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# General Principles of the Field Theory and Inclusive Reactions at High Energies

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## ABSTRACT

The authors have studied physical consequences for inclusive processes which follow from unitarity and analyticity over angular variables in a number of papers [1,2,10,13,23,24]. In the present paper we shall summarize the main results, obtained here and give detailed discussions of physical consequences which deal with the behaviour of differential cross sections for two-particle inclusive processes, quadratic multiplicity, pionization phenomenon, crossing inclusive processes at high energies, etc.

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$$\frac{d\,\sigma_{ab \to cd}}{d^3k_c\, d^3k_d} = \sum_j \frac{d\sigma_{ab \to c + d + B_j}}{d^3k_c\, d^3k_d}$$

$$d^3k = \frac{d\,\vec{k}}{(2\pi)^3\; 2E} \qquad (1.4)$$

The introduced differential cross sections are related to the cross sections of the corresponding processes via the relation:

$$\sigma_{ab \to c + A_j}(s) = \int d^3k_c \frac{d\sigma_{ab \to c + A_j}}{d^3k_c} \qquad (1.5)$$

$$\sigma_{ab \to c + d + B_j}(s) = \int d^3k_c\, d^3k_d \frac{d\sigma_{ab \to c + d + B_j}}{d^3k_c\, d^3k_d} \qquad (1.6)$$

iii) Distribution functions for inclusive processes (I) and (II) are

$$f_{ab\to c}(s,\vec{k}_c) = \sum_j n_c^j \frac{d\,\sigma_{ab\to c+A_j}}{d^3k_c} \qquad (1.7)$$

$$f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d) = \sum_j n_c^j\, n_d^j \frac{d\,\sigma_{ab\to c+d+B_j}}{d^3k_c\, d^3k_d} \qquad (1.8)$$

where $n_c^j$ is the number of particles of "c"-kind, produced in channel (III), and $n_c^j$ and $n_d^j$ are the numbers of particles of "c"-kind and "d"-kind, produced in channel (IV).

The distribution functions $f_{ab\to c}(s,\vec{k}_c)$ and $f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)$ are normalized in the following way:

$$\int f_{ab\to c}(s,\vec{k}_c)\, d^3k_c = \langle n_c \rangle\, \sigma_{ab\to c}(s) \qquad (1.9)$$

## 1. Physical Characteristics for Inclusive Processes

To describe the inclusive processes

$$a + b \to c + \sum_j A_j \qquad \text{(I)}$$

$$a + b \to c + d + \sum_j B_j \qquad \text{(II)}$$

where $A_j$ and $B_j$ are some fixed hadron groups, and $\sum_j$ denotes summing over all possible channels, it is possible to introduce the following characteristics:

i) The total cross section for inclusive process (I) is

$$\sigma_{ab \to c}^{(s)} = \sum_j \sigma_{ab \to c + A_j}^{(s)} \qquad (1.1)$$

and the total cross section for inclusive process (II) is

$$\sigma_{ab \to cd}^{(s)} = \sum_j \sigma_{ab \to c + d + B_j}^{(s)} \qquad (1.2)$$

where $\sigma_{ab \to c + A_j}^{(s)}$ and $\sigma_{ab \to c + d + B_j}^{(s)}$ are the total cross sections for the processes

$$a + b \to c + A_j \qquad \text{(III)}$$

$$a + b \to c + d + B_j \qquad \text{(IV)}$$

respectively.

ii) Differential cross sections for inclusive processes (I) and (II) are

$$\frac{d\,\sigma_{ab \to c}}{d^3 K_c} = \sum_j \frac{d\,\sigma_{ab \to c + A_j}}{d^3 K_c} \qquad (1.3)$$

where

$$\sigma_{ab\to cd}(s,V_i) = \int_{V_i} \frac{d\sigma_{ab\to cd}}{d^3k_c\, d^3k_d}\, d^3k_c\, d^3k_d \tag{1.13}$$

and

$$\langle n_c n_d\rangle_{V_i} = \frac{1}{\sigma_{ab\to cd}(s,V_i)} \int_{V_i} f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)\, d^3k_c\, d^3k_d \ . \tag{1.14}$$

From (1.12) it is clear that

$$\langle n_c n_d\rangle = \sum_{i=1}^{N} \langle n_c n_d\rangle_{V_i} \frac{\sigma_{ab\to cd}(s,V_i)}{\sigma_{ab\to cd}(s)} \ . \tag{1.15}$$

Subdomains $V_i$ of the phase volume should be chosen from physical reasons (see sections 4 and 6) which would correspond to the given mechanism of particle generation in inclusive process.

It should be noted that the behaviour of average $\langle n_c n_d\rangle$ and $\langle n_c n_d\rangle_{V_i}$ may greatly differ from each other. As generally speaking the processes with small multiplicity but large production cross section, as well as those with large multiplicity and small production cross section, may make contributions to sum (1.15).

$$\int f_{ab \to cd}(s,\vec{k}_c,\vec{k}_d)\, d^3k_c\, d^3k_d = \langle n_c n_d \rangle\, \sigma_{ab \to cd}(s) \tag{1.10}$$

where $\langle n_c \rangle$ is the average value of multiplicity for the particles of "c"-kind in reaction (I), and $\langle n_c n_d \rangle$ is the average value of the multiplicity product for the particles of "c"- and "d"-kind in reaction (II). It should be noted that if in reaction (II) detected particles are of the same kind, i.e. if $c = d$, then in formulae (1.8) and (1.10) the following change $n_c n_d \to n_c(n_c - 1)$ is to be made.

Due to the fact that particle generation in reactions (I) and (II) may develop in different parts of phase space in different ways (e.g., via different production mechanisms: pionization, diffraction dissociation, etc.), then one may expect that the distribution function in different parts of the phase space will be of different asymptotic behaviour. When studying such a characteristic as average multiplicity (when integration is performed through the whole phase space of the detected particles), the effects, related to different mechanisms of particle generation, are smeared out. If one wants to observe these effects it would be useful to introduce average multiplicity of particles into the subspaces of the phase volume. Such a characteristic of particle generation may give a more detailed information on particle production in inclusive processes.

By the example of inclusive process (II) we will give a definition of the new characteristic [1,2] average multiplicity in some subdomain $V_i$ of a two-particle phase volume.

A full two-particle volume for particles "c" and "d" in reaction (II) $V_{cd}$ we will represent as a sum of subdomains $V_i$

$$V_{cd} = \sum_{i=1}^{N} V_i \tag{1.11}$$

Then from (1.10) we find that

$$\langle n_c n_d \rangle\, \sigma_{ab\to cd}(s) = \sum_{i=1}^{N} \int_{V_i} d^3k_c\, d^3k_d\, f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d) = \sum_{i=1}^{N} \langle n_c n_d \rangle_{V_i}\, \sigma_{ab\to cd}(s,V_i) \tag{1.12}$$

$$T_{ab\to c+d+B_j}(s,\vec{n},\xi) = \int_{X_L}^{\infty} dx \int d\vec{u}\ \frac{\Psi(x,\vec{u},s,\xi)}{x - (\vec{n}\cdot\vec{u})} \tag{2.1}$$

Here $\vec{n}$ is a unit vector along the momentum $\vec{p}_a$. In the co-ordinate system, where momentum $\vec{k}_c$ is directed along the axis $z$, and the momentum $\vec{k}_d$ lies in the plane $xoz$, $\vec{n}$ is of the form $\vec{n} = (\sin\Theta\cos\varphi, \sin\Theta\sin\varphi, \cos\Theta)$ and $\vec{u}$ is an arbitrary unit vector, and $X_L$ is the semi-major axis of the Lehmann ellipse.

$$X_L = \left[1 + \frac{(m_1^2 - m_a^2)(m_2^2 - m_b^2)}{\vec{p}_a^{\,2}\left[s - (m_1 - m_2)^2\right]}\right]^{\frac{1}{2}} \tag{2.2}$$

In (2.2) $\vec{p}_a$ is a 3-momentum of "a"-particle in the c.m.s. From (2.1) it follows that the amplitude $T_{ab\to c+d+B_j}(s,\vec{n},\xi)$ is analytical over variables $z = \cos\Theta$ and $\omega = e^{i\varphi}$ in the domain determined by the condition

$$(|1 + z||\omega| + |1 - z|\frac{1}{|\omega|})(|1 + z|\frac{1}{|\omega|} + |1 - z||\omega|) < 4\, X_L^2 \tag{2.3}$$

with exception of the point $z$, which belongs to the segments $[-X_L, -1]$ and $[1, X_L]$.

From (2.3) it follows that at physical values of the variable $\omega$ the amplitude $T_{ab\to c+d+B_j}(s,\vec{n},\xi)$ will be analytical over $z$ in the domain (2.4)

$$|1 + z| + |1 - z| < 2\, X_L \tag{2.4}$$

excluding point $z$, belonging to the segments $[-X_L, -1]$ and $[1, X_L]$.

From (2.3) it also follows that for physical values the variable $z = \cos\Theta$ the domain of analyticity over the variable $\omega$ is not less than

$$r_L^- < |\omega| < r_L^+ \tag{2.5}$$

## 2. Analyticity of Inelastic Process Amplitudes and Inclusive Cross Sections over Angular Variables

In this section we will present the results dealing with analytical properties of inelastic process amplitudes and differential cross sections over angular variables, which follow from the general principles of theory.

The importance of studying the analytical properties of the amplitude over angular variables has been noted by N.N. Bogolubov [3)]. He was the first to study the analytical properties of the imaginary part of the elastic amplitude in momentum transfer (see also ref. 4).

Further steps in studying elastic and binary processes were made by Lehmann [5)], Bross, Epstein, Glaser [6)], Martin [7)] and Sommer [8)].

Analytical properties of the amplitudes for the processes of the type $a + b \to c + d + e$ over angular variables were considered on the basis of Dyson representation in paper [9)], a more detailed study of this problem for inelastic and inclusive processes is given in papers [2,10-13)].

Following papers [2,10-13)], we shall consider the process

$$a + b \to c + d + B_j \qquad \text{(IV)}$$

where, as before, $B_j$ stands for a hadron group.

In the c.m.s. we will introduce the variables: $s = (p_a + p_b)^2$ is the system total energy squared; $\cos\Theta$ where $\Theta$ is the angle between momenta $\vec{p}_a$ and $\vec{k}_c$; $\omega = e^{i\varphi}$ where $\varphi$ is the angle between the planes $(\vec{p}_a, \vec{k}_c)$ and $(\vec{k}_c, \vec{k}_d)$; $\xi$ is a set of variables, which define the relative configuration of particle momenta in the final state.

With Dyson representation for the amplitude of process IV we can write down the following representation therefore, taking into account analyticity of the imaginary part of the amplitude for elastic scattering

$$a + b \to a + b$$

over the variable $z = \cos\Theta$ in the Martin ellipse, we find (see [2,13])

$$\sum_{m=-\ell}^{m=\ell} \int d\Gamma_j \, |T_\ell^m(s,\xi,j)|^2 \leqslant \frac{R(s)}{\sqrt{\ell}} \exp\{-\ell \, \ln(X_M + \sqrt{X_M^2 - 1})\} \qquad (2.11)$$

where $R(s)$ is degree of increase of the amplitude imaginary part; and $X_M$ is the semi-major axis of the Martin ellipse and equal to [7,8)]

$$X_M = 1 + \frac{2m_\pi^2}{|\vec{p}_a|^2}$$

Using the estimation for $d^\ell_{m,n}$ $(\mathrm{Re}\Theta + i\,\mathrm{Im}\Theta)$ on the ellipse with the semi-major axis $X = \mathrm{ch}\,\mathrm{Im}\Theta$

$$|d^\ell_{m,n}(\mathrm{Re}\Theta + i\,\mathrm{Im}\Theta)| \leqslant (\mathrm{ch}\,\mathrm{Im}\Theta + \sqrt{\mathrm{ch}^2\mathrm{Im}\Theta - 1})^\ell$$

and Bunyakovsky-Schwartz inequality, we will find out that series (2.9) converges absolutely in the domain, where

$$\frac{X + \sqrt{X^2 - 1}}{(X_M + \sqrt{X_M^2 - 1})^{\frac{1}{2}}} < 1 \qquad (2.12)$$

Whereof we find that differential cross section $\dfrac{d\sigma_{ab\to c+d+B_j}}{d\cos\Theta}$ as the function of the complex variable $\cos\Theta$ is analytical in the ellipse with foci at the points $\pm I$ and semi-major

$$X = \sqrt{\frac{X_M + 1}{2}} \qquad (2.13)$$

It should also be noted that the domain where the differential cross section of the form where

$$r_L^{\pm} = X_L \pm \sqrt{X_L^2 - 1} \tag{2.6}$$

As representation (2.1) is also valid for $T^*_{ab\to c+d+B_j}(s,\vec{n},\xi)$, then it will be an analytic function over the variables $z$ and $\omega$ in the domain (2.3).

It may be shown that analytical properties of the amplitude lead to analyticity of differential cross section of the form

$$\frac{d\sigma_{ab\to c+d+B_j}}{d\cos\Theta} \tag{2.7}$$

over the variable $z = \cos\Theta$ in the Lehmann ellipse (see [2,9,10]).

Indeed, as it was shown in papers [2,10-13], differential cross section (2.7) as the function of the complex variable $z = \cos\Theta$ is analytic in a wider domain.

Indeed, in decomposing the amplitude over the Wigner functions

$$T_{ab\to c+d+B_j}(s,\cos\Theta,e^{i\varphi},\xi) = \left(2\frac{\sqrt{s}}{|\vec{p}_a|}\right)^{\frac{1}{2}} \sum_{\ell=o}^{\infty} \sum_{m=-\ell}^{\ell} (2\ell+1)\, T_\ell^m(s,\xi,j)\, d^\ell_{m,o}(\Theta)\, e^{im\varphi} \tag{2.8}$$

we can write down differential cross section (2.7) in the form

$$\frac{d\sigma_{ab\to c+d+B_j}}{d\cos\Theta} = \frac{2\pi}{\vec{p}_a^{\,2}} \sum_{\ell,\ell'} \sum_{m} (2\ell+1)(2\ell'+1) d^\ell_{m,o}(\Theta) d^{\ell'}_{m,o}(\Theta) \int d\Gamma_j\, T^{*m}_{\ell'}(s,\xi,j)\, T^m_\ell(s,\xi,j) \tag{2.9}$$

The magnitudes $T_\ell^m(s,\xi,j)$ and $d\Gamma_j$ enter the unitarity conditions for partial amplitudes in the following way

$$\mathrm{Im}\, f_\ell(s) = |f_\ell(s)|^2 + \sum_j \sum_{m=-\ell}^{m=\ell} \int d\,\Gamma_j |T_\ell^m(s,\xi,j)|^2 + \ldots \tag{2.10}$$

$$\frac{d\sigma_{ab\to cd}}{d\cos\Theta\, d\varphi} = \frac{1}{|\vec{p}_a|^2} \sum_{\ell,m} \sum_{\ell',m'} (2\ell+1)(2\ell'+1)\, d^{\ell}_{m,o}(\Theta)\, d^{\ell'}_{m',o}(\Theta)\, \cdot$$

$$\cdot e^{i(m-m')\varphi} \cdot \sum_j \int d\Gamma_j\, \overset{*}{T}{}^{m'}_{\ell'}(s,\xi,j)\, T^{m}_{\ell}(s,\xi,j) \qquad (2.16)$$

From unitarity condition (2.10) and Bunyakovsky-Schwartz inequality, as well as from analyticity of the imaginary part of elastic amplitude over $\cos\Theta$ in the Martin ellipse, it follows [2,12,13] that series (2.16) converges in the domain

$$\frac{1}{2}\left(|\omega| + \frac{1}{|\omega|}\right)\left(|1 - z| + |1 + z|\right) < 2X \qquad (2.17)$$

therefore the function $\frac{d\sigma_{ab\to cd}}{d\cos\Theta d\varphi}$ is analytic over $z$ and $\omega$ in domain (2.17) with exception for the points of the segments $[-X, -1]$ and $[1, X]$ over the variable $z$.

In conclusion, we would like to note that the analyticity properties and estimations for inelastic and inclusive cross sections obtained above, are strict consequences of the main postulate of the field theory. The analyticity domains, obtained on the basis of the main principles of theory, possess one common feature, i.e., with energy growth they shrink ($X_L \to 1$, $r^{\pm}_L \to 1$, $X_M \to 1$, $X \to 1$) and at $s \to \infty$ they degenerate into "domains", consisting of physical points only.

One should pay attention to the following fact. The found domains of analyticity over $z$ and $\omega$ (2.13) and (2.17) were obtained with the help of the series and therefore domains (2.13) and (2.17) were fully determined by nearest singularities ("thresholds") in t-channel. Of course, these domains are not the natural domains of holomorphicity for the functions $\frac{d\sigma_{ab\to cd}}{d\cos\Theta}$ and $\frac{d\sigma_{ab\to cd}}{d\cos\Theta\, d\varphi}$, and one may hope that a more detailed study will allow one to exhibit an existence of analyticity domains independent of $s$ in the neighbourhood of the physical points in the planes $z$ and $\omega$.

$$\frac{d\sigma_{ab\to cd}}{d\cos\Theta} = \sum_j \frac{d\sigma_{ab\to c+d+B_j}}{d\cos\Theta}$$

is analytical over the variable $z = \cos\Theta$ , is also an ellipse with semi-major (2.13). This statement is proved similarly, if one takes into account that on the basis of unitarity condition (2.3) there takes place an inequality

$$\sum_j \sum_{m=-\ell}^{m=\ell} \int d\Gamma_j |T_\ell^m(s,\xi,j)|^2 \leqslant \frac{R(s)}{\sqrt{\ell}} \exp\{-\ell \,\ell n(X_M + \sqrt{X_M^2 - 1})\} \qquad (2.11a)$$

Here one should pay attention to a very important fact that the obtained analyticity of inelastic cross sections leads to the fact that position of real singularity over momentum transfer on the contrary with the case of analyticity in the Lehmann ellipse does not depend on $s$ and starts from point $t = m_\pi^2$ .

Let us enlist here some estimations for differential cross sections. These estimations are consequences of the analyticity over the variable $z = \cos\Theta$ proved above.

$$\left.\frac{d\sigma_{ab\to c+d+B_j}}{d\cos\Theta}\right|_{\Theta=o} \leqslant \frac{s}{16\, m_\pi^2}\, \sigma_{ab\to c+d+B_j}(s)\ \ell n^2 R(s) , \qquad (2.14)$$

$$\left.\frac{d\sigma_{ab\to cd}}{d\cos\Theta}\right|_{\Theta=o} \leqslant \frac{s}{16\, m_\pi^2}\, \sigma_{ab\to cd}(s)\ \ell n^2 R(s). \qquad (2.15)$$

It should be mentioned that similar inequality was obtained in [7,14,15] for elastic processes.

We will proceed now to study analytical properties over angular variables for differential cross section of inclusive process (II) of the form $\frac{d\sigma_{ab\to cd}}{d\cos\Theta\, d\varphi}$ .

Using decomposition (2.8), it is easy to obtain that

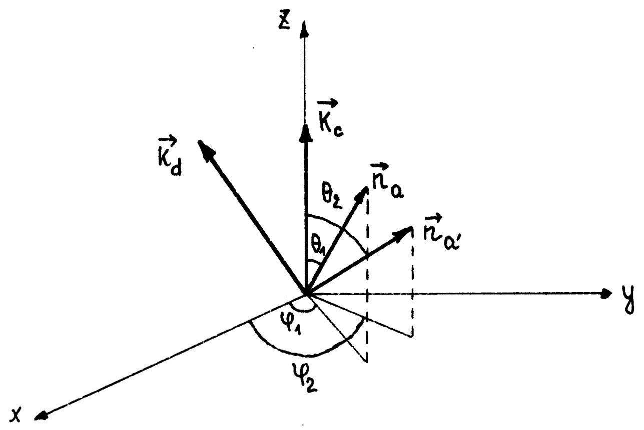


Fig. 1

It is obvious that

$$\frac{d\sigma_{ab\to cd}}{d\cos\Theta\, d\varphi} = \frac{1}{2\sqrt{s}\,|\vec{p}_a|}\,\Phi(s,z_1,\omega_1;\ z_2,\omega_2)\Bigg|_{\substack{z_1=z_2=\cos\Theta\\ \omega_1=\omega_2=e^{i\varphi}}} \tag{3.3}$$

It should be noted that the contribution of inclusive process (II) to the imaginary part of elastic process is expressed through the function $\Phi(s,z_1,\omega_1;\ z_2,\omega_2)$ in the following way

$$\mathrm{Im}\, T^{(II)}_{ab\to ab}(s,\vec{n}_a,\vec{n}_a') = \frac{1}{8\pi}\int \frac{\Phi(s,z_1,\omega_1;\ z_2,\omega_2)}{\sqrt{1-z_1^2-z_2^2-z^2+2z_1z_2z}}\,\frac{d\omega_1}{i\omega_1}\,dz_1dz_2 \tag{3.4}$$

where $z = (\vec{n}_a.\vec{n}_a')$.

The variable $\omega_2$ is the function of the variables $z,z_1,z_2,\omega_1$ and is found from the equation

$$z = z_1z_2 + \frac{1}{2}\sqrt{1-z_1^2}\,\sqrt{1-z_2^2}\,\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right)$$

## 3. The Upper Bound for Decrease of Differential Cross Section for Inclusive Processes in Large Angle Region

In this section we shall put forward a hypothesis on the domains of analyticity over angular variables and find the upper bound for decrease of differential cross section for inclusive processes in the region of large angles.

It is more convenient to formulate our assumption about analyticity for the functions

$$\Phi(s,z_1,\omega_1;\ z_2,\omega_2) = 2\,\frac{\sqrt{s}}{|\vec{p}_a|}\sum_{m_1=-\infty}^{\infty}\omega_1^{m_1}\sum_{m_2=-\infty}^{\infty}\omega_2^{-m_2}\ .$$

$$\cdot\sum_{\ell_1=|m_1|}^{\infty}(2\ell_1+1)\,d^{\ell_1}_{m_1,o}(z_1)\sum_{\ell_2=|m_2|}^{\infty}(2\ell_2+1)\,d^{\ell_2}_{m_2,o}(z_2)\sum_j C^{m_1,m_2}_{\ell_1,\ell_2}(s,j) \qquad (3.1)$$

where

$$C^{m_1,m_2}_{\ell_1,\ell_2}(s,j) = \int d\Gamma_j\ \overset{*}{T}{}^{m_2}_{\ell_2}(s,\xi,j)\ T^{m_1}_{\ell_1}(s,\xi,j) \qquad (3.2)$$

From the unitarity condition and Bunyakovski-Schwartz inequality it follows that

$$\left|C^{m_1,m_2}_{\ell_1,\ell_2}(s,j)\right| \leqslant \sqrt{\operatorname{Im} f_{\ell_1}(s)\ \operatorname{Im} f_{\ell_2}(s)} \qquad (3.2a)$$

In (3.1) the variables are equal to $z_1 = \cos\Theta_1$, $\omega_1 = e^{i\varphi_1}$, $z_2 = \cos\Theta_2$, $\omega_2 = e^{i\varphi_2}$ where $(\Theta_1,\varphi_1)$ and $(\Theta_2,\varphi_2)$ are spherical co-ordinates of unit vectors $\vec{n}_a = \frac{\vec{p}_a}{|\vec{p}_a|}$ and $\vec{n}'_a = \frac{\vec{p}'_a}{|\vec{p}'_a|}$ respectively, in the co-ordinate system constructed on the vectors $\vec{k}_c$ and $\vec{k}_d$ (see Fig. 1).

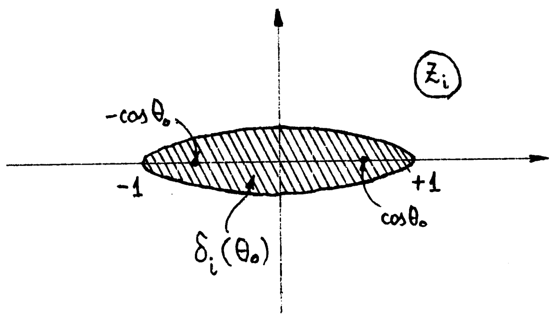


Fig. 2

The union of the domains $\partial_i$ and $\delta_i(\Theta_o)$ will be designated by $D_i$.

Similarly, let $g_1(\varphi_o)$ and $g_2(\varphi_o)$ be some s-independent neighbourhoods of the points $\omega_1 = e^{i\varphi_1}$ and $\omega_2 = e^{i\varphi_2}$ respectively; where $\varphi_o \leqslant \varphi_i \leqslant \pi-\varphi_o$; $\pi+\varphi_o \leqslant \varphi_i \leqslant 2\pi-\varphi_o$ $(i=1,2)$, $(\varphi_o = \mathrm{const} \neq 0,\pi,2\pi)$.

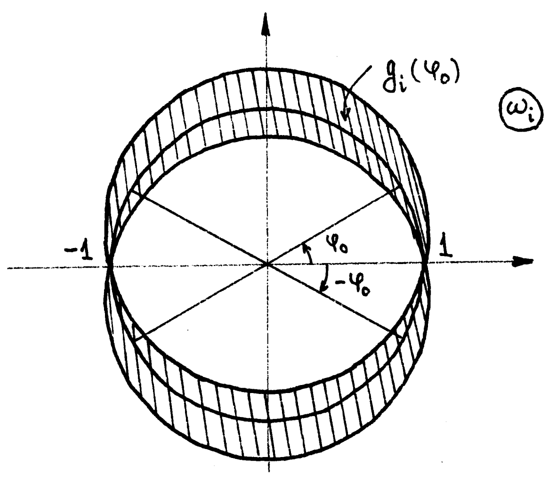


Fig. 3

The union of the domains $h_i$ and $g_i(\varphi_o)$ will be denoted by $G_i$.

For the function $\Phi(s,z_1,\omega_1;\ z_2,\omega_2)$ general principles of field theory give the following polydomain of analyticity over variables $z_1$, $\omega_1$, $z_2$, $\omega_2$ :

$$\Delta_i\ :\ \left\{\frac{1}{2}\left(|\omega_i| + \frac{1}{|\omega_i|}\right)\left(|1-z_i| + |1+z_i|\right) < 2x\right\}\quad i=1,2 \tag{3.5}$$

with the exception of the points $z_i \in [-x,-1]$ and $z_i \in [1,x]$. This statement is easily proved by finding the domain of convergence of series (3.1) on the basis of inequality (2.11a).

From (3.5) it follows that at physical values of $\omega_i$ (i=1,2) the function $\Phi(s,z_1,e^{i\varphi_1};\ z_2,e^{i\varphi_2})$ is analytical over $z_1$ and $z_2$ in the product of the domains of the type

$$\partial_i\ :\ \left\{|1-z_i| + |1+z_i| < 2x\right\}\quad i=1,2 \tag{3.6}$$

Similarly, at physical values of $z_1 = \cos\Theta_1$ and $z_2 = \cos\Theta_2$ the function $\Phi(s,\cos\Theta_1,\omega_1;\ \cos\Theta_2,\omega_2)$ over the variables $\omega_1$ and $\omega_2$ is analytic in the product of the domains of the form

$$h_i\ :\ \left\{\frac{1}{2}\left(|\omega_i| + \frac{1}{|\omega_i|}\right) < x\right\}\quad i=1,2. \tag{3.7}$$

The function $\Phi(s,z_1,\omega_1;\ z_2,\omega_2)$ which is analytic in polydomain of the form $\Delta_i$ (i=1,2) with coefficients of decomposition (3.2) and polynomially bounded in $S$ in $\Delta_1 \otimes \Delta_2$ is designated by H. Moreover, the H-class is called a physical one.

It should be noted that requirements of the basic postulates of theory (relativistic invariance, unitarity, causality and polynomial boundedness) lead to functions belonging to H-class.

Let $\delta_1(\Theta_o)$ and $\delta_2(\Theta_o)$ be some s-independent fixed neighbourhood of $-\cos\Theta_o \leqslant z_1 \leqslant \cos\Theta_o$ and $-\cos\Theta_o \leqslant z_2 \leqslant \cos\Theta_o$ ($\Theta_o$ = const $\neq 0,\pi$) respectively (see Fig. 2 ).

where n is an arbitrarily large, but fixed number, then for large $S$ we have

$$\left|F(s,\cos\Theta,e^{i\varphi};\ \cos\Theta,e^{i\varphi})\right|_{\substack{\Theta_0 \leq \Theta \leq \pi-\Theta_0 \\ \varphi_0 \leq \varphi \leq \pi-\varphi_0 \\ \pi+\varphi_0 \leq \varphi \leq 2\pi-\varphi_0}} \leq \text{const}\ \frac{\ln^4(\frac{s}{s_o})\ \ln^6\ln(\frac{s}{s_o})}{\sin^4\Theta\ \sin^6\varphi} \qquad (3.12)$$

<u>If we assume</u> that the function $\Phi(s,z_1,\omega_1;\ z_2,\omega_2)$ introduced above, which belongs to the physical class H, satisfies the conditions a) and b) of Theorem I, then for the differential cross section of the inclusive process II we have

$$\left.\frac{d\sigma_{ab\to cd}}{d\cos\Theta\ d\varphi}\right|_{\substack{\Theta_0 \leq \Theta \leq \pi-\Theta_0 \\ \varphi_0 \leq \varphi \leq \pi-\varphi_0 \\ \pi+\varphi_0 \leq \varphi \leq 2\pi-\varphi_0}} \leq \frac{\text{const}}{S}\ \frac{\ln^9(\frac{s}{s_o})}{\sin^4\Theta\ \sin^6\varphi} \qquad (3.13)$$

Similarly, <u>if we assume</u> that the function $\Phi(s,z_1,\omega_1;\ z_2,\omega_2)$, belonging to class H, satisfies the conditions of Theorem II, then for the differential cross section of the inclusive process II we find

$$\left.\frac{d\sigma_{ab\to cd}}{d\cos\Theta\ d\varphi}\right|_{\substack{\Theta_0 \leq \Theta \leq \pi-\Theta_0 \\ \varphi_0 \leq \varphi \leq \pi-\varphi_0 \\ \pi+\varphi_0 \leq \varphi \leq 2\pi-\varphi_0}} \leq \frac{\text{const}}{S}\ \frac{\ln^4(\frac{s}{s_o})\ \ln^6\ln(\frac{s}{s_o})}{\sin^4\Theta\ \sin^6\varphi} \qquad (3.14)$$

Note that analyticity of the function $\Phi(s,z_1,\omega_1;\ z_2,\omega_2)$ in the neighbourhood of the physical points $\delta_i(\Theta_o)$ and $g_i(\varphi_o)$ (i=1,2) is just a hypothesis. We do not know whether this assumption follows from the main postulates of theory. On the other hand, this hypothesis leads to certain physical consequences which can be checked experimentally.

The following theorem can be proved by the method developed in [2].

<u>Theorem I</u> : If the function $F(s,z_1,\omega_1;\ z_2,\omega_2)$ in the domain $\Delta_1 \otimes \Delta_2$ allows an expansion of the form (3.1) with the coefficients of the expansion of modulus less than 1 and satisfies the conditions:

a) For the points $|\omega_i| = 1$ (i=1,2) it is analytic in $z_1$ and $z_2$ in the polydomain $D_1 \otimes D_2$ and polynomially bounded in it over $S$.

$$\left. |F(s,z_1,e^{i\varphi_1};\ z_2,e^{i\varphi_2})| \right|_{z_1,z_2 \in D_1 \otimes D_2} \leqslant \text{const}\ \left(\frac{s}{s_o}\right)^{N_1} \qquad (3.8)$$

b) For the points $-1 \leqslant z_i \leqslant 1$ (i=1,2) it is analytical in $\omega_1$ and $\omega_2$ in the polydomain $G_1 \otimes G_2$ and polynomially bounded over $S$ in it

$$\left. |F(s,\cos\Theta_1,\omega_1;\ \cos\Theta_2,\omega_2)| \right|_{\omega_1,\omega_2 \in G_1 \otimes G_2} \leqslant \text{const}\ \left(\frac{s}{s_o}\right)^{N_2} \qquad (3.9)$$

then for large $S$ the estimation

$$\left. |F(s,\cos\Theta,e^{i\varphi};\ \cos\Theta,e^{i\varphi})| \right|_{\substack{\Theta_o \leqslant \Theta \leqslant \pi-\Theta_o \\ \varphi_o \leqslant \varphi \leqslant \pi-\varphi_o \\ \pi+\varphi_o \leqslant \varphi \leqslant 2\pi-\varphi_o}} \leqslant \frac{\text{const}\ \ln^9\left(\frac{s}{s_o}\right)}{\sin^4\Theta\ \sin^6\varphi} \qquad (3.10)$$

is valid.

<u>Theorem II</u> : If the function $F(s,z_1,\omega_1;\ z_2,\omega_2)$ satisfies the conditions of Theorem I and if instead of (3.9) for the points $1 \leqslant z_i \leqslant 1$, $\omega_i \in G_i$ the following inequality is true

$$\left. |F(s,\cos\Theta_1,\omega_1;\ \cos\Theta_2,\omega_2)| \right|_{\omega_1,\omega_2 \in G_1 \otimes G_2} \leqslant \text{const}\ \ln^n\left(\frac{s}{s_o}\right) \qquad (3.11)$$

On the other hand (see (1.12)),

$$\int_{V_1} f_{ab\to cd}(\vec{k}_c,\vec{k}_d)\, d^3k_c\, d^3k_d = \langle n_c n_d\rangle_{V_1}\, \sigma_{ab\to cd}(s,V_1) \quad . \tag{4.4}$$

From (4.2), (4.3) and (4.4) it follows that

$$\langle n_c n_d\rangle_{V_1} \geqslant \text{const}\, \frac{s}{\ln^9(\frac{s}{s_o})} \quad . \tag{4.5}$$

If, instead of (3.13), we use inequality (3.14), then we will obtain

$$\langle n_c n_d\rangle_{V_1} \geqslant \text{const}\, \frac{s}{\ln^4(\frac{s}{s_o})\ \ln^6\ \ln(\frac{s}{s_o})} \quad . \tag{4.6}$$

From (4.5) and (4.6) we see that pionization can exist only in the case when average multiplicity reaches the values close to the limited one. Note, that

$$\max \langle n_c n_d\rangle_{V_1} \sim \frac{s}{m_c m_d} \quad . \tag{4.7}$$

From (4.3), (4.4) and (4.7) it also follows that, if pionization does exist in nature, then

$$\sigma_{ab\to cd}(s,V_1) \geqslant \frac{\text{const}}{s} \quad . \tag{4.8}$$

## 4. Pionization Phenomenon

Nowadays wide discussions are being held on the so-called pionization phenomenon. Following [16], we will take the following definition of pionization.

If, in hadron collisions at extremely high energies, particles are produced in the c.m.s. with bounded momenta (independent of initial energy) and if as $s \to \infty$ their distribution function differs from zero, i.e.

$$\lim_{s\to\infty} f_{ab\to c,d,\ldots}(s,\vec{k}_c,\vec{k}_d,\ldots) = f_{ab\to c,d,\ldots}(\vec{k}_c,\vec{k}_d,\ldots) \qquad (4.1)$$

then we call this phenomenon pionization.

In this section, we will consider two-particle distribution function and the problem of consistency of the hypothesis on pionization with the estimations on the inclusive cross section, obtained on the basis of analyticity and unitarity. As $V_1$ we will denote the subdomain of the phase volume of "c" and "d" particles of the following form

$$V_1 : \begin{cases} |\vec{k}_c| \leqslant p_c, \quad |\vec{k}_d| \leqslant p_d \\ 0 \neq \Theta_o \leqslant \Theta \leqslant \pi-\Theta_o \;;\quad 0 \neq \varphi_o \leqslant \varphi \leqslant \pi-\varphi_o \;;\quad \pi+\varphi_o \leqslant \varphi \leqslant 2\pi-\varphi_o \end{cases}$$

Here, $p_c$ and $p_d$ are some fixed numbers.

From (3.13) it is clear that

$$\sigma_{ab\to cd}(s,V_1) = \int_{V_1} d^3k_c\, d^3k_d \frac{d\sigma_{ab\ cd}}{d^3k_c\, d^3k_d} \leqslant \text{const}\, \frac{\ell n^9(\frac{s}{s_o})}{s} \; . \qquad (4.2)$$

If pionization exists, then at $s \to \infty$ due to (4.1)

$$\int_{V_1} f_{ab\to cd}(\vec{k}_c,\vec{k}_d)\, d^3k_c\, d^3k_d \geqslant \text{const} \neq 0 \qquad (4.3)$$

In accordance with the hypothesis of limiting fragmentation the distribution functions, e.g., for one and two particles, do not depend on the system energy and are functions of the form

$$f_{ab\to c}(p_c^{\parallel},\vec{p}_c^{\perp}); \quad f_{ab\to cd}(p_c^{\parallel},\vec{p}_c^{\perp};\ p_d^{\parallel},\vec{p}_d^{\perp}) \tag{5.1}$$

where $\vec{p}_c$ and $\vec{p}_d$ are the momenta of "c" and "d" particles in the laboratory co-ordinate system, $p_c^{\parallel}$, $p_d^{\parallel}$ and $\vec{p}_c^{\perp}$, $\vec{p}_d^{\perp}$ are longitudinal and transverse components of these momenta. In the c.m.s. limiting distribution, (5.1) may be written down in the form

$$f_{ab\to c}(x_c,\vec{k}_c^{\perp}); \quad f_{ab\to cd}(x_c,\vec{k}_c^{\perp};\ x_d,\vec{k}_d^{\perp}) \tag{5.2}$$

where the variables $x_c$ and $x_d$ are given by

$$x_c = \frac{2k_c^{\parallel}}{\sqrt{s}} \quad ; \quad x_d = \frac{2k_d^{\parallel}}{\sqrt{s}} \ . \tag{5.3}$$

In (5.3) $k_c^{\parallel}$ and $k_d^{\parallel}$ are longitudinal components of the momenta $\vec{k}_c$ and $\vec{k}_d$ respectively with respect to the axis of collision in the c.m.s. From (5.2) and (5.3) it is obvious that at scale transformations of the form $k^{\parallel} \to \lambda k^{\parallel}$ and $s \to \lambda^2 s$ the distribution function does not change, i.e. scale invariance takes place.

## 5. Limiting Fragmentation. Scale Invariance and Automodelity

Experimental data, obtained in the cosmic rays and on the powerful accelerators, allowed to make a very important conclusion that transverse momenta of produced particles are bounded and apparently depend weakly on the initial energy. On the other hand, the experimental data obtained at the Serpukhov accelerator [17)] have exhibited for the first time that the ratio of kaon and anti-proton production probabilities to the production probability of $\pi$ mesons depends only on the ratio of the produced particle momentum to its maximum value $\frac{p}{p_{max}}$.

Experimentally observed scale invariance at high energies resulted in arising of a number of model treatment. For the first time, such type of model was already earlier considered [18)]. Basing on the experimental data, Yang and collaborators [19)] have developed a limiting fragmentation model; to explain the same phenomena, Feynman [20)] has proposed a parton model. Tavkhelidze and collaborators [21,22)] have worked out an approach which is based on the generalized analysis of dimensionalities and which allows one to describe a number of facts, dealing with dynamics of strong interactions.

To describe hadron-hadron collisions at high energies (due to meza-like extended structure of hadrons) the following particle production mechanism was proposed [19)].

Particles with bounded momenta independent of energy (target fragments) and particles whose momentum increases with total energy growth (projectile fragments) are produced during collisions in the laboratory system of co-ordinates (where one of hadrons is at rest).

It is assumed that at $s \to \infty$ the distribution function of the target fragments tends to some limiting distributions, not equal to zero and independent of the system total energy. It should be noted that projectile fragments do not enter any limiting distribution. Investigation of projectile fragments requires consideration of the projectile co-ordinate (where incident hadron is at rest). Similarly, at $s \to \infty$, an existence of limiting distributions of projectile fragments is assumed in the projectile system.

$$\mathrm{Im}\, T_{ab\to ab}(s, \vec{n}_a \cdot \vec{n}_a') =$$

$$= \frac{(2\pi)^4}{2} \sum_j \int \prod_{i=1}^{j} d^3k_i \, \langle s, \vec{n}_a' | T | k_1 \ldots k_j \rangle \langle k_1 \ldots k_j | T | s, \vec{n}_a \rangle \cdot$$

$$\cdot\, \delta(p_a + p_b - \sum_{i=1}^{j} k_i) \, . \tag{6.1}$$

Here

$$d^3k_i = \frac{d\vec{k}}{(2\pi)^3 \, 2E} \tag{6.2}$$

$\vec{n}_a$ and $\vec{n}_a'$ are unit vectors in the direction of the momenta $\vec{p}_a$ and $\vec{p}_a'$, respectively.

A matrix element of T-operator is normalized in the following way:

$$\sigma_j(s) = \frac{(2\pi)^4}{4\sqrt{s}\,|\vec{p}_a|} \int \prod_{i=1}^{j} d^3k_i \, |\langle k_1 \ldots k_j | T | s, \vec{n}_a \rangle|^2 \delta(p_a + p_b - \sum_{i=1}^{j} k_i) \, . \tag{6.3}$$

Expanding the amplitude over angular variables (see (2.8)) and substituting it into unitarity relation (6.1), we obtain

$$\mathrm{Im}\, f_\ell(s) = |f_\ell(s)|^2 + \sum_j \sum_{m=-\ell}^{m=\ell} \int d\Gamma_j \, |T_\ell^m(s, \xi, j)|^2 + \ldots \tag{6.4}$$

For a given j-th channel of the reaction, the energy conservation law may be presented in the following way

$$S = (E_a + E_b)^2 = \left( \sum_{i=1}^{n_c^j} E_c^i + \sum_{i=1}^{n_d^j} E_d^i + \ldots + \sum_{i=1}^{n_f^j} E_f^i \right)^2 \tag{6.5}$$

$n_c^j$ is the number of "c" particles in the j-th channel of the reaction; $E_c^i$ is the energy of the i-th particle of "c"-kind. Multiplication by "s" of both sides of

## 6. Particle Momentum Cut-Off in Inclusive Processes in the Region of Large Angles [24)]

Weak dependence on initial energy for the transverse components of detected particle momenta is a characteristic of multi-particle processes. It is evident that, if the transverse momentum of a produced particle is bounded at some energy-independent value M, then fast particles ($p \sim p_{max}(s)$) in the c.m.s. are produced in the angular range

$$0 \leqslant \Theta^2 \lesssim \frac{M^2}{s}$$

and momenta of the particles produced in the large-angle range are bounded by some constant value.

In most model descriptions this fact is used as a postulate [18)-22)] and the corresponding phenomenon has no proper theoretical explanation.

In the given section, on the basis of analyticity and unitarity, we study the behaviour of average values of momentum products for detected particles in binary inclusive processes at high energies. It is done with the method developed in [2),12),13)] and described in sections 2 and 3 of this paper.

In particular, a conclusion is made here, that in the given range of angles ($\Theta \neq 0,\pi$; $\varphi \neq 0,\pi,2\pi$) mean values for products of longitudinal as well as transverse component moments of "c" and "d" particles, produced in binary inclusive processes, cannot increase more rapidly than $\ell n^{\gamma}(\frac{s}{s_o})$. It is also shown that, if the mean values of multiplicity products $n_c$ and $n_d$ increase as a power of energy, then the absolute contribution of large angles to the mean value $\langle E_c E_d \rangle$ decreases with increasing energy. As the probability of producing fast particles seems to be considerable, it follows from analyticity and unitarity that production of fast particles takes place in the region of small angles.

In the c.m.s. the unitarity condition can be written in the form:

$$\langle E_c E_d \rangle (\Theta,\varphi) =$$

$$= \frac{2\pi}{N(s)} \int E_c E_d \, f_{ab\to cd}(s,|\vec{k}_c|,|\vec{k}_d|,\Theta,\varphi,\psi) \frac{|\vec{k}_c|^2 d|\vec{k}_c| \, |\vec{k}_d| d|\vec{k}_d| d\cos\psi}{(2\pi)^6 \, 4 E_c E_d} . \tag{6.11}$$

Here, $\Theta$ and $\varphi$ are angular variables of initial particle "a", $\psi$ is the angle between momenta $\vec{k}_c$ and $\vec{k}_d$. It should be noted that

$$\langle E_c E_d \rangle = \int \langle E_c E_d \rangle (\Theta,\varphi) d\cos\Theta \, d\varphi . \tag{6.12}$$

If use is made of (2.8), then for the function $\langle E_c E_d \rangle (\Theta,\varphi)$ one can write the following expansion

$$N(s) \langle E_c E_d \rangle (\Theta,\varphi) = \frac{1}{|\vec{p}_a|^2} \sum_{\ell,m} \sum_{\ell',m'} (2\ell+1)(2\ell'+1) d^{\ell}_{m,o}(\Theta) d^{\ell'}_{m',o}(\Theta)$$

$$e^{-i(m'-m)\varphi} \sum_j n^j_c n^j_d \int d\Gamma_j \, E_c E_d \, \overset{*}{T}{}^{m'}_{\ell'}(s,\xi,j) \, T^{m}_{\ell}(s,\xi,j) . \tag{6.13}$$

It is convenient to introduce the function

$$F_{ab\to cd}(s,\cos\Theta_1,e^{i\varphi_1};\cos\Theta_2,e^{i\varphi_2}) = \frac{2}{\sqrt{s}\,|\vec{p}_a|} \sum_{\ell_1,m_1} \sum_{\ell_1,m_1} (2\ell_1+1)(2\ell_2+1) d^{\ell_1}_{m_1,o}(\Theta_1) d^{\ell_2}_{m_2,o}(\Theta_2) \cdot$$

$$\cdot \, e^{im_1\varphi_1 - im_2\varphi_2} \sum_j n^j_c n^j_d \int d\Gamma_j E_c E_d \, T^{m_1}_{\ell_1}(s,\xi,j) \, \overset{*}{T}{}^{m_2}_{\ell_2}(s,\xi,j) \tag{6.14}$$

in order to study the character of the increase of $\langle E_c E_d \rangle (\Theta,\varphi)$ with energy. It should be noted that

$$N(s) \langle E_c E_d \rangle (\Theta,\varphi) = \frac{\sqrt{s}}{2|\vec{p}_a|} F_{ab\to cd}(s,\cos\Theta,e^{i\varphi};\cos\Theta,e^{i\varphi}) . \tag{6.15}$$

It may be shown, with the help of unitarity and inequality (6.8), that the function $F_{ab\to cd}(s,\cos\Theta_1,e^{i\varphi_1};\cos\Theta_2,e^{i\varphi_2})$ for physical values of $\varphi_1$ and $\varphi_2$

the unitarity condition (6.4) for a partial wave amplitude, keeping into account (6.5) in the integrand, gives after some elementary calculations

$$\mathrm{Im}\, f_\ell(s) = |f_\ell(s)|^2 +$$

$$+ \frac{2}{S} \sum_j n_c^j n_d^j \int d\Gamma_j\, E_c E_d \sum_{m=-\ell}^{\ell} |T_\ell^m(s,\xi,j)|^2 +$$

$$+ \frac{1}{S} \sum n_c^j (n_c^j - 1) \int d\Gamma_j\, E_c^{(1)} E_c^{(2)} \sum_{m=-\ell}^{m=\ell} |T_\ell^m(s,\xi,j)|^2 + \ldots \quad . \tag{6.6}$$

From (6.6) it follows that

$$\frac{1}{S} \sum_j n_c^j n_d^j \int d\Gamma_j\, E_c E_d \sum_{m=-\ell}^{m=\ell} |T_\ell^m(s,\xi,j)|^2 \leqslant \frac{1}{8} \,. \tag{6.7}$$

On the other hand, from analyticity of the imaginary part of an elastic amplitude in $\cos\Theta$ in the Martin ellipse, it follows

$$\frac{1}{S} \sum_j n_c^j n_d^j \int d\Gamma_j\, E_c E_d \sum_{m=-\ell}^{m=\ell} |T_\ell^m(s,\xi,j)|^2 \leqslant R(s)\, e^{-\ell \frac{4m_\pi}{\sqrt{s}}} \quad . \tag{6.8}$$

Let us consider the mean value of the energy product for "c" and "d" particles

$$\langle E_c E_d \rangle = \frac{1}{N(s)} \int E_c E_d\, f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)\, d^3k_c\, d^3k_d \tag{6.9}$$

where

$$N(s) = \langle n_c n_d \rangle_t\, \sigma_{tot}(s) = \int f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)\, d^3k_c\, d^3k_d \quad . \tag{6.10}$$

If, in the c.m.s. a co-ordinate system is chosen so that the z-axis is directed along the momentum $\vec{k}_c$ and the XOZ plane goes through the momenta $\vec{k}_c$ and $\vec{k}_d$, then it is possible to introduce the quantity

Integration of inequalities (6.19) and (6.20) in the domain

$$V_2 : \left\{ 0<\Theta_0 \leq \Theta \leq \pi-\Theta_0 ; \quad 0<\varphi_0 \leq \varphi \leq \pi-\varphi_0 , \quad \pi+\varphi_0 \leq \varphi \leq 2\pi-\varphi_0 \right\} \qquad (6.21)$$

leads to

$$\langle E_c E_d \rangle_{V_2} = \int_{V_2} \langle E_c E_d \rangle (\Theta,\varphi) d\cos\Theta d\varphi \leq \text{const} \frac{\ln^9 (\frac{s}{s_0})}{\sigma_{tot}^{(s)} \langle n_c n_d \rangle_t} \qquad (6.22)$$

$$\langle E_c E_d \rangle_{V_2} = \int_{V_2} \langle E_c E_d \rangle (\Theta,\varphi) d\cos\Theta d\varphi \leq \text{const} \frac{\ln^4 (\frac{s}{s_0}) \ln^6 \ln (\frac{s}{s_0})}{\sigma_{tot}^{(s)} \langle n_c n_d \rangle_t} \qquad (6.23)$$

respectively.

From this it follows that, if the average value of the multiplicity product $\langle n_c n_d \rangle_t$ increases as a power of energy and the total cross section has the lower bound $\sigma_{tot}^{(s)} \geq \ln^{-r} (\frac{s}{s_0})$ (where $r$ is some fixed number), then the absolute contribution of the $V_2$ angular domain to the average value of the magnitude $\langle E_c E_d \rangle$ tends to zero with increasing energy.

From (6.22) an inequality for average values of the products of longitudinal and transverse momenta of "c" and "d" particles follows:

$$\langle p_c^{\perp} p_d^{\perp} \rangle_{V_2} \leq \text{const} \frac{\ln^9 (\frac{s}{s_0})}{\sigma_{tot}^{(s)} \langle n_c n_d \rangle_t} \qquad (6.24)$$

$$\langle p_c^{\parallel} p_d^{\parallel} \rangle_{V_2} \leq \text{const} \frac{\ln^9 (\frac{s}{s_0})}{\sigma_{tot}^{(s)} \langle n_c n_d \rangle_t} \qquad (6.25)$$

is analytic in variables $z_1 = \cos\Theta_1$ and $z_2 = \cos\Theta_2$ in the Martin-type ellipse with the exception of the cuts $[-x, -1]$ and $[1, x]$. Again, for the physical values of the variables $z_1 = \cos\Theta_1$ and $z_2 = \cos\Theta_2$ the function $F_{ab\to cd}(s,z_1,\omega_1;\ z_2,\omega_2)$ is analytic over variables $\omega_1 = e^{i\varphi_1}$ and $\omega_2 = e^{i\varphi_2}$ in the rings

$$x - \sqrt{x^2 - 1} < |\omega_i| < x + \sqrt{x^2 - 1} \quad . \tag{6.16}$$

If we assume that function $F_{ab\to cd}(s,z_1,\omega_1;\ z_2,\omega_2)$ satisfies the conditions of Theorem I, then it is possible to obtain the following inequality

$$\left| F_{ab\to cd}(s,\cos\Theta,e^{i\varphi};\ \cos\Theta,e^{i\varphi}) \right|_{\substack{\Theta_o \leqslant \Theta \leqslant \pi-\Theta_o \\ \varphi_o \leqslant \varphi \leqslant \pi-\varphi_o \\ \pi+\varphi_o \leqslant \varphi \leqslant 2\pi-\varphi_o}} \leqslant \text{const} \frac{\ln^9(\frac{s}{s_o})}{\sin^4\Theta \sin^6\varphi} \ . \tag{6.17}$$

This inequality may be improved if the function $F_{ab\to cd}(s,z_1,\omega_1;\ z_2,\omega_2)$ satisfies a more strict requirement of Theorem II. In this case we shall have

$$\left| F_{ab\to cd}(s,\cos\Theta,e^{i\varphi};\ \cos\Theta,e^{i\varphi}) \right|_{\substack{\Theta_o \leqslant \Theta \leqslant \pi-\Theta_o \\ \varphi_o \leqslant \varphi \leqslant \pi-\varphi_o \\ \pi+\varphi_o \leqslant \varphi \leqslant 2\pi-\varphi_o}} \leqslant \text{const} \frac{\ln^4(\frac{s}{s_o})\ln^6\ln(\frac{s}{s_o})}{\sin^4\Theta \sin^6\varphi} \tag{6.18}$$

On the basis of inequalities (6.17) and (6.18) from equality (6.15) we have respectively

$$N(s) \langle E_c E_d \rangle (\Theta,\varphi) \leqslant \text{const} \frac{\ln^9(\frac{s}{s_o})}{\sin^4\Theta \sin^6\varphi} \tag{6.19}$$

$$N(s) \langle E_c E_d \rangle (\Theta,\varphi) \leqslant \text{const} \frac{\ln^4(\frac{s}{s_o})\ln^6\ln(\frac{s}{s_o})}{\sin^4\Theta \sin^6\varphi} \ . \tag{6.20}$$

and

$$\int_{V_2} f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)\,\frac{d\,\vec{k}_c}{2(2\pi)^3}\cdot\frac{d\,\vec{k}_d}{2(2\pi)^3} \leq \text{const}\ \ell n^4\left(\frac{s}{s_o}\right)\ell n^6\,\ell n\left(\frac{s}{s_o}\right) \qquad (6.30)$$

respectively.

It should also be noted that, if pionization really takes place, then the distribution function of particles of bounded momenta does not depend on the system energy and, consequently, the pionization hypothesis does not contradict inequalities (6.29) and (6.30).

In the same way from (6.23) it follows

$$<p_c^{\perp}\, p_d^{\perp}>_{V_2} \leqslant \text{const}\, \frac{\ell n^4(\frac{s}{s_o})\,\ell n^6 \ell n(\frac{s}{s_o})}{\sigma_{tot}^{(s)} <n_c n_d>_t} \qquad (6.26)$$

$$<p_c''\, p_d''>_{V_2} \leqslant \text{const}\, \frac{\ell n^4(\frac{s}{s_o})\,\ell n^6 \ell n(\frac{s}{s_o})}{\sigma_{tot}^{(s)} <n_c n_d>_t} \quad . \qquad (6.27)$$

It should be noted that the total interaction cross section and average value of multiplicity product behave so that *)

$$\sigma_{tot}^{(s)} <n_c n_d>_t \geqslant \text{const} \neq 0 \qquad (6.28)$$

Then from inequalities (6.24) and (6.25) it follows that average values for longitudinal and transverse components of momenta in $V_2$ angular range cannot increase with energy more rapidly than $\ell n^9(s/s_o)$.

Thus, in the range of large angles, defined by inequality (6.21), a cut-off of produced particle momentum takes place. Using the expression for the magnitude $<E_c E_d>(\Theta,\varphi)$ via the distribution function (see formula (6.12)) we can write inequalities (6.22) and (6.23) in the forms

$$\int_{V_2} f_{ab\to cd}(s,\vec{k}_c,\vec{k}_d)\, \frac{d\,\vec{k}_c}{2(2\pi)^3}\, \frac{d\,\vec{k}_d}{2(2\pi)^3} \leqslant \text{const}\, \ell n^9(\frac{s}{s_o}) \qquad (6.29)$$

*) Note that, if pionization [1),16)] does exist, then inequality (6.28) is always true.

where $d^2\sigma_V^j / dt\, dw^2$ is the differential cross section for process (VII), and $\sigma_{tot}^{ab}(s)$ the total cross section for interaction on collision of two particles "a + b".

The average multiplicity for inclusive reaction (VI) is determined in a similar way.

In works [10,25] it has been stated that for any fixed $t$, $w^2$ asymptotic equality takes place

$$\frac{d^2\sigma_V^j}{dt\, dw^2} \simeq \frac{d^2\sigma_{VI}^j}{dt\, dw^2} \tag{7.3}$$

$$\frac{d^2\sigma_V}{dt\, dw^2} \simeq \frac{d^2\sigma_{VI}}{dt\, dw^2} \quad . \tag{7.4}$$

From formula (7.3) and definition (7.2) it follows that at high energies

$$\sigma_{tot}^{ab} \langle N_V(s,t,w^2)\rangle \simeq \sigma_{tot}^{\tilde{a}b'} \langle N_{VI}(s,t,w^2)\rangle \ . \tag{7.5}$$

In particular, for the inclusive process with $\tilde{b} \equiv b'$ (under the assumption that the Pomeranchuk theorem is fulfilled $\sigma_{tot}^{ab} \sim \sigma_{tot}^{ab}$ ) we have

$$\langle N_V(s,t,w^2)\rangle \simeq \langle N_{VI}(s,t,w^2)\rangle \ . \tag{7.6}$$

As it follows from derivation of formula (7.3), given in paper [10,25], relations (7.6) may be integrated in any fixed interval of the variables $t$ and $w^2$ (independent of energy s). Let these intervals of variables $t$ and $w^2$ be $\Delta t$ and $\Delta w^2$, respectively. We shall then have the following equalities

$$\langle N_V(s,\Delta t,\Delta w^2)\rangle \simeq \langle N_{VI}(s,\Delta t,\Delta w^2)\rangle \tag{7.7}$$

for average multiplicities in the indicated domains.

## 7. Asymptotic Equality for Average Multiplicity in Crossing Inclusive Reactions

In the present section we should like to pay attention to the fact that from general considerations, based on the properties of analyticity and crossing symmetry, some conclusions on average multiplicity in crossing inclusive reactions at high energies may be made [24].

Let us consider the processes of the type

$$a + b \rightarrow b' + \sum_j x_j \qquad \text{(V)}$$

$$\tilde{a} + b' \rightarrow b + \sum_j \tilde{x}_j \ . \qquad \text{(VI)}$$

We shall introduce the following variables: $s = (p_a + p_b)^2$ is the total energy squared in the c.m.s. $t = (p_b - p_{b'})^2$ and $u = (p_a - p_{b'})^2$ are transferred momenta squared; $w^2 = (p_a + p_b - p_{b'})^2$ is the missing mass squared for the reaction in channel (I). All the other independent variables, which describe some definite (j)-channel of the process

$$a + b \rightarrow b' + x_j \qquad \text{(VII)}$$

will be designated via $\xi_j$. The crossing condition for the amplitudes of processes V and VI has the form

$$T_V^j\,(u,t,w^2,\xi_j) = \overset{*}{T}{}^j_{VI}\,(s,t,w^2,\xi_j)\ , \qquad (7.1)$$

where

$$s + t + u = m_a^2 + m_b^2 + m_{b'}^2 + w^2\ .$$

Let us determine average multiplicity in inclusive process (V)

$$\langle N_V(s,t,w^2)\rangle = \frac{1}{\sigma_{tot}^{ab}} \sum_j N^j \frac{d^2\sigma_V^j}{dt\, dw^2} \qquad (7.2)$$

If, along with reaction (VI), we consider its charge-conjugated one

$$a + \tilde{b}' \to \tilde{b} + \sum_j x_j \qquad \text{(VIII)}$$

then from analogous reasoning it follows:

$$\sigma^{ab}_{tot} \langle N_V(s,t,w^2)\rangle \simeq \sigma^{a\tilde{b}'}_{tot} \langle N_{VIII}(s,t,w^2)\rangle \ . \qquad (7.8)$$

In particular, for $b \equiv b'$

$$\langle N_V(s,t,w^2)\rangle \simeq \langle N_{VIII}(s,t,w^2)\rangle \qquad (7.9)$$

and for fixed intervals $\Delta t$ and $\Delta w^2$

$$\langle N_V(s,\Delta t,\Delta w^2)\rangle \simeq \langle N_{VIII}(s,\Delta t,\Delta w^2)\rangle \ . \qquad (7.10)$$

If similarly to (7.2), one determines average multiplicities for charged particles only, then one can obtain absolutely similar asymptotic equalities for average multiplicities for charged particles in reactions (V), (VI), (VIII).

For example, in reactions

$$p + p \to p + \ldots$$

$$\tilde{p} + p \to \tilde{p} + \ldots$$

for fixed intervals $\Delta t$, $\Delta w^2$, average multiplicities for hyperons should be similar and equal to analogous average multiplicity of antihyperons in the reaction

$$\tilde{p} + p \to p + \ldots \ .$$

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