frac{g^2(t,t_2^\pm)}{(\alpha(t)-\alpha_{c,2})^2} h(j,t,t_2^\pm) = 1 \tag{3.27}$$

Since no new physics should be obtained by j-plane considerations that cannot as well be derived in the energy plane, the only solution to (3.26) is a Rp amplitude with $h \equiv 1$ for any j. Thus, unlike the more general Eq. (3.19), the diagonalized form (3.26) serves as an integral equation for an Rp amplitude with a restricted, simple form. As we shall show later on, the situation for RR amplitudes is such that contradictions arise if one is not careful on this point.

Another way to overcome the complication of non-zero lower limits of integration in Eq. (3.16) is to extract from the homogeneous term $\tilde{T}_{R_1a}$, the two-cluster term $\tilde{A}^{(2)}_{Ra}$ 15):

$$\tilde{T}_{R_1a}(j-\alpha_{c,1}-1,t,t_1^\pm) = \tilde{A}^{(2)}_{R_1a}(j-\alpha_{c,1}-1,t,t_1^\pm) + \tilde{S}_{R_1a}(j-\alpha_{c,1}-1,t,t_1^\pm). \tag{3.28}$$

For $\tilde{S}_{R_1a}$ one easily finds the expression

$$\tilde{S}_{R_1a}(j-\alpha_{c,1}-1,t,t_1^\pm) = \int d\phi_2\eta_2 \int_0^{\bar{s}} ds_2\,(s_2^\perp)^{\alpha_{c,1}+1}(s_2)^{-j+\alpha_{c,2}} A_{R_1R_2}(s_2,s_2^\perp,t,t_i^\pm)$$
$$\cdot \int_{\bar{s}/s_2}^{\infty} dx\, x^{-j-1+\alpha_{c,2}} \int_{\bar{s}}^{\infty} ds_3\,(s_3)^{-j+\alpha_{c,2}} A_{R_2a}(s_3,t,t_2^\pm). \tag{3.29}$$

Here the lower limit of the $s_3$ integration is independent of x and $s_2$, due to the extraction of the two-cluster term. The effect of the NDC becomes: $s_2 < \bar{s}$, $s_3 > \bar{s}$, $s_{23}/s_3 > \bar{s}$. We clearly have

$$\int_{\bar{s}}^{\infty} ds_3 (s_3)^{-j+\alpha_{c,2}} A_{R_2 a}(s_3, t, t_2^{\pm}) = \tilde{A}_{R_2 a}(j-\alpha_{c,2}-1, t, t_2^{\pm}) - \tilde{A}^{(1)}_{R_2 a}(j-\alpha_{c,2}-1, t, t_2^{\pm}) \quad (3.30)$$

The integral equation thus becomes:

$$\left[\tilde{A}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm}) - \tilde{A}^{(1)}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm})\right] = \tilde{A}^{(2)}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm})$$
$$+ \int d\phi_2 \eta_2 \int_0^{\bar{s}} ds_2 (s_2^{\perp})^{\alpha_{c,1}+1} (s_2)^{-j+\alpha_{c,2}} A_{R_1 R_2}(s_2, s_2^{\perp}, t, t_i^{\pm}) \quad (3.31)$$
$$\cdot \int_{\bar{s}/s_2}^{\infty} dx\, x^{-j-1+\alpha_{c,2}} \cdot \left[\tilde{A}_{R_2 a}(j-\alpha_{c,2}-1, t, t_2^{\pm}) - \tilde{A}^{(1)}_{R_2 a}(j-\alpha_{c,2}-1, t, t_2^{\pm})\right].$$

After the x and $s_2$ integrations (the latter again becomes a "good" FMSR) we find:

$$\left[\tilde{A}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm}) - \tilde{A}^{(1)}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm})\right] = \tilde{A}^{(2)}_{R_1 a}(j-\alpha_{c,1}-1, t, t_1^{\pm})$$
$$+ g(t, t_1^{\pm}) (\bar{s})^{\alpha(t)-j} \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \frac{g(t, t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})} \cdot \quad (3.32)$$
$$\left[\tilde{A}_{R_2 a}(j-\alpha_{c,2}-1, t, t_2^{\pm}) - \tilde{A}^{(1)}_{R_2 a}(j-\alpha_{c,2}-1, t, t_2^{\pm})\right].$$

Thus an ordinary integral equation with a factorized kernel is found for $\tilde{A}_{Ra} - \tilde{A}^{(1)}_{Ra}$ rather than for $\tilde{A}_{Ra}$. The inhomogeneous term is $\tilde{A}^{(2)}_{Ra}$ -- the two-cluster term. Using the explicit factorized form of $\tilde{A}^{(2)}_{Ra}$ as calculated in Appendix B and employing standard techniques one can solve for $\tilde{A}_{R_1 a} - \tilde{A}^{(1)}_{R_1 a}$ and find the solution (3.25). Thus, independently of the detailed form of $A_{R_1 a}$, the solution of (3.32) is a cut-free amplitude provided the bootstrap constraint (3.14) is obeyed. The difference between this integral equation and the ones commonly found in literature is that on the right-hand side of Eq. (3.32) $\tilde{A}^{(2)}_{R_1 a}$ is, in general, not cancelled by the $\tilde{A}^{(1)}_{R_1 a}$ piece of the integral (in usual CGL [21] type equations $\tilde{A}^{(2)}$ is a factorized product of Mellin transforms). Only for the restricted example with $h(j,t,t_1^{\pm}) \equiv 1$ does such a cancellation occur, and Eq. (3.26) is obtained.

c) Equations of Fig. 4b in energy and J-plane

The second way of constructing the unitarity sum for a Reggeon-particle amplitude results in the equation depicted in Fig. 4b (here, cluster assignment is begun at the right end of the multiperipheral chain). In the energy plane the equation is:

$$A_6(s,s_{23},t,t_1^{\pm}) = A_6^{(1)}(s,s_{23},t,t_1^{\pm}) + T_6'(s,s_{23},t,t_1^{\pm}) \quad . \tag{3.33}$$

Here, similar to Eq. (3.5) we have (i = 1,2)

$$T_6'(s,s_{23},t,t_1^{\pm}) = \theta(s_{23}-\bar{s})\frac{1}{s_{23}}\int_0^{\bar{s}} ds_3 \int_0^{s_{23}/\bar{s}} ds_2 \int d\phi_2\, \gamma_{bc}(t_1^-)\,\gamma_{b'c'}(t_1^+)$$
$$\cdot\, \Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+))\,\eta_2\,(-s_{12})^{\alpha(t_1^-)}(-s_{12}')^{\alpha(t_1^+)}(s_{23})^{\alpha(t_2^-)+\alpha(t_2^+)} \tag{3.34}$$
$$\cdot\, A_{R_1R_2}(s_2,s_2^{\perp},t,t_i^{\pm})\,A_{R_2a}(s_3,t,t_2^{\pm}) \quad .$$

Repeating the procedure leading to Eq. (3.8) we find at the Rp level:

$$T'_{Ra}(s_{23},t,t_1^{\pm}) = \theta(s_{23}-\bar{s})\,(s_{23})^{-\alpha_{c,1}-1}\int d\phi_2\,\eta_2\,(s_{23})^{\alpha_{c,2}}$$
$$\cdot\int_0^{\bar{s}} ds_3 \int_0^{s_{23}/\bar{s}} ds_2\,(s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1R_2}(s_2,s_2^{\perp},t,t_i^{\pm})\,A_{R_2a}(s_3,t,t_2^{\pm}) \quad . \tag{3.35}$$

Again, the $s_2$ and $s_3$ integrals can be done using FMSR (2.16), (2.21). The result is simply

$$T'_{Ra} = T_{Ra} \tag{3.36}$$

It is thus proven that the same result holds starting the NDC from either end of the multiperipheral chain.

In the J-plane things are not as trivial. One has (similar to Eqs. (3.15)-(3.16)):

$$\tilde{T}'_{R_1a}(J,t,t_1^\pm) = \int d\phi_2 \eta_2 \int_0^{\bar{s}} ds_3 (s_3)^{-J-1+\alpha_{c,2}-\alpha_{c,1}} A_{R_2a}(s_3,t,t_2^\pm)$$

$$\cdot \int_{\bar{s}/s_3}^{\infty} dx\, x^{-J-2+\alpha_{c,2}-\alpha_{c,1}} \int_{\bar{s}/s_3x}^{\infty} ds_2 (s_2)^{-J-1+\alpha_{c,2}-\alpha_{c,1}} (s_2^\perp)^{\alpha_{c,1}+1} \tag{3.37}$$

$$\cdot A_{R_1R_2}(s_2,s_2^\perp,t,t_i^\pm) .$$

In analogy to (3.17) we now define

$$(s_2^\perp)^{\alpha_{c,1}+1} A_{R_1R_2}(s_2,s_2^\perp,t,t_i^\pm) \equiv \int \frac{dJ'}{2\pi i} (s_2)^{J'} \tilde{A}^{as}_{R_1^*R_2}(J',t,t_i^\pm) . \tag{3.38}$$

$\tilde{A}^{as}_{R_1^*R_2}$ is the same creature as occurring in Eq. (2.20), with the asymmetry this time being due to the legs on the left-hand side ($t_1^\pm$). Here the x and $s_2$ integrations are straightforward, converting the $s_3$ integral into a FMSR of the type (2.16). Again, shifting $J = j - \alpha_{c,1} - 1$ and $J' = j' - \alpha_{c,2} - 1$ we obtain the analogue of Eq. (3.19)

$$\tilde{T}'_{R_1a}(J,t,t_1^\pm) = (\bar{s})^{\alpha(t)-j} \gamma_{\alpha a'}(t) \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \cdot$$

$$\int \frac{dj'}{2\pi i (j'-\alpha_{c,2})(j-j')} \cdot \frac{g(t,t_2^\pm)}{(\alpha(t)-\alpha_{c,2})} \tilde{A}^{as}_{R_1^*R_2}(j'-\alpha_{c,2}-1,t,t_i^\pm) . \tag{3.39}$$

$\tilde{A}^{as}_{R_1^*R_2}$ has a zero at $J' = -1$ (see discussion in Section 2.3), or at $j' = \alpha_{c,2}$. Therefore, its most general form is

$$\tilde{A}^{as}_{R_1^*R_2}(j'-\alpha_{c,2}-1,t,t_i^\pm) = \pi \frac{g(t,t_1^\pm)\, g(t,t_2^\pm)}{\Gamma(\alpha(t)+1)} \cdot \frac{1}{j'-\alpha(t)}$$

$$\cdot \frac{(j'-\alpha_{c,2})}{(\alpha(t)-\alpha_{c,2})} H(j',t,t_i^\pm) . \tag{3.40}$$

$H(j',t,t_i^{\pm})$ is again a smooth function of $j'$ satisfying $H(j' = \alpha(t),t,t_i^{\pm}) \equiv 1$. Closing the $j'$ contour to the left we again pick up only the pole at $j' = \alpha(t)$ and find

$$\tilde{T}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) = \bar{\kappa}\,\frac{\gamma_{aa'}(t)\, g(t,t_1^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-j}}{j-\alpha(t)} \cdot I(\alpha,t)$$

$$= \tilde{T}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) \quad . \tag{3.41}$$

Here $I(\alpha,t)$ is defined by Eq. (3.14). This general result does not depend on the detailed form of $H(j',t,t_i)$.

Notice that in this approach one does not end up with an integral equation for $A_{Ra}$. Rather, Eq. (3.39) constitutes a consistency check on RR amplitudes. Another point worth mentioning is the delicacy of closing the $j'$ contour in the correct way. Had we closed the contour to the right (see discussion preceding Eq. (3.27)), we would have obtained again an equation diagonalized in $j$ (analogue of Eq. (3.26)). The latter would again lead to a highly restricted form for the solution (3.42), namely with $H(j',t,t_1^{\pm}) \equiv 1$ for all $j'$. That this is not easy to satisfy for RR amplitudes can be seen as follows. According to our definitions, $A^{as}_{R_1^*R_2}(j-\alpha_{c,2}-1,t,t_i^{\pm})$ has a zero at $j' = \alpha_{c,2}$. As discussed in Section 2, it is not inconceivable that it may have a zero at $j' = \alpha_{c,1}$ as well. Let us assume for the moment that this is indeed the case. One then has

$$H(j',t,t_i^{\pm}) = \frac{j'-\alpha_{c,1}}{\alpha(t)-\alpha_{c,1}}\, G(j',t,t_i^{\pm}) \quad , \tag{3.42}$$

with $G(j' = \alpha(t),t,t_i^{\pm}) \equiv 1$. Closing the $j'$ contour to the right in Eq. (3.39) one picks up the pole at $j' = j$ and finds $\tilde{T}_{R_1 a}(j-\alpha_{c,1}-1,t,t_1^{\pm}) \propto (j-\alpha_{c,1})$. Hence $\tilde{T}_{R_1 a}$ vanishes when $j \to \alpha_{c,1}$. As a result $\tilde{A}_{R_1 a} = \tilde{A}^{(1)}_{R_1 a} + \tilde{T}_{R_1 a}$ does not vanish there; this is inconsistent with the analyticity properties of $\tilde{A}_{R_1 a}$ and leads to cuts in $j$. This example is only meant to indicate that, unlike the case of Rp amplitudes, with RR scattering one expects a more complicated j-plane structure so that closing the $j'$ integration contour in the correct sense is essential.

If we try to solve the integral equation implied by Fig. 4b by our second method, namely, by separating the two-cluster term (analogous to Eq. (3.28)):

$$\tilde{T}'_{R_1 a} = \tilde{A}^{(2)'}_{R_1 a} + \tilde{S}'_{R_1 a} \qquad , \tag{3.43}$$

one can easily write an equation for $\tilde{S}'_{R_1 a}$ (analogous to Eqs. (3.29) and (3.31)):

$$\tilde{S}'_{R_1 a}(j-\alpha_{c,1}-1, t, t_i^{\pm}) = \int d\phi_2 \eta_2 \int_0^{\bar{s}} ds_3 (s_3)^{-j+\alpha_{c,2}} A_{R_2 a}(s_3, t, t_2^{\pm})$$
$$\cdot \int_{\bar{s}/s_3}^{\infty} dx\, x^{-j-1+\alpha_{c,2}} \int_{\bar{s}}^{\infty} ds_2 (s_2^{\pm})^{\alpha_{c,1}+1} (s_2)^{-j+\alpha_{c,2}} A_{R_1 R_2}(s_2, s_2^{\pm}, t, t_i^{\pm}) \tag{3.44}$$

$$= \gamma_{aa'}(t)\, (\bar{s})^{\alpha(t)-j} \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \frac{g(t,t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})} \cdot$$
$$\left[\tilde{A}^{as}_{R_1^* R_2}(j-\alpha_{c,2}-1, t, t_i^{\pm}) - \tilde{A}^{(1)as}_{R_1^* R_2}(j-\alpha_{c,2}-1, t, t_i^{\pm})\right] .$$

We obviously cannot "solve" this equation as, again, the right-hand side involves RR amplitudes. However, using the high-energy behaviour of $A_{RR}$

$$\tilde{A}^{as}_{R_1^* R_2}(j-\alpha_{c,2}-1, t, t_i^{\pm}) - \tilde{A}^{(1)as}_{R_1^* R_2}(j-\alpha_{c,2}-1, t, t_i^{\pm})$$
$$= \pi \frac{g(t,t_1^{\pm})\, g(t,t_2^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-j}}{j-\alpha(t)} \qquad , \tag{3.45}$$

we find:

$$\tilde{A}_{R_1 a} - \tilde{A}^{(1)}_{R_1 a} = \tilde{A}^{(2)'}_{R_1 a} +$$
$$\pi \frac{g(t,t_1^{\pm})\, \gamma_{aa'}(t)}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{2\alpha(t)-2j}}{j-\alpha(t)} \cdot \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \frac{g^2(t,t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})} . \tag{3.46}$$

Notice that, trivially, $\tilde{A}^{(2)'}_{R_1 a} = \tilde{A}^{(2)}_{R_1 a}$. Hence, using the explicit form of the two-cluster term, found in Appendix B, Eq. (3.46) becomes identical to Eq.(3.23). This procedure merely provides a consistency check on RR amplitudes. Thus, no inconsistency is ever found in the study of Rp amplitudes if care is taken of the correct limits of integration and of the analytic properties of Reggeon amplitudes. A pure pole solution is consistent and the only output constraint is the $\bar{s}$-independent bootstrap condition (3.14).

## 3.3 Integral equations for Reggeon-Reggeon amplitudes

To complete our consistency study, we look at the integral equations for RR scattering. Starting in the energy plane with the eight-point function of Fig. 2 one has an equation depicted in Fig. 5:

$$A_8 = A_8^{(1)} + T_8 \quad , \tag{3.47}$$

where $T_8$ is the homogeneous term. In analogy to Eq. (3.5), one now writes an integral expression for $T_8$. Extracting the appropriate kinematical factors (see Table 1) yields an integral equation for the RR amplitude (similar to Eq. (3.8)):

$$\begin{aligned} &A_{R_1R_3}(s_{23}, s_{23}^{\perp}, t, t_1^{\pm}, t_3^{\pm}) - A_{R_1R_3}^{(1)}(s_{23}, s_{23}^{\perp}, t, t_1^{\pm}, t_3^{\pm}) \equiv T_{R_1R_3} = \\ &\theta(s_{23} - \bar{s})(s_{23})^{-\alpha_{c,1} - \alpha_{c,3} - 2} \int d\phi_2 \, \eta_2(s_{23})^{\alpha_{c,2}} \cdot \\ &\int_0^{\bar{s}} ds_2 \int_0^{s_{23}/\bar{s}} ds_3 \, (s_2^{\perp})^{\alpha_{c,1}+1} A_{R_1R_2}(s_2, s_2^{\perp}, t, t_1^{\pm}, t_2^{\pm}) \cdot \\ &\qquad (s_3^{\perp})^{\alpha_{c,3}+1} A_{R_2R_3}(s_3, s_3^{\perp}, t, t_2^{\pm}, t_3^{\pm}) \end{aligned} \tag{3.48}$$

On the right-hand side we now have two RR amplitudes each appearing in a "good" FMSR (fixed pole free type -- see Eq. (2.21)). The result is

$$\begin{aligned} &A_{R_1R_3} - A_{R_1R_3}^{(1)} = \\ &\pi \frac{g(t, t_1^{\pm}) g(t, t_3^{\pm})}{\Gamma(\alpha(t)+1)} (s_{23})^{\alpha(t) - \alpha_{c,1} - \alpha_{c,3} - 2} \, I(\alpha, t) \end{aligned} \tag{3.49}$$

This exactly agrees with the high-energy behaviour of RR amplitudes, provided the bootstrap condition (3.14) is satisfied. Hence, nothing new is found by looking at RR scattering. Moreover, the bootstrap is consistent with an $A_{RR}$ which is symmetric in the external legs although the counting procedure is asymmetric.

In turning to the J-plane, various Mellin transforms may be considered (see Section 2 and Appendix A). Let us begin with the asymmetric transform defined in Eq. (2.20). We have

$$\tilde{A}^{as}_{R_1 R_3^*} - \tilde{A}^{(1)\,as}_{R_1 R_3^*} = \int_{\bar{s}}^{\infty} d s_{23}\, (s_{23})^{-J-1}\, (s_{23}^{\perp})^{\alpha_{c,3}+1} \cdot T_{R_1 R_3}(s_{23}, s_{23}^{\perp}, t, t_1^{\pm}, t_3^{\pm}) \tag{3.50}$$

Taking $s_{23}^{\perp} \cong s_{23}$ for $s_{23} \geq \bar{s}$ we use our standard techniques (see derivation of Eq. (3.19)) to find an integral equation (not diagonal in j):

$$\begin{aligned} &\tilde{A}^{as}_{R_1 R_3^*}(j-\alpha_{c,1}-1, t, t_1^{\pm}, t_3^{\pm}) - \tilde{A}^{(1)\,as}_{R_1 R_3^*}(j-\alpha_{c,1}-1, t, t_1^{\pm}, t_3^{\pm}) = \\ &(\bar{s})^{\alpha(t)-j}\, g(t,t_1^{\pm})\, \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2\, \eta_2 \int \frac{dj'}{2\pi i\,(j'-\alpha_{c,2})(j-j')} \cdot \\ &\frac{g(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})}\, \tilde{A}^{as}_{R_2 R_3^*}(j'-\alpha_{c,2}-1, t, t_2^{\pm}, t_3^{\pm}) \,. \end{aligned} \tag{3.51}$$

$\tilde{A}^{as}_{R_2 R_3^*}$ has a zero at $j' = \alpha_{c,2}$ (see Eq. (3.42)). Again by closing the j' contour of integration to the left we find that the cut is killed and, independently of the detailed form of $\tilde{A}^{as}_{R_2 R_3^*}$,

$$\begin{aligned} &\tilde{A}^{as}_{R_1 R_3^*}(j-\alpha_{c,1}-1, t, t_1^{\pm}, t_3^{\pm}) - \tilde{A}^{(1)\,as}_{R_1 R_3^*}(j-\alpha_{c,1}-1, t, t_1^{\pm}, t_3^{\pm}) = \\ &\pi\, \frac{g(t,t_1^{\pm})\, g(t,t_3^{\pm})}{\Gamma(\alpha(t)+1)} \cdot \frac{(\bar{s})^{\alpha(t)-j}}{j-\alpha(t)} \,, \end{aligned} \tag{3.52}$$

provided the usual bootstrap condition (3.14) holds.

Notice that each term on the left-hand side of Eq. (3.52) is asymmetric. However, the shift $J \to j - \alpha_{c,1} - 1$ and the neglect of $p_{\perp}^2$ in $s_{23}$ for $s_{23} > \bar{s}$ result in a symmetric form for the difference.

Another possible (symmetric) transform, not discussed beforehand, is the following:

$$\tilde{A}_{R_1R_3}(J,t,t_1^{\pm},t_3^{\pm}) \equiv \int_0^{\infty} ds_{23}\,(s_{23})^{-J-1} A_{R_1R_3}(s_{23},s_{23}^{\perp},t,t_1^{\pm},t_3^{\pm}) \quad (3.53)$$

It is easy to show that in the approximation $s_{23}^{\perp} \simeq s_{23}$ (for $s_{23} > \bar{s}$) this transform and the other symmetric types discussed in Section 2 also yield self-consistency namely,

$$\tilde{A}^{sy}_{R_1R_3}(j) - \tilde{A}^{(1)sy}_{R_1R_3}(j) = \tilde{A}^{\perp}_{R_1R_3}(j) - \tilde{A}^{(1)\perp}_{R_1R_3}(j) =$$
$$\tilde{A}_{R_1R_3}(j-\alpha_{c,1}-\alpha_{c,3}-2) - \tilde{A}^{(1)}_{R_1R_3}(j-\alpha_{c,1}-\alpha_{c,3}-2) \quad (3.54)$$

all being equal to the right-hand side of Eq. (3.51), and hence also to the right-hand side of Eq. (3.52). These relations do not constitute integral equations for the symmetric transforms, since the right-hand side of Eq. (3.52) includes the asymmetric transform. To derive integral equations for each of them, we again separate the two cluster terms to obtain, for example (analogue of (3.32))

$$[\tilde{A}^{sy}_{R_1R_3}(j) - \tilde{A}^{(1)sy}_{R_1R_3}(j)] = \tilde{A}^{(2)sy}_{R_1R_3} +$$
$$g(t,t_1^{\pm})(\bar{s})^{\alpha(t)-j}\frac{\pi}{\Gamma(\alpha(t)+1)}\int d\phi_2\,\eta_2\,\frac{g(t,t_2^{\pm})}{(j-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})}\cdot \quad (3.55)$$
$$[\tilde{A}^{sy}_{R_2R_3}(j) - \tilde{A}^{(1)sy}_{R_2R_3}(j)]$$

Given the form of the two-cluster term in Appendix B, the solution of Eq. (3.55) is straightforward. Its form is analogous to Eq. (3.25), and, using the bootstrap condition (3.14), becomes identical to the right-hand-side of (3.52). Similar equations (diagonal in j) obviously hold for $\tilde{A}^{\perp}_{R_1R_3}$, $\tilde{A}_{R_1R_3}$ and $\tilde{A}^{as}_{R_1R_3^*}$ with the appropriate shifts in j. However, for the symmetric transforms, the $\tilde{A}^{(2)}$ term can <u>never</u> be cancelled by the $\tilde{A}^{(1)}$ contribution to the integral (see, e.g. Eq. (3.55)). As for the asymmetric transform $A^{as}_{R_1R_3^*}(j - \alpha_{c,1} - 1)$, in general, the cancellation is not possible, except for the restricted possibility

$$\tilde{A}^{as}_{R_1 R_3^*}(j-\alpha_{c,1}-1) = \pi \frac{g(t,t_1^\pm)\,g(t,t_3^\pm)}{\Gamma(\alpha(t)+1)} \cdot \frac{1}{j-\alpha(t)} \; \frac{j-\alpha_{c,2}}{\alpha(t)-\alpha_{c,2}} \qquad (3.56)$$

(i.e., $H \equiv 1$ in Eq. (3.40)).

In conclusion, no inconsistencies arise in the application of planar unitarity to RR amplitudes.

### 3.4 Cut cancellation in the energy plane

Apart from the various integral equations and self-consistency checks satisfied by planar Reggeon amplitudes, it is instructive to examine how planar self-consistency works term by term in the unitarity sum.

Consider, for example, the case of a RR amplitude

$$A_{R_\ell R_r}(M^2, M_\perp^2, t, t_\ell^\pm, t_r^\pm) \quad .$$

The two-cluster term, calculated in Appendix B, has the form

$$A^{(2)}_{R_\ell R_r} = \pi \frac{g(t,t_\ell^\pm)\,g(t,t_r^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{-\alpha_{c,\ell}-\alpha_{c,r}-2} \quad .$$

$$\Big\{ \theta(\bar{s}^2 - M^2)\,\theta(M^2-\bar{s})\,(M^2)^{\alpha(t)}\, I(\alpha,t) \; + \qquad (3.57)$$

$$\theta(M^2-\bar{s}^2)\,\frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_1 \eta_1 \frac{g^2(t,t_1^\pm)}{(\alpha(t)-\alpha_{c,1})^2} \frac{(M^2)^{\alpha_{c,1}}}{(\bar{s})^{2(\alpha(t)-\alpha_{c,1})}} \Big\} \quad .$$

The first term has the "pure pole" form in the limited range $\bar{s} < M^2 < \bar{s}^2$. The second term, which takes over at $M^2 > \bar{s}^2$ has an explicit cut in the J-plane (we remind the reader that all our energy variables are scaled in units of $s_0 = 1/\alpha' \simeq 1$).

The calculation of the three-cluster term is more tedious but, again, straightforward, giving

$$A^{(3)}_{R_e R_r} = \pi \frac{g(t,t_e^{\pm})\, g(t,t_r^{\pm})}{\Gamma(\alpha(t)+1)} (M^2)^{-\alpha_{c,e}-\alpha_{c,r}-2} \cdot$$

$$\Big\{ \theta(\bar{s}^3 - M^2)\,\theta(M^2 - \bar{s}^2)\,(M^2)^{\alpha(t)}\, I^2(\alpha,t) \; -$$

$$\theta(M^2-\bar{s}^2) \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_1 \eta_1 \frac{g^2(t,t_1^{\pm})}{(\alpha(t)-\alpha_{c,1})^2} (M^2)^{\alpha_{c,1}} (\bar{s})^{2(\alpha(t)-\alpha_{c,1})} \cdot \tag{3.58}$$

$$I(\alpha,t) \; +$$

$$\theta(M^2-\bar{s}^3) \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_1 \eta_1 \frac{g^2(t,t_1^{\pm})}{(\alpha(t)-\alpha_{c,1})^2} \cdot \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \cdot$$

$$\frac{g^2(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})^2} \Big[ (M^2)^{\alpha_{c,1}} (\bar{s})^{3(\alpha(t)-\alpha_{c,1})} \frac{\alpha(t)-\alpha_{c,2}}{\alpha_{c,1}-\alpha_{c,2}} \; + \; (M^2)^{\alpha_{c,2}} (\bar{s})^{3(\alpha(t)-\alpha_{c,2})} \frac{\alpha(t)-\alpha_{c,1}}{\alpha_{c,2}-\alpha_{c,1}} \Big] \Big\} \, .$$

By induction one proves that the form of the N+1 cluster term is

$$A^{(N+1)}_{R_e R_r} = \pi \frac{g(t,t_e^{\pm})\, g(t,t_r^{\pm})}{\Gamma(\alpha(t)+1)} (M^2)^{-\alpha_{c,e}-\alpha_{c,r}-2} \cdot$$

$$\Big\{ \theta(\bar{s}^{N+1} - M^2)\,\theta(M^2-\bar{s}^N)\,(M^2)^{\alpha(t)}\, I^N(\alpha,t)$$

$$- \theta(M^2-\bar{s}^N)\, C_N(M^2,t)\, I(\alpha,t) \tag{3.59}$$

$$+ \theta(M^2-\bar{s}^{N+1})\, C_{N+1}(M^2,t) \Big\} \, ,$$

with

$$C_{N+1}(M^2,t) = \sum_{i=1}^{N} \Big\{ \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_i \eta_i \frac{g^2(t,t_i^{\pm})}{(\alpha(t)-\alpha_{c,i})^2}$$

$$(M^2)^{\alpha_{c,i}} (\bar{s})^{(N+1)(\alpha(t)-\alpha_{c,i})} \cdot \tag{3.60}$$

$$\prod_{\substack{k=1 \\ k\neq i}}^{N} \Big( \frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_k \eta_k \frac{g^2(t,t_k)}{(\alpha(t)-\alpha_{c,k})^2} \cdot \frac{\alpha(t)-\alpha_{c,k}}{\alpha_{c,i}-\alpha_{c,k}} \Big) \Big\} \, .$$

The C terms will have explicit cuts in the J plane.

At a given energy $M^2$, the maximum number of clusters that our NDC allows is

$$N_{max} = \left[ \frac{\ln M^2}{\ln \bar{s}} \right] + 1 .$$

This gives

$$(\bar{s})^{N_{max}-1} \leq M^2 < (\bar{s})^{N_{max}} .$$

The value of the amplitude is given by

$$A_{R_\ell R_r}(M^2 > \bar{s}) = \sum_{N=2}^{N_{max}} A^{(N)}_{R_\ell R_r} . \tag{3.61}$$

At the given value of M all the "pure pole" terms ($\sim (M^2)^{\alpha(t)-\alpha_{c,\ell}-\alpha_{c,r}-2}$) vanish due to their $\theta$ functions, except for the last one (with $\theta(\bar{s}^{N_{max}} - M^2)\,\theta(M^2 - \bar{s}^{N_{max}-1}) \equiv$ $\equiv 1$). Moreover, provided the bootstrap condition (3.14) is obeyed, all cut terms cancel in pairs, except for the last one coming from $A^{(N_{max})}_{R_\ell R_r}$:

$$A_{R_\ell R_r}(M^2 > \bar{s}) = \pi \frac{g(t,t_\ell^\pm)\, g(t,t_r^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{\alpha(t)-\alpha_{c,\ell}-\alpha_{c,r}-2} +$$

$$\pi \frac{g(t,t_\ell^\pm)\, g(t,t_r^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{-\alpha_{c,\ell}-\alpha_{c,r}-2} .$$

$$\Big\{ \theta(M^2-\bar{s})\, C_2 + [-\theta(M^2-\bar{s})C_2 + \theta(M^2-\bar{s}^2)\, C_3] \tag{3.62}$$

$$+ [-\theta(M^2-\bar{s}^2)C_3 + \theta(M^2-\bar{s}^3)\, C_4]$$

$$+ \cdots\cdots [-\theta(M^2-\bar{s}^{N_{max}-1})C_{N_{max}-1}$$

$$+ \theta(M^2-\bar{s}^{N_{max}})\, C_{N_{max}}] \Big\} =$$

$$\pi \frac{g(t,t_\ell^\pm)\, g(t,t_r^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{-\alpha_{c,\ell}-\alpha_{c,r}-2} .$$

$$[(M^2)^{\alpha(t)} + \theta(M^2-\bar{s}^{N_{max}})\, C_{N_{max}}] = \pi \frac{g(t,t_\ell^\pm)\, g(t,t_\ell^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{\alpha(t)-\alpha_{c,\ell}-\alpha_{c,r}-2} .$$

The last step in Eq. (3.62) follows from the definition of $N_{max}$ $(M^2 < (\bar{s})^{N_{max}})$. Thus, at any given $M^2 > \bar{s}$ the pure pole term is contributed by those phase-space configurations that require the maximum allowed number ($N_{max}$) of clusters (populated bins). The cut terms which appear in each k-cluster term are cancelled successively.

Mellin transforming the N+1 cluster term (using e.g., the symmetric transform defined in Eq.(2.23)) one can prove by induction that

$$\tilde{A}^{(N+1)\,sy}_{R_\ell R_r}(J,t,t_\ell^\pm,t_r^\pm) \equiv \int_0^\infty dM^2 (M^2)^{-J-1} (M_\perp^2)^{\alpha_{c,\ell}+\alpha_{c,r}+2} \cdot A^{(N+1)}_{R_\ell R_r}(M^2, M_\perp^2, t, t_\ell^\pm, t_r^\pm) =$$

$$\pi \frac{g(t,t_\ell^\pm)\, g(t,t_r^\pm)}{\Gamma(\alpha(t)+1)} \frac{(\bar{s})^{\alpha(t)-J}}{J-\alpha(t)} \cdot \tag{3.63}$$

$$\left( \frac{\pi\, \bar{s}^{\alpha(t)-J}}{\Gamma(\alpha(t)+1)} \int d\phi_1 \eta_1 \frac{g^2(t,t_1^\pm)}{(J-\alpha_{c,1})(\alpha(t)-\alpha_{c,1})} \right)^{N-1} \cdot$$

$$\frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \eta_2 \frac{g^2(t,t_2^\pm)}{(J-\alpha_{c,2})(\alpha(t)-\alpha_{c,2})} \left[ \frac{J-\alpha_{c,2}}{\alpha(t)-\alpha_{c,2}} - \bar{s}^{\,\alpha(t)-J} \right] .$$

This form has an explicit cut in the J plane. It is precisely the N + 1 cluster term one obtains from the series expansion of the explicit solution of Eq. (3.55) for $\tilde{A}^{sy}_{R_\ell R_r} - \tilde{A}^{(1)sy}_{R_\ell R_r}$. In fact, by a straightforward summation of the geometric series (whose N-th term is Eq. (3.63)), or by using the simple recursion relation between $\tilde{A}^{(N+1)sy}_{R_\ell R_r}(J)$ and $\tilde{A}^{(N)sy}_{R_\ell R_r}(J)$ one can easily derive Eq. (3.55) and its solution (analogous to (3.25)). Thus the geometric series adds up to a pure pole in the J-plane.

## 3.5 Pitfalls in the rapidity formulation

A number of works on the planar bootstrap have used the rapidity language. At first sight, rapidity seems a natural variable for multiperipheral cluster models. Particles are arranged along the rapidity axis in clusters of sizes $(L_i)$ up to a maximum extent $\bar{L}$; Reggeons are exchanged across the gaps between clusters. Kinematics and NDC are easily translatable into this language. Despite its intuitive appeal, the rapidity formulation may easily lead to pitfalls that spoil planar self-consistency.

The basic problem is the simulation of the correct analyticity properties of amplitudes, in particular, FMSR. The naive approach, whereby simple Regge behaviour is assumed for amplitudes down to threshold (L = 0), is incompatible with a self-consistent bootstrap (even if a NDC is applied). In other words, for a RR amplitude whose asymptotic behaviour in rapidity is

$$A_{R_\ell R_r}(L,t) \underset{L \gg \bar{L}}{\longrightarrow} F_\ell(t) F_r(t) e^{L\alpha(t)} \quad , \tag{3.64}$$

the corresponding FMSR is not given by

$$\int_0^{\bar{L}} dL\, e^{-\alpha_{c,e} L} A_{R_\ell R_r}(L,t) = F_\ell(t) F_r(t) \frac{e^{\bar{L}(\alpha(t)-\alpha_{c,e})} - 1}{\alpha(t) - \alpha_{c,e}} \quad . \tag{3.65}$$

Instead, the analogue of the "good" FMSR discussed in Section 2 should be imposed:

$$\int_0^{\bar{L}} dL\, e^{-\alpha_{c,e} L} A_{R_\ell R_r}(L,t) = F_\ell(t) F_r(t) \frac{e^{\bar{L}(\alpha(t)-\alpha_{c,e})}}{\alpha(t) - \alpha_{c,e}} \quad . \tag{3.66}$$

This peculiar form, in which the lower limit piece is missing, mocks up the low-energy behaviour of the amplitude at threshold (L = 0). The physical explanation is that a zero size cluster is not empty. It includes, at least, the single stable particle intermediate state.

Keeping track of these analyticity requirements while maintaining strict no-double-counting in the unitarity integral is extremely delicate in the rapidity picture. Only a particular order of integration over clusters and gaps readily yields self-consistency[15]. Other orders of integration require care in keeping track of the L = 0 singularities. Otherwise, double-counting errors are committed.

## 4. REMARKS AND CONCLUSIONS

In this paper we have studied the important interplay between analyticity and proper counting of events in planar unitarity. This interplay is essential for guaranteeing self-consistency (i.e., pure pole solutions to the bootstrap) of planar amplitudes. Abandoning either element destroys full consistency (i.e., no cuts) although consistency at the pole may be possible.

The notion of clusters used in this paper is that of mere mathematical objects -- essentially dividing phase space into bins (sets of particles) over which one averages using the known (or the assumed) analytic properties of Reggeon amplitudes. It is thus only appropriate that the results are independent of the cluster mass cut-off $\bar{s}$. The only constraint resulting from this approach is the bootstrap condition (3.14), originally derived in Ref. 10). A numerical study of this constraint has shown[11),14)] that it yields reasonable values for the Regge trajectory and the triple Reggeon coupling.

An extension of our results to models with physical clusters (e.g., narrow resonances) seems worthwhile, in the hope that entities of phenomenological consequence may be constrained. For such an approach (see also Ref. 3)) one expects the results to depend on the cluster size cut-off. The symmetric NDC (3.1) is generally assumed to be appropriate in this case[8),19),20)]. However, with this NDC, it has been shown[15)] that cut cancellation is possible for $\bar{s} = 1$ (in units of $s_0$). For $\bar{s} > 1$ cut cancellation is still an open problem.

Let us conclude with a few remarks.

1) Revealing the analyticity properties of Reggeon amplitudes is a delicate matter. Using the dual model example as a guide we have extracted certain "good" FMSR which are expected to be satisfied by appropriately defined Reggeon amplitudes.

2) Proper counting of events leads to the appearance of these "good" FMSR in the unitarity summation.

3) Another consequence of the NDC is the non-trivial limits of integration over Reggeon amplitudes in the integral equations. As a result, the j-plane integral equations are not diagonal in j if the single-cluster term is chosen as the inhomogeneous term.

4) It is possible to write integral equations which are diagonal in j for the amplitudes minus their single-cluster terms (the two-cluster contribution then becomes the driving term).

5) The solution of the bootstrap yields the precise form of the high-energy part of the amplitude (a pure Regge pole) and not the low-energy (single-cluster) part. Moreover, the low-energy part is only constrained by FMSR. Its full j plane content never enters the bootstrap. The mechanism of cut-killing is that of "promotion" of a cut into a pole, unlike the cut cancellation in the AFS case[22)23)]. Here at a given energy $M^2 > \bar{s}$ we have a sum of multi-cluster terms, all positive definite adding up to a pure pole (i.e., pure $(M^2)^\alpha$ behaviour).

All non-leading terms cancel successively, and the $(M^2)^\alpha$ piece remains. This piece comes solely from the term with maximum number ($N_{max}$) of clusters. Notice that the $N_{max}$ configuration includes the events, which in the ordinary multiperipheral model[22),24)], are the most probable ones -- those of uniform particle distribution (in rapidity). It is the latter configuration which in the usual multiperipheral picture gives the leading ($(M^2)^\alpha$) term. Here, by properly including all possible events (not only the most probable ones) we are able to eliminate the undesired non-leading terms.

6) The use of rapidity in formulating multiperipheral models is convenient. Rapidity enables one to construct a simple intuitive picture. However, the rapidity language may obscure important features of the problem. The main aspect is the analytic structure of amplitudes which is naturally studied in terms of invariant $(\text{mass})^2$ variables. In particular, the appropriate form of FMSR has to be imposed in the rapidity formulation. As a consequence, the integration over clusters and gaps becomes a delicate matter[15)].

7) Multi-Regge kinematics are essential to our approach. Moreover, we had to assume simple Reggeon exchange across gaps between clusters even when these gaps were small and we have kept leading Reggeons everywhere, which is probably quantitatively incorrect (one should add, e.g., $\pi$-exchange), but did not seem to affect the question of self-consistency of a pole solution.

Finally, $t_{min}$ effects have been neglected. A rough estimate indicates that $t_{min}$ effects build up singularities which are lower (by at least one unit) from the cut we want to cancel. $t_{min}$ effects will influence quantitatively the bootstrap attempted in Ref. 11) and indications are that they improve the agreement over a range of t[25)].

8) The low-energy behaviour of planar amplitudes plays a crucial role in attaining full-consistency at the planar level. These amplitudes may then be used as input for the calculation of the Pomeron (cylinder)[8),14),25)]. Although off-

hand one might think that only the high-energy behaviour of planar amplitudes is relevant to the calculation of the Pomeron parameters, this is not obvious. Iterating approximate forms for planar amplitudes may lead to the accumulation of serious quantitative errors in the calculated Cylinder amplitude[15].

## NOTES ADDED

After this work had been written, a paper of I.G. Halliday (Imperial College preprint ICTP/75/12, 1976) came to our attention, where FMSR for RR scattering are discussed in a $\lambda\phi^3$ example. The author's conclusion is rather pessimistic concerning the possibility of obtaining naive FMSR for RR scattering, of which our dual model example provides an explicit realization. It seems to us, however, that Halliday's analysis has not fully exploited planarity, which is crucial to our analysis. This could be the reason why we can get around some of his problems.

Subsequently, we also received a paper by P. Hoyer, N.A. Törnqvist and B.R. Webber (LBL preprint-4854, 1976) in which FMSR for RR scattering are discussed. It is argued that naive FMSR can be written down for certain double discontinuities of $A_{RR}$. Our explicit calculations indicate that FMSR free of fixed pole contributions might also exist for those single discontinuities of $A_{RR}$ which enter in the unitarity equation.

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## APPENDIX A

### Reggeon-Reggeon Scattering in the Dual Model

The dual amplitude $B_8$ may be written as a multiple beta transform[17)] over the nine "energy variables" shown in Fig. 2. By taking the leading poles in the Reggeon legs, four integrations may be trivially done to yield:

$$B_8 \cong g^6 (-\alpha' s_1)^{\alpha(t_1^-)} (-\alpha' s_1')^{\alpha(t_1^+)} (-\alpha' s_2)^{\alpha(t_2^-)} (-\alpha' s_2')^{\alpha(t_2^+)} \cdot$$
$$\frac{1}{(2\pi i)^5} \int_{-i\infty}^{+i\infty} d\tau_{M^2} \int_{-i\infty}^{+i\infty} d\tau \int_{-i\infty}^{+i\infty} d\tau' \int_{-i\infty}^{+i\infty} d\bar{\tau} \int_{-i\infty}^{+i\infty} d\bar{\tau}'$$
$$\Gamma(-\tau)\Gamma(-\tau')\Gamma(-\bar{\tau})\Gamma(-\bar{\tau}')(-1)^{\tau+\tau'+\bar{\tau}+\bar{\tau}'}(M^2)^{-\tau-\tau'-\bar{\tau}-\bar{\tau}'} \cdot \tag{A.1}$$
$$\Gamma(-\alpha(t_1^-)+\tau+\bar{\tau})\,\Gamma(-\alpha(t_1^+)+\tau'+\bar{\tau}')\,\Gamma(-\alpha(t_2^-)+\tau+\bar{\tau}')\,\Gamma(-\alpha(t_2^+)+\tau'+\bar{\tau}) \cdot$$
$$B_4(-\tau_{M^2}, -\alpha(M^2)) \cdot$$
$$B_4\left(-\alpha(t)+\alpha_{c,1}+\alpha_{c,2}+2-\tau-\tau'-\bar{\tau}-\bar{\tau}'+\tau_{M^2},\ 1+\bar{\tau}+\bar{\tau}'\right) \cdot$$

The last $B_4$ in Eq. (A.1) can be expanded in poles of $\tau_{M^2}$ with the numerator terms of the expansion-polynomials in $\bar{\tau}$ and $\bar{\tau}'$. The $\tau_{M^2}$ integration is now easily done.

By exploiting the expansion

$$x^r = \frac{1}{\Gamma(-r)} \sum_{m=0}^{\infty} g_m(-r)\, B(-r+m, x+1) \quad , \tag{A.2}$$

where $g_m(-r)$ is a polynomial in r of degree m, we may combine

$$\Gamma(-\bar{\tau})(\bar{\tau})^r = \sum_{\ell=0}^{r} (-1)^r \frac{r!}{\ell!}\, g_{r-\ell}(-r)\, \Gamma(-\bar{\tau}+\ell) \tag{A.3}$$

(and similarly for $\Gamma(-\bar{\tau}')\ (\bar{\tau}')^{r'}$). After introducing $\beta \equiv \tau + \tau' + \bar{\tau} + \bar{\tau}'$ and

$\gamma \equiv \tau + \bar{\tau}'$, the three integrations over $\tau,\tau'$ and $\gamma$ may be performed explicitly. The RR amplitude extracted from $B_4$ according to Table 1 may now be written as

$$A_{R_1R_2}(M^2, \mu_\perp^2, t, t_1^\pm, t_2^\pm) = g^2 \sum_{K=0}^{\infty} \sum_{\ell,\ell'=0}^{K} C_K^{\ell\ell'} \cdot$$

$$\frac{\Gamma(-\alpha(t_1^-)+\ell)}{\Gamma(-\alpha(t_1^-))} \frac{\Gamma(-\alpha(t_1^+)+\ell')}{\Gamma(-\alpha(t_1^+))} \frac{\Gamma(-\alpha(t_2^-)+\ell')}{\Gamma(-\alpha(t_2^-))} \frac{\Gamma(-\alpha(t_2^+)+\ell)}{\Gamma(-\alpha(t_2^+))} \cdot \tag{A.4}$$

$$\int_{-i\infty}^{+i\infty} \frac{d\beta\,(-1)^\beta}{2\pi i} \frac{\Gamma(-\beta+\ell+\ell')\Gamma(\beta-\alpha_{c,1}-1)\Gamma(\beta-\alpha_{c,2}-1)}{\Gamma(-\alpha_{c,1}-1+\ell+\ell')\Gamma(-\alpha_{c,2}-1+\ell+\ell')} (\mu_\perp^2)^{-\beta} \cdot$$

$$B_4(-\beta-\alpha(t)+\alpha_{c,1}+\alpha_{c,2}+2+K, -\alpha(M^2)) .$$

Here $C_K^{\ell\ell'}$ are numerical factors resulting from the various expansions (i.e., the index K runs over poles in $\tau_{M^2}$ mentioned after Eq. (A.1)).

To calculate the asymmetric Mellin transform $\tilde{A}^{as}_{R_1R_2^*}(J,t,t_1^\pm,t_2^\pm)$ of Eq. (2.20) we first expand in powers of $P_\perp^2 \equiv (\vec{p}_c + \vec{p}_d)^2$ about $P_\perp^2 = 0$. Then, integrating over $M^2$ using

$$\int_0^\infty dx\, x^\rho \frac{1}{2i\pi} \mathrm{Disc}_x\, B_4(-\omega,-x) = \frac{1}{\Gamma(-\rho)} \sum_{m=0}^{\infty} \frac{g_m(-\rho)}{\rho-\omega-1+m} , \tag{A.5}$$

we find

$$\tilde{A}^{as}_{R_1R_2^*} = g^2 \sum_{K=0}^{\infty} \sum_{\ell,\ell'=0}^{K} C_K^{\ell\ell'} \frac{\Gamma(-\alpha(t_1^-)+\ell)}{\Gamma(-\alpha(t_1^-))} \frac{\Gamma(-\alpha(t_1^+)+\ell')}{\Gamma(-\alpha(t_1^+))} \cdot$$

$$\frac{\Gamma(-\alpha(t_2^-)+\ell')}{\Gamma(-\alpha(t_2^-))} \frac{\Gamma(-\alpha(t_2^+)+\ell)}{\Gamma(-\alpha(t_2^+))} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (P_\perp^2)^n \cdot \tag{A.6}$$

$$\sum_{m=0}^{\infty} \frac{1}{(J-\alpha(t)+\alpha_{c,1}+1+n+K+m)\Gamma(-\alpha_{c,1}-1+\ell+\ell')\Gamma(-\alpha_{c,2}-1+\ell+\ell')}$$

$$\int_{-i\infty}^{+i\infty} \frac{d\beta\,(-1)^\beta}{2\pi i} \frac{\Gamma(-\beta+\ell+\ell')\Gamma(\beta-\alpha_{c,1}-1)\Gamma(\beta-\alpha_{c,2}-1+n)}{\Gamma(\beta+J-\alpha_{c,2}+n)} g_m(\beta+J-\alpha_{c,2}+n) .$$

After combining the polynomial $g_m(\beta + J - \alpha_{c,2} + n)$ with $\Gamma(-\beta + \ell + \ell')$ via Eq. (A.3), the $\beta$ integration may be explicitly performed. The resulting term-by-term j-dependence of (A.6) is:

$$\frac{(J-\alpha_{c,2}+n+\ell+\ell')^{q'}\,\Gamma(J+\alpha_{c,1}+2-\ell-\ell'-q)}{\Gamma(J+1)\,\Gamma(J+1+\alpha_{c,1}-\alpha_{c,2}+n)\,(J-\alpha(t)+\alpha_{c,1}+1+n+K+m)} \qquad \text{(A.7)}$$

,

where $0 \leq q, q' \leq m$. Thus, at $J = -1, -2, -3, \ldots$, the series for $\tilde{A}^{as}_{R_1 R_2^*}$ vanishes term-by-term. This result does not depend on the lower order poles in $M^2$ (i.e., $K \neq 0$) or on the non-leading terms in the $P_\perp^2$ expansion (i.e. $n \neq 0$). We shall assume that the series itself has a zero at $J = -1, -2, -3 \ldots$ . Note that the leading ($n = 0$) term, also has a zero at $J = \alpha_{c,2} - \alpha_{c,1} - 1$ although (A.6) does not vanish there for $n \neq 0$. The poles in the numerator factor $(J + \alpha_{c,1} + 2 - \ell - \ell' - g)$ are clearly spurious -- they can be at arbitrarily high values of J. One can easily trace the origin of these poles to divergences from large values of $\beta$ in (A.6). But this large $\beta$ behaviour cannot be trusted because various $B_4$'s in the beta-transform (A.1) have been replaced by high sub-energy forms ( i.e., $B_4(-\tau,\alpha(s)) \to \Gamma(-\tau)(\alpha' s)^\tau$). Notice that the poles which come from the high $M^2$ part of the RR amplitude are explicit in the denominator of (A.7).

Since $\tilde{A}^{as}_{R_1 R_2^*}$ vanishes at $J = -1, -2, -3, \ldots$, we take the high $M^2$ form of the RR amplitude (A.4) to derive the following asymmetric FMSR having no fixed pole piece:

$$\int_0^{\bar{s}} dM^2 (M^2)^n (M_\perp^2)^{\alpha_{c,2}+1} \frac{1}{2i} \mathrm{Disc}_{M^2} A_{R_1 R_2}(M^2, M_\perp^2, t, t_1^\pm, t_2^\pm)$$

$$\cong \pi g^2 \frac{\Gamma(\alpha(t)+1)}{\Gamma(\alpha(t)-\alpha_{c,1})} \; \frac{\Gamma(\alpha(t)+1)}{\Gamma(\alpha(t)-\alpha_{c,2})} \; \frac{1}{\Gamma(\alpha(t)+1)} \; \frac{(\bar{s})^{\alpha(t)-\alpha_{c,1}+n}}{\alpha(t)-\alpha_{c,1}+n}, \qquad \text{(A.8)}$$

$$(n = 0, 1, 2, \ldots) \, .$$

This is a special case of Eq. (2.21), which is used throughout the paper.

To study the symmetric Mellin transform $\tilde{A}^{sy}_{R_1 R_2}(J,t,t_1^\pm,t_2^\pm)$ of Eq. (2.23), one replaces the integral in Eq. (A.6) by

$$\int_{-i\infty}^{+i\infty} \frac{d\beta(-1)^\beta}{2\pi i} \frac{\Gamma(-\beta+\ell+\ell')\,\Gamma(\beta-\alpha_{c,1}-1)\,\Gamma(\beta-\alpha_{c,2}-1)}{\Gamma(\beta+J-\alpha_{c,1}-\alpha_{c,2}-1+n)} \left\{ \frac{\Gamma(\beta-\alpha_{c,1}-\alpha_{c,2}-2+n)}{\Gamma(\beta-\alpha_{c,1}-\alpha_{c,2}-2)} \cdot g_m(\beta+J-\alpha_{c,1}-\alpha_{c,2}-1+n) \right\} . \qquad \text{(A.9)}$$

Unlike the asymmetric case, the $n \neq 0$ terms now pose a problem. After absorbing the polynomial in $\beta$ in the curly brackets of Eq. (A.9) into $\Gamma(-\beta + \ell + \ell')$ through use of (A.3), we find the term-by-term J-dependence of $\tilde{A}^{sy}_{R_1 R_2}$ is:

$$\frac{(J-1-\alpha_{c,1}-\alpha_{c,2}+\ell+\ell'+n)^{r'}\,\Gamma(J+1+n-\ell-\ell'-r)}{\Gamma(J-\alpha_{c,1}+n)\,\Gamma(J-\alpha_{c,2}+n)\,(J-\alpha(t)+n+k+m)} \tag{A.10}$$

where $0 \leqslant r' \leqslant m$ and $0 \leqslant r \leqslant m + n$. For $P_\perp = 0$ (i.e., $n = 0$) each term in the expansion of $\tilde{A}^{sy}_{R_1 R_1}$ has an explicit zero at $J = \alpha_{c,i} - s$ ($i = 1,2$; $s = 0,1,2,3 \ldots$). However, in general (i.e. $P_\perp \neq 0$) there are no term-by-term zeros for any value of J. Of course we cannot rule out a "magic" cancellation in resumming the various expansion coefficients.

## APPENDIX B

### Two-cluster term

Using the dual model as a guide, we establish the form of the two-cluster term for the RR amplitude. The expression is identical in form to Eq. (3.48) with one modification (see also Fig. 5): $s_3$ is limited not only by the NDC ($s_3 < s_{23}/\bar{s}$) but also by the cluster size cut-off ($s_3 < \bar{s}$). We find

$$
\begin{aligned}
A^{(2)}_{R_1R_3} &= \theta(s_{23}-\bar{s})(s_{23})^{-\alpha_{c,1}-\alpha_{c,3}-2}\int d\phi_2\, \eta_2\,(s_{23})^{\alpha_{c,2}} \cdot \\
&\int_0^{s} ds_2 \Big[\int_0^{\bar{s}} ds_3\, \theta(s_{23}-\bar{s}^2) + \int_0^{s_{23}/\bar{s}} ds_3\, \theta(\bar{s}^2-s_{23})\Big](s_2^{\perp})^{\alpha_{c,1}+1}(s_3^{\perp})^{\alpha_{c,3}+1} \cdot \\
&A_{R_1R_2}(s_2, s_2^{\perp}, t, t_1^{\pm}, t_2^{\pm})\, A_{R_2R_3}(s_3, s_3^{\perp}, t, t_2^{\pm}, t_3^{\pm}) \quad .
\end{aligned}
\tag{B.1}
$$

Now we use FMSR of the type (2.21). Note that although the upper limit of the $s_3$ integration in the second term of (B.1) may be small, we nevertheless use the FMSR to yield the leading behaviour.

$$
\begin{aligned}
A^{(2)}_{R_1R_3} &= \bar{\pi}\,\frac{g(t,t_1^{\pm})\,g(t,t_3^{\pm})}{\Gamma(\alpha(t)+1)}(s_{23})^{-\alpha_{c,1}-\alpha_{c,3}-2} \cdot \\
&\Big\{\theta(\bar{s}^2-s_{23})\,\theta(s_{23}-\bar{s})\,(s_{23})^{\alpha(t)}\, I(\alpha,t) + \\
&\theta(s_{23}-\bar{s}^2)\,\frac{\bar{\pi}}{\Gamma(\alpha(t)+1)}\int d\phi_2\,\eta_2\,\frac{g^2(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})^2}(s_{23})^{\alpha_{c,2}}(\bar{s})^{2\alpha(t)-2\alpha_{c,2}}\Big\} ,
\end{aligned}
\tag{B.2}
$$

where $I(\alpha,t)$ is defined in Eq. (3.14). Note that a condition for no cluster overlap is $\bar{s} > s_0 = 1/\alpha' \simeq 1$.

The symmetric Mellin transform of Eq. (2.23) becomes

$$
\begin{aligned}
\tilde{A}^{(2)sy}_{R_1R_3} &= \bar{\pi}\,\frac{g(t,t_1^{\pm})\,g(t,t_3^{\pm})}{\Gamma(\alpha(t)+1)}\Big\{\Big[\frac{\bar{s}^{\alpha(t)-J} - \bar{s}^{2(\alpha(t)-J)}}{J-\alpha(t)}\Big] I(\alpha,t) + \\
&\frac{\bar{\pi}}{\Gamma(\alpha(t)+1)}\,\bar{s}^{2(\alpha(t)-J)}\int d\phi_2\,\eta_2\,\frac{g^2(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})^2(J-\alpha_{c,2})}\Big\} ,
\end{aligned}
$$

which may be written (using the explicit form of $I(\alpha,t)$)

$$\tilde{A}^{(2)\,sy}_{R_1R_3} = \pi \frac{g(t,t_1^{\pm})\,g(t,t_3^{\pm})}{\Gamma(\alpha(t)+1)} \frac{(\bar{s})^{\alpha(t)-J}}{J-\alpha(t)} \; .$$

$$\frac{\pi}{\Gamma(\alpha(t)+1)} \int d\phi_2 \, \eta_2 \frac{g^2(t,t_2^{\pm})}{(\alpha(t)-\alpha_{c,2})(J-\alpha_{c,2})} \left\{ \frac{J-\alpha_{c,2}}{\alpha(t)-\alpha_{c,2}} - \bar{s}^{\,\alpha(t)-J} \right\} \quad \text{(B.3)}$$

We emphasize that this form of the two-cluster term depends crucially on analyticity via the FMSR and the NDC. It is not of the standard form

$$(\bar{s})^{\alpha(t)-J} \int d\phi_2 \, \eta_2 \, \tilde{F}_{R_1R_2}(J,t,t_1^{\pm},t_2^{\pm};\bar{s}) \frac{1}{J-\alpha_{c,2}} \tilde{F}_{R_2R_3}(J,t,t_2^{\pm},t_3^{\pm};\bar{s})$$

where $\tilde{F}_{RR}$ is some cut-off Mellin transform over a Reggeon-Reggeon amplitude.

To obtain the two-cluster term for Reggeon-particle or particle-particle scattering, replace the appropriate Reggeon leg(s) of (B.2) by particles. For example, to continue $R_3(t_3^{\pm})$ to particle a, replace $\alpha_{c,3} \to -1$ and $g(t,t_3^{\pm}) \to \gamma_{aa}'(t)$.

Table 1

Definition of Reggeon amplitudes

| | |
|---|---|
| $A_{ab\to a'b'} = \gamma_{aa'}(t)\,\gamma_{bb'}(t)\,\Gamma(-\alpha(t))\,(-s)^{\alpha(t)}$ | $\mathrm{Im}\,A_{ab\to a'b'} \xrightarrow[s\to\infty]{} \pi \dfrac{\gamma_{aa'}(t)\gamma_{bb'}(t)}{\Gamma(\alpha(t)+1)}\, s^{\alpha(t)}$ |
| $A_{abc\to a'b'c'} = \gamma_{bc}(t_1^-)\gamma_{b'c'}(t_1^+)\Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+)) \cdot (-s)^{\alpha(t_1^-)}(-s')^{\alpha(t_1^+)} A_{R(t_1^-)a\to R(t_1^+)a'}(M^2,t,t_1^\pm)$ | $\frac{1}{2i}\,\mathrm{Disc}_{M^2}\, A_{R(t_1^-)a\to R(t_1^+)a'} \xrightarrow[M^2\to\infty]{} \dfrac{\pi\,\gamma_{aa'}(t)\, g(t,t_1^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{\alpha(t)-\alpha_{c,1}-1}$ |
| $A_{abcd\to a'b'c'd'} = \gamma_{bd}(t_1^-)\gamma_{b'd'}(t_1^+)\gamma_{ac}(t_2^-)\gamma_{a'c'}(t_2^+) \cdot \Gamma(-\alpha(t_1^-))\Gamma(-\alpha(t_1^+))\Gamma(-\alpha(t_2^-))\Gamma(-\alpha(t_2^+)) \cdot (-s_1)^{\alpha(t_1^-)}(-s_1')^{\alpha(t_1^+)}(-s_2)^{\alpha(t_2^-)}(-s_2')^{\alpha(t_2^+)} \cdot A_{R(t_1^-)R(t_2^-)\to R(t_1^+)R(t_2^+)}(M^2,M_\perp^2,t,t_1^\pm,t_2^\pm)$ | $\frac{1}{2i}\,\mathrm{Disc}_{M^2}\, A_{R(t_1^-)R(t_2^-)\to R(t_1^+)R(t_2^+)} \xrightarrow[M^2\to\infty]{} \dfrac{\pi\, g(t,t_1^\pm)\, g(t,t_2^\pm)}{\Gamma(\alpha(t)+1)} (M^2)^{\alpha(t)-\alpha_{c,1}-\alpha_{c,2}-2}$ |

REFERENCES

1) G. Veneziano, Nucl.Phys. B74, 365 (1974), Phys. Letters 52B, 220 (1974); See also H.M. Chan, J.E. Paton and T.S. Tsun, Nucl. Phys. B86, 479 (1975).

2) R.G. Roberts in Proceedings of the 14th Scottish Universities Summer School in Physics, 1973, eds. R.L. Crawford and R. Jennings (Academic Press, 1974);
R.C. Brower, C.E. DeTar and J.H. Weis, Phys. Reports 14C, 257 (1974).

3) G. Veneziano, CERN preprint TH.2200 (1976).

4) For a review see, e.g. Dual Theory, collection of Physics Reports reprints (edited by M. Jacob).

5) For a review see, e.g. H.D.I. Abarbanel, J.D. Bronzan, R.L. Sugar and A.R. White, Phys. Reports 21C, 119 (1975).

6) G. 't Hooft, Nucl. Phys. B72, 461 (1974), Nucl. Phys. B75, 461 (1974);
C.G. Callan, N. Coote and D.J. Gross, Phys. Rev. D13, 1649 (1976).

7) G.F. Chew and C. Rosenzweig, Nucl. Phys. B104, 290 (1976);
Chan Hong-Mo, J. Kwiecinski and R.G. Roberts, Phys. Letters 60B, 367 (1976);
Chan Hong-Mo, K. Konishi, J. Kwiecinski and R.G. Roberts, Phys. Letters 60B, 469 (1976);
C. Schmid, D.M. Webber and C. Sorensen, ETH preprint (1976).

8) P. Aurenche, Ng Sing Wai, J.E. Paton, Tsou Sheung Tsun, Chan Hong-Mo, A. Gula, T. Inami and R.G. Roberts, Rutherford preprint RL-75-084 (1975);
N. Papadopoulos, C. Schmid, C. Sorensen and D.M. Webber, Nucl. Phys. B101, 189 (1975);
G.F. Chew and C. Rosenzweig, Phys. Letters 58B, 93 (1975);
B.R. Webber, Cavendish (Cambridge) preprint HEP 76/5 (1976);
Chan Hong-Mo, Ken-Ichi Konishi, J. Kwiecinski and R.G. Roberts, Rutherford preprint RL-76-056 (1976).

9) G. Marchesini and G. Veneziano, Phys. Letters 56B, 271 (1975);
M. Ciafaloni, G. Marchesini and G. Veneziano, Nucl. Phys. B98, 472, 493 (1975);
G. Marchesini, CERN preprint TH.2069 (1975).

10) C. Rosenzweig and G. Veneziano, Phys. Letters 52B, 335 (1974).

11) M.M. Schaap and G. Veneziano, Nuovo Cim. Letters 12, 204 (1975).

12) J. Kwiecinski, Nuovo Cim. Letters 3, 619 (1972);
M.B. Einhorn, J. Ellis and J. Finkelstein, Phys. Rev. D5, 2063 (1972);
M. Bishari, G.F. Chew and J. Koplik, Nucl. Phys. B72, 61 (1974).

13) M. Bishari and G. Veneziano, Phys. Letters 58B, 445 (1975).

14) M. Bishari, Phys. Letters 59B, 461 (1975).

15) J.R. Freeman and Y. Zarmi, to be published in Nucl. Phys. (1976).

16) J. Kwiecinski, Nucl. Phys. B97, 475 (1975).

17) See, e.g., G. Veneziano, Phys. Reports 9C, 199 (1974) and references therein.

18) P. Hoyer, N.A. Törnqvist and B.R. Webber, Phys. Letters 61B, 191 (1976).

19) J. Kwiecinski and N. Sakai, Nucl. Phys. B106, 44 (1976).

20) J. Finkelstein and J. Koplick, Columbia preprint CO-2271-72 (1975).

21) G.F. Chew, M.L. Goldberger and F.E. Low, Phys. Rev. Letters 22, 208 (1969);
I.G. Halliday, Nuovo Cim. 60, 177 (1969).

22) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cim. 26, 896 (1962).

23) S. Mandelstam, Nuovo Cim. 30, 1127, 1148 (1963);
J.C. Polkinghorne, Phys. Letters 4, 24 (1963)

24) For a review see M.L. Goldberger in Proceedings of the International School of Physics "Enrico Fermi" Course LIV, 1971, ed. R. Gatto (Academic Press, 1972).

25) G.F. Chew and C. Rosenzweig, Phys. Rev. D12, 3907 (1975);
N. Sakai, Nucl. Phys. B99, 167 (1975);
L.A.P. Balazs, Phys. Rev. D11, 1071 (1975).

## FIGURE CAPTIONS

Fig. 1 : Kinematics of six-point function and definition of Rp amplitude.

Fig. 2 : Kinematics of eight-point function and definition of RR amplitude. Solid lines - Beta transformed "energy" variables; dashed lines - untransformed variables.

Fig. 3 : Binning intermediate state particles into clusters according to asymmetric NDC (3.2).

Fig. 4 : Planar unitarity equation for six-pointed function.
a. Cluster assignment begins at Reggeon side.
b. Cluster assignment begins at particle leg.

Fig. 5 : Planar unitarity equation for eight-point function.

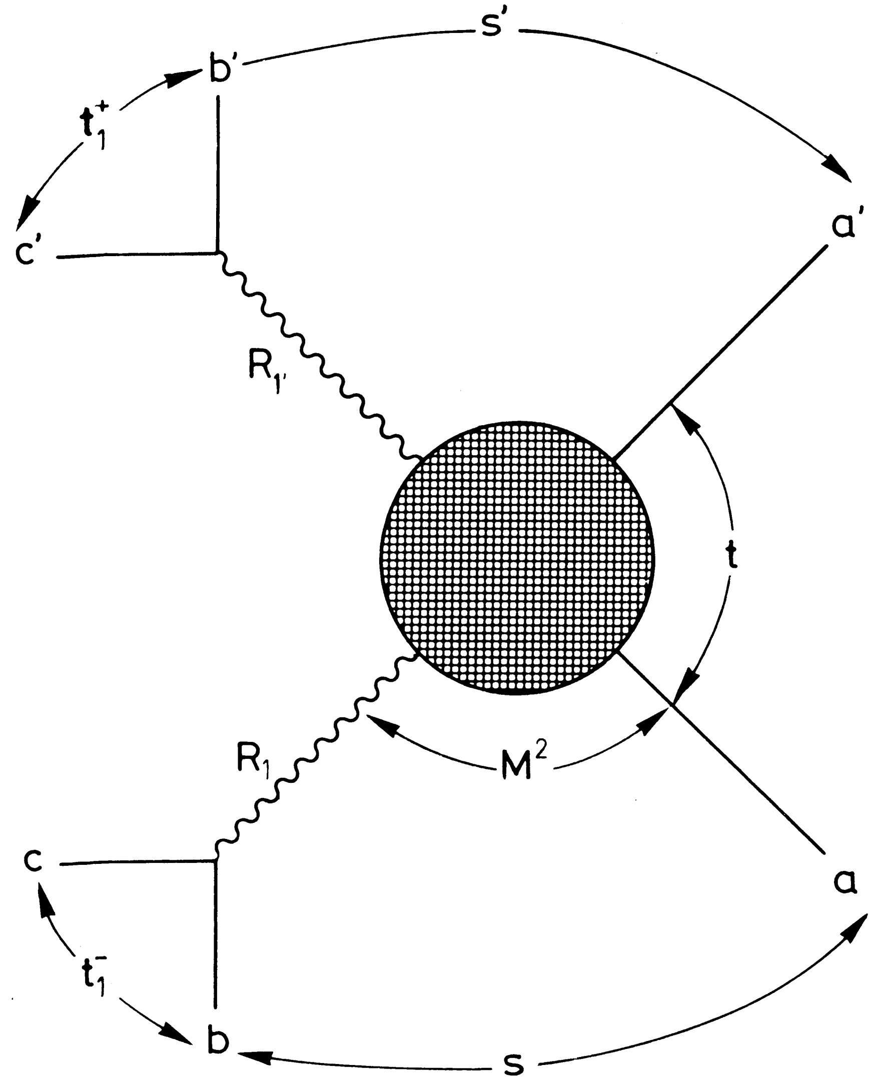


FIG. 1

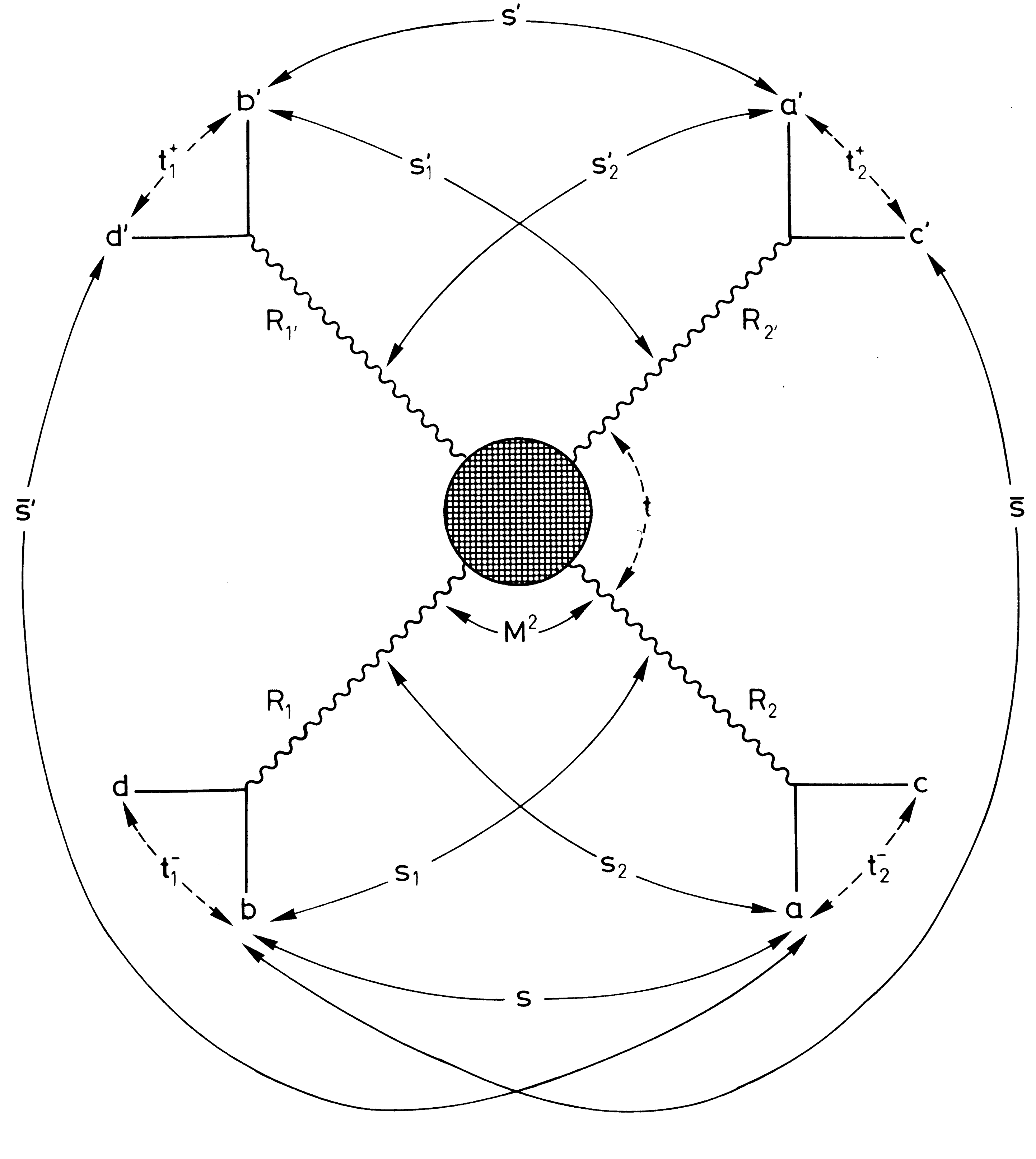


FIG. 2

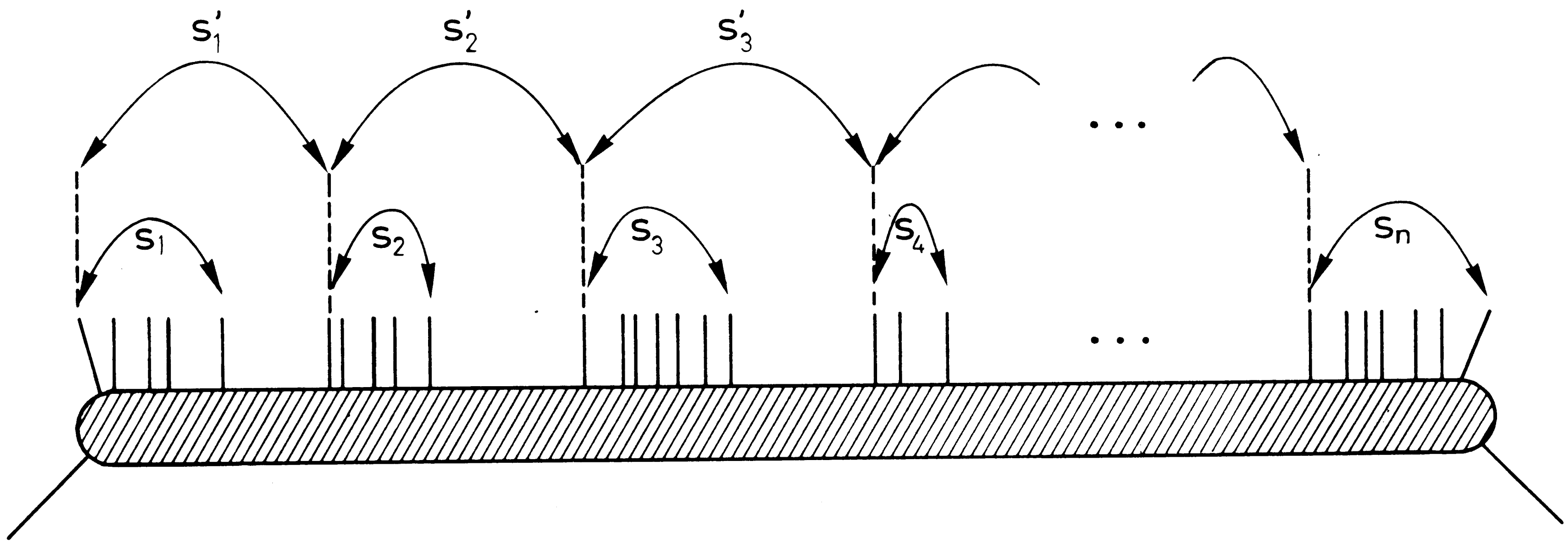


FIG. 3

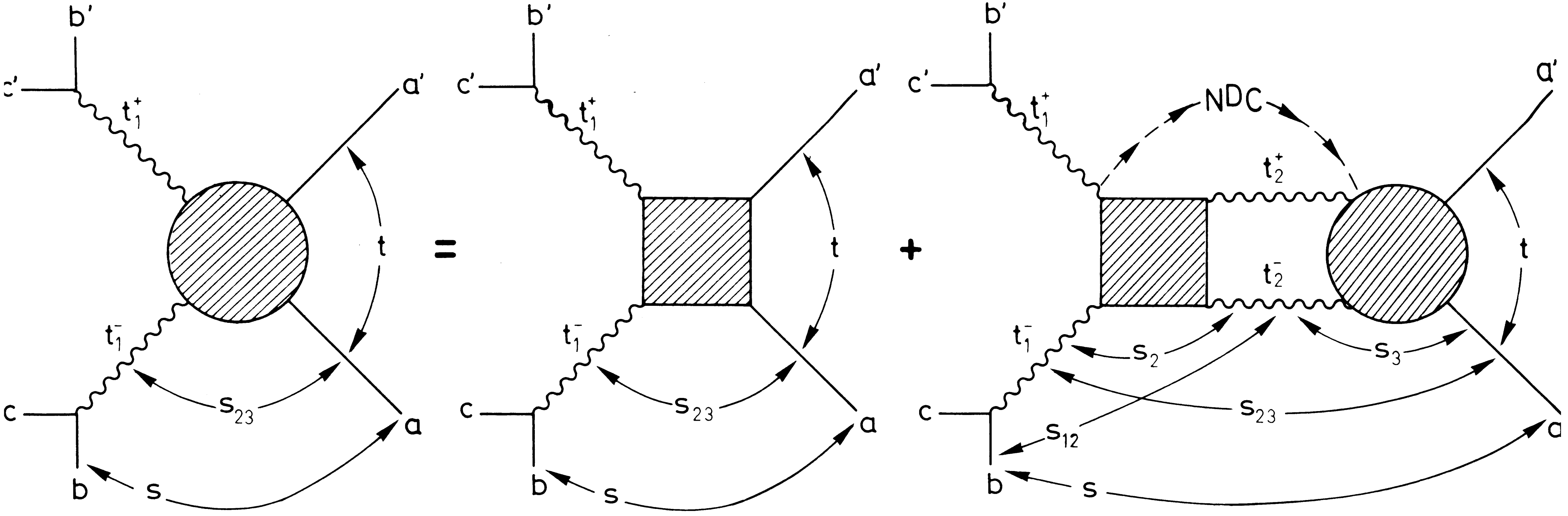


FIG. 4a

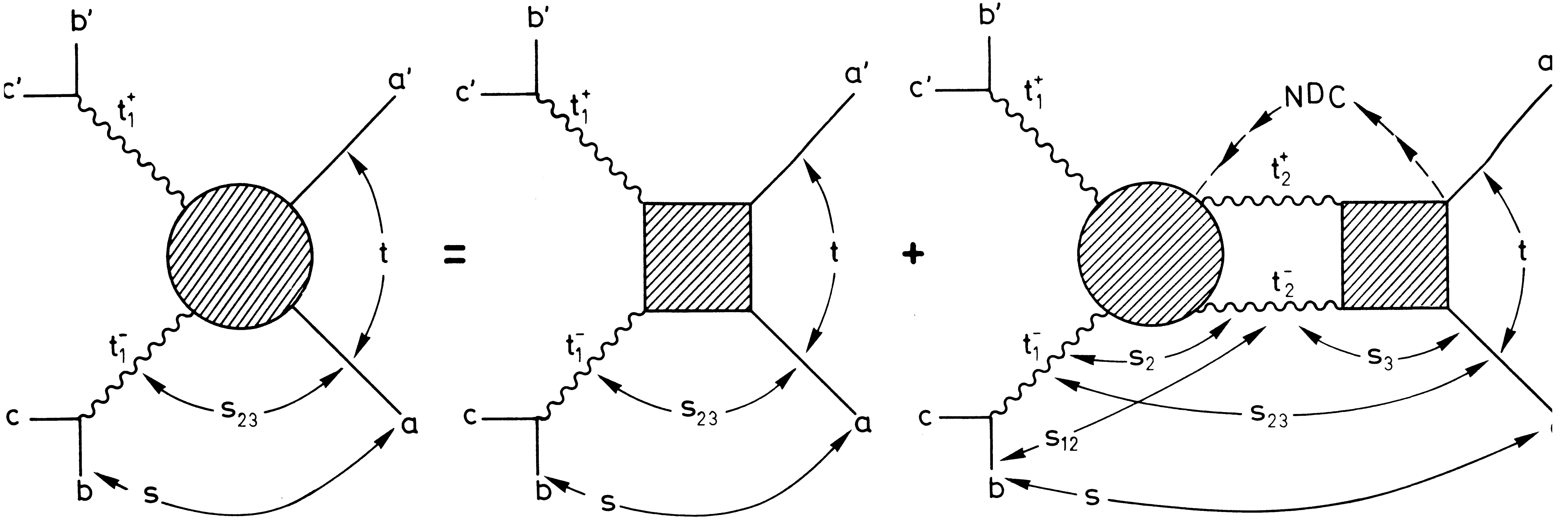


FIG. 4b

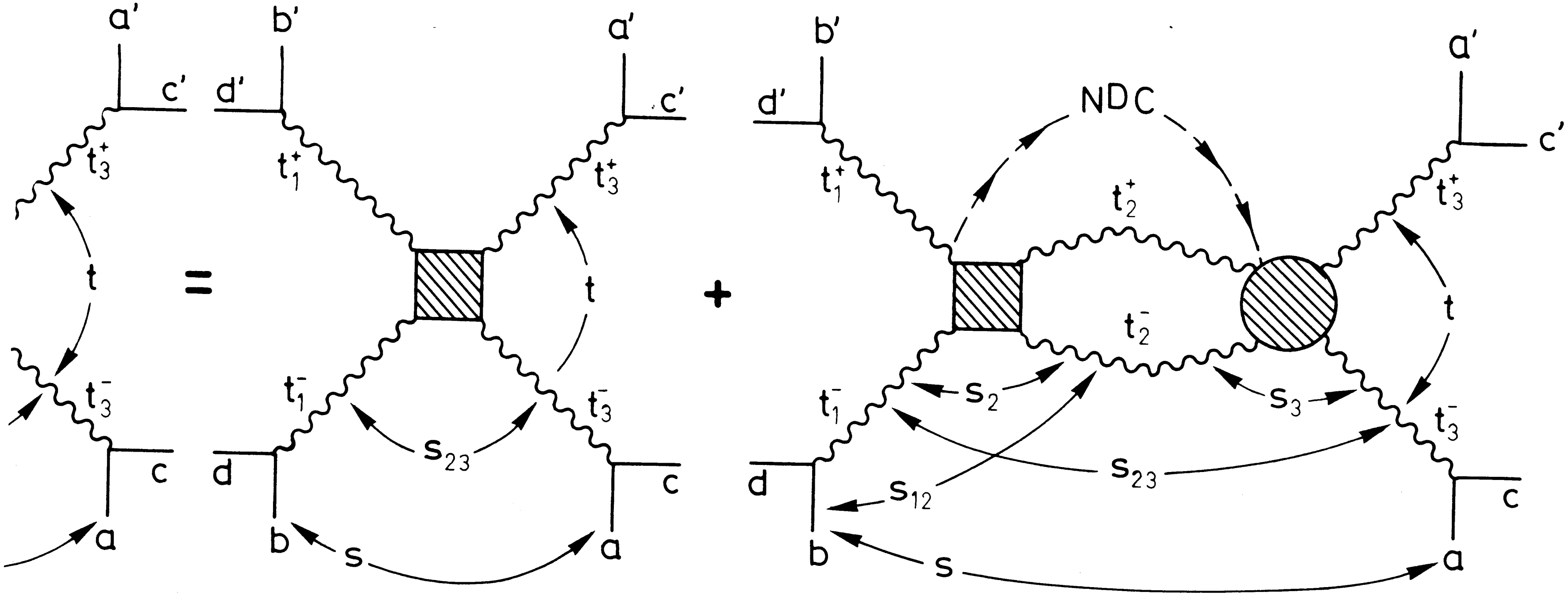


FIG. 5