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HOW DOES BETATRON MISMATCHING AFFECT BEAM SIZE AND BEAM DENSITY ?

by

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1. INTRODUCTION

We work in the normalized (x, x') phase plane in which betatron oscillations with amplitude a are represented by a circle of radius a.

A matched beam of emittance πR^2 is represented by the area of a circle of radius R. A mismatched beam with the same emittance is given by an ellipse with axes zR and R/z. $z \geq 1$ is the mismatch parameter. This is shown in Fig. 1.

After filamentation has taken place betatron oscillations will occupy a circle of radius zR.

We shall answer the following questions :

i) What is the density distribution inside this circle for several given density distributions at the beginning, and for a given mismatch parameter z ?

ii) What is the visible density distribution as seen by a detector which detects the spatial density only ?

iii) What is the effect of a mismatch on the luminosity of the ISR ?

2. ANALYSIS

We use both polar co-ordinates (γ, φ) and Cartesian co-ordinates (x, x') in the normalized phase plane.

Let the normalized density distribution of a matched beam be called $f(r)$. The mismatched beam will then have a distribution function $F(r, \varphi, z)$ which is given by :

$$F(r, \varphi, z) = f\left[(rz \cos \varphi)^2 + \left(\frac{r}{z} \sin \varphi\right)^2\right] \qquad (1)$$

Filamentation will smooth out the azimuthal variations of the density distribution function and yield a new distribution function $\bar{F}$:

$$\bar{F}(r, z) = \frac{1}{2\pi} \int_0^{2\pi} F(r, \varphi, z)\, d\varphi \qquad (2)$$

It is also interesting to calculate how much of a mismatched filamented beam fits into a given radius. This is given by the integrated distribution function D :

$$D(r, z) = 2\pi \int_0^r y\, \bar{F}\,(y, z)\, dy \qquad (3)$$

The "visible" distribution function $G(x, z)$ is obtained by integrating $\bar{F}$ over all slopes x' :

$$G(x, z) = \int_{-\infty}^{\infty} \bar{F}\left(\sqrt{x^2 + x'^2}\,,\; z\right) dx' \qquad (4)$$

Finally, the luminosity of the ISR, assuming perfect overlap of the two colliding beams, is proportional to L(z) :

$$L(z) = \int_{-\infty}^{+\infty} G(x, z)\, dx \qquad (5)$$

3. NUMERICAL EXAMPLES

The functions defined above were calculated for uniform, parabolic and Gaussian density distributions in the injected beam. In all cases analytical formulae which are given below were used as far as possible. The remaining functions were computed by numerical integration. The accuracy in the integration routine GAUSS 2 [*)] is good and the relative error is less than 10^{-4}. H(x) is the Heaviside step function; H(x) = 1 for x ≥ o and H(x) = o for x < o.

3.1 Uniform beam

$$f(r) = \frac{1}{\pi} H(1 - r) \qquad (6)$$

$$\bar{F}(r, z) = \frac{H(z - r)}{\pi} \left[1 - \frac{2\alpha}{\pi} H(r - \frac{1}{z})\right] \qquad (7)$$

*) CERN 6600 PROGRAM LIBRARY D 103

where

$$\alpha = \sin^{-1}\left(\frac{z^2 - 1/r^2}{z^2 - 1/z^2}\right)^{1/2} \tag{8}$$

For $x \leq \frac{1}{z}$

$$G(x, z) = \frac{2}{\pi}\left(\frac{1}{z^2} - x^2\right)^{1/2} + 2\int_{\sqrt{\frac{1}{z^2} - x^2}}^{\sqrt{z^2 - x^2}} \bar{F}\left(\sqrt{x^2 + x'^2}, z\right) dx' \tag{9}$$

For $\frac{1}{z} < x \leq z$

$$G(x, z) = 2\int_0^{\sqrt{z^2 - x^2}} \bar{F}\left(\sqrt{x^2 + x'^2}, z\right) dx' \tag{10}$$

3.2 Parabolic beam

$$f(r) = \frac{2}{\pi}(1 - r^2)\, H(1 - r) \tag{11}$$

$$\bar{F}(r, z) = \frac{2H(z - r)}{\pi}\left\{\left[1 - \frac{2\alpha}{\pi} H(r - \frac{1}{z})\right]\left[1 - \frac{r^2}{2}(z^2 + \frac{1}{z^2})\right] + \frac{r^2}{\pi}\left(z^2 - \frac{1}{r^2}\right)^{1/2}\left(\frac{1}{r^2} - \frac{1}{z^2}\right)^{1/2} H\left(r - \frac{1}{z}\right)\right\} \tag{12}$$

α is again given by (8).

For $x \leq \frac{1}{z}$

$$G(x, z) = \frac{2}{3\pi} \left(\frac{1}{z^2} - x^2\right)^{1/2} \left[5 - \frac{1}{z^4} - 2x^2\left(z^2 + \frac{1}{z^2}\right)\right]$$

$$+ 2 \int_{\sqrt{\frac{1}{z^2} - x^2}}^{\sqrt{z^2 - x^2}} \bar{F}\left(\sqrt{x^2 + x'^2}, z\right) dx' \qquad (13)$$

For $\frac{1}{z} < x \leq z$

$$G(x, z) = 2 \int_{0}^{\sqrt{z^2 - x^2}} \bar{F}\left(\sqrt{x^2 + x'^2}, z\right) dx' \qquad (14)$$

3.3 Gaussian beam

$$f(r) = \frac{1}{\pi} e^{-r^2} \qquad (15)$$

4. RESULTS AND CONCLUSIONS

Graphs of the distribution function for z = o and z = 1.5 are shown in Figs. 2 to 7. The luminosity is plotted as a function of z in Fig. 8. A programme for computing these functions (....) and tables of them are available to those interested. As is already indicated by the choice of z = 1.5 in all the figures, betatron mismatching is fairly uncritical as far as beam density in the middle of the beam and luminosity are concerned. Beam size oscillations by a factor $z^2 = 1.21$ which correspond to the expected accuracy of the beam profile monitors on the first turn in the ISR result in a loss of luminosity by only a few percent.

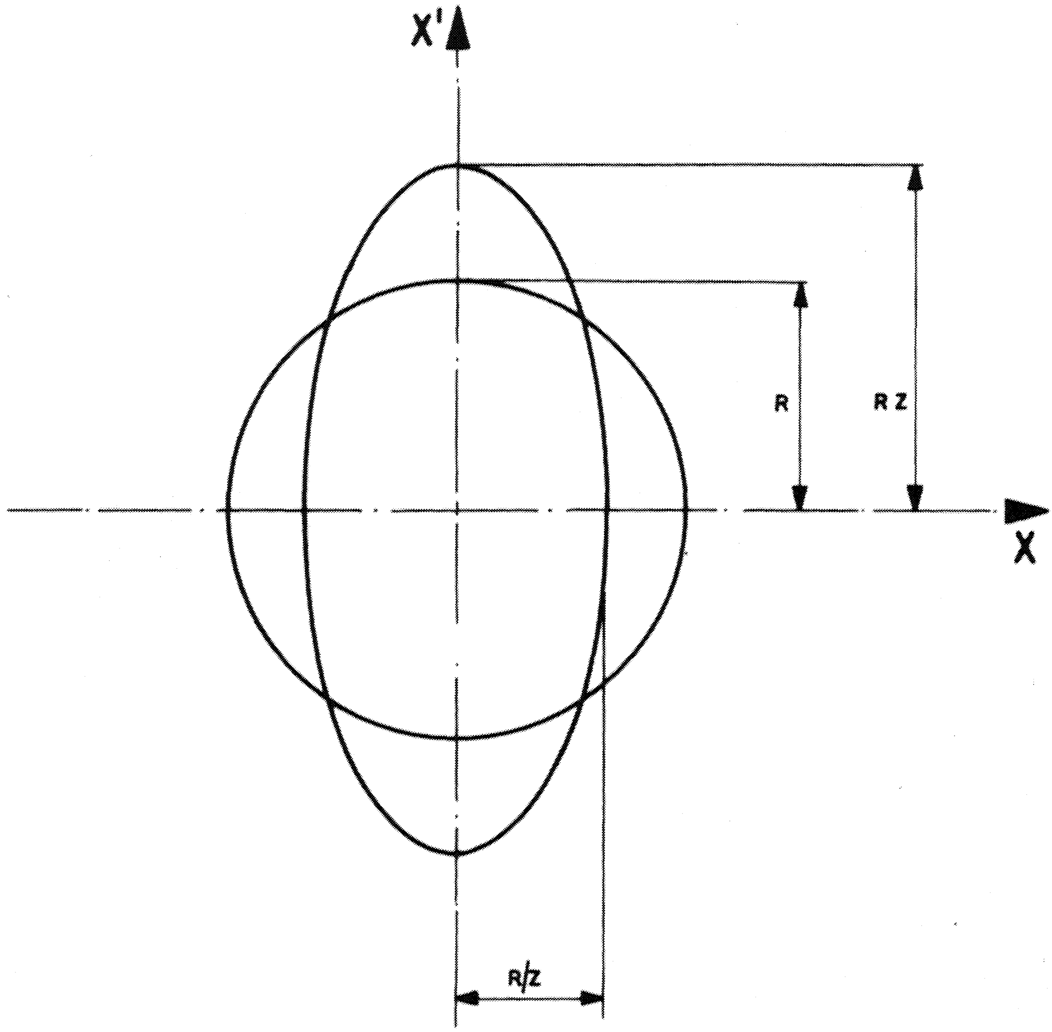
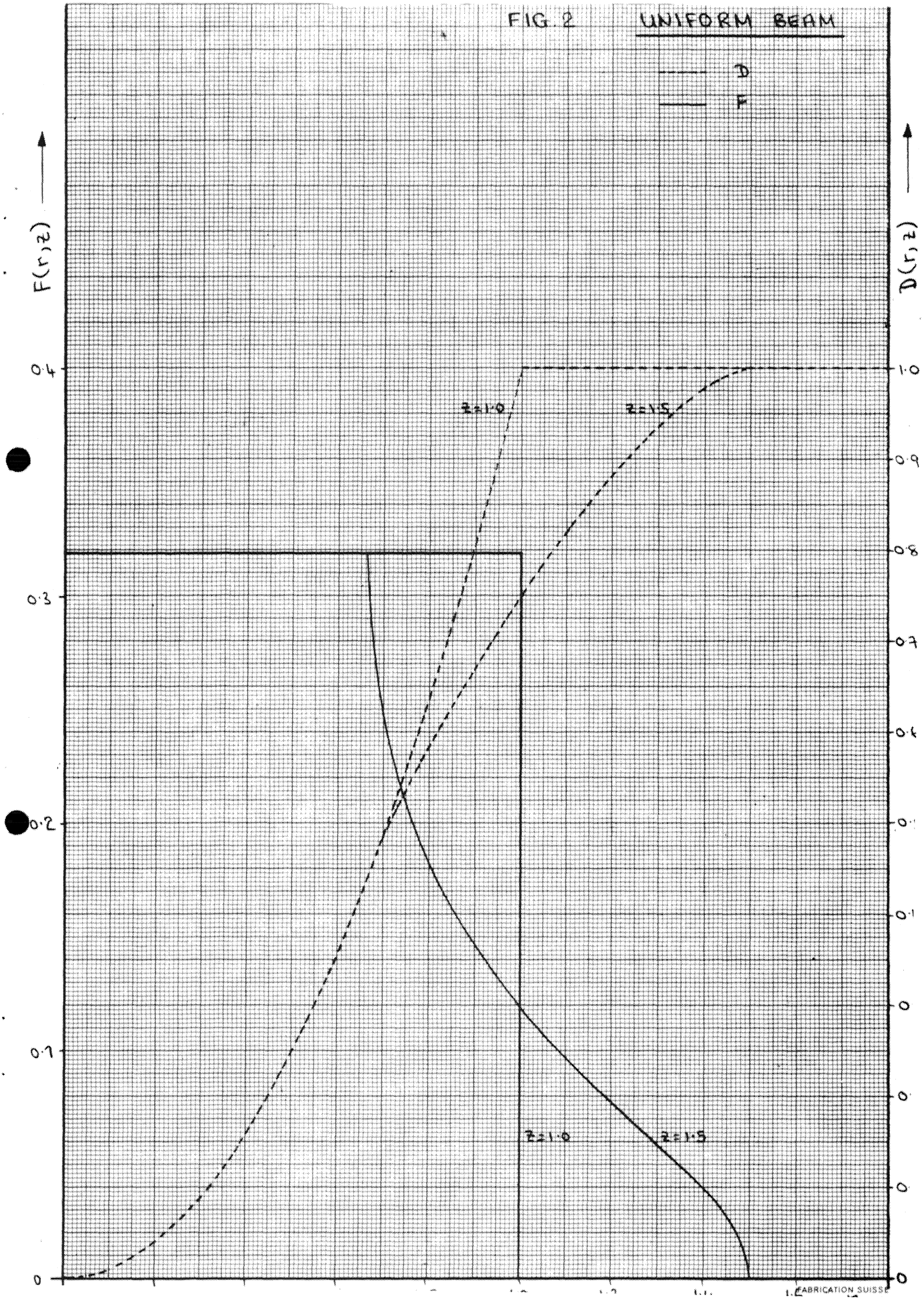


Fig.1

FIG. 2



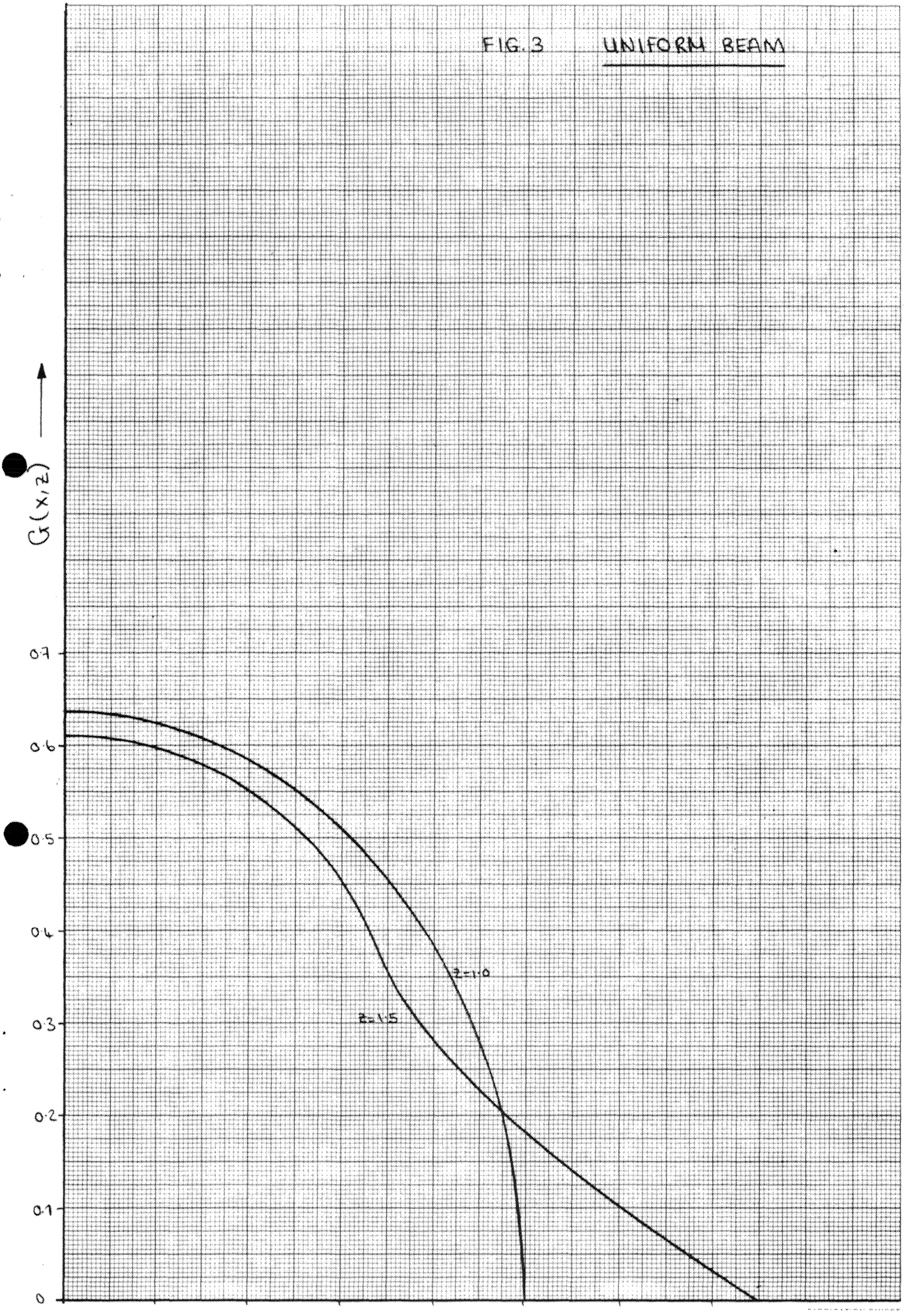


FIG. 3 UNIFORM BEAM

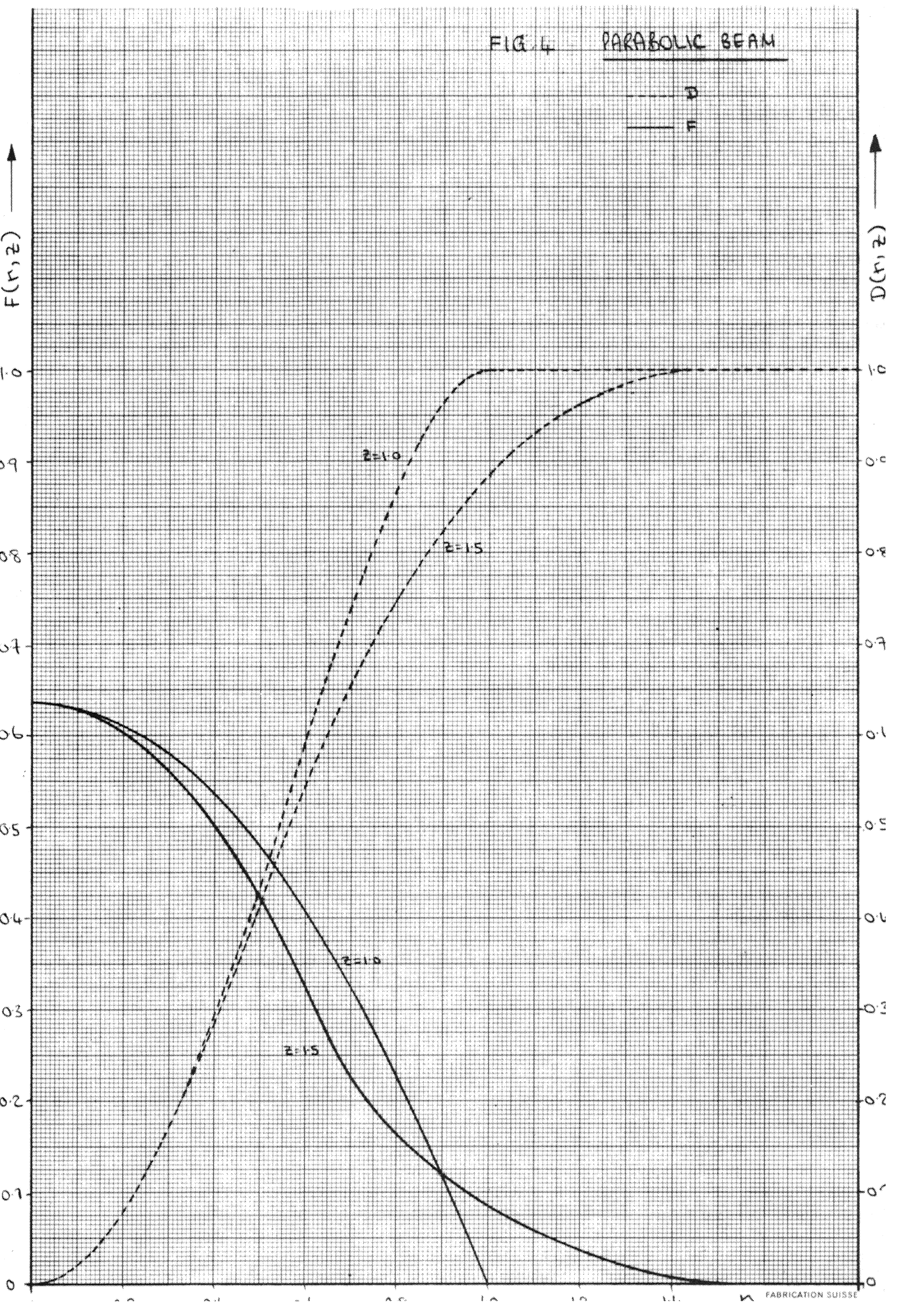


FIG. 4 PARABOLIC BEAM
D
F
F(r, z)
D(r, z)
z=1.0
z=1.5
z=1.0
z=1.5
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
r
FABRICATION SUISSE

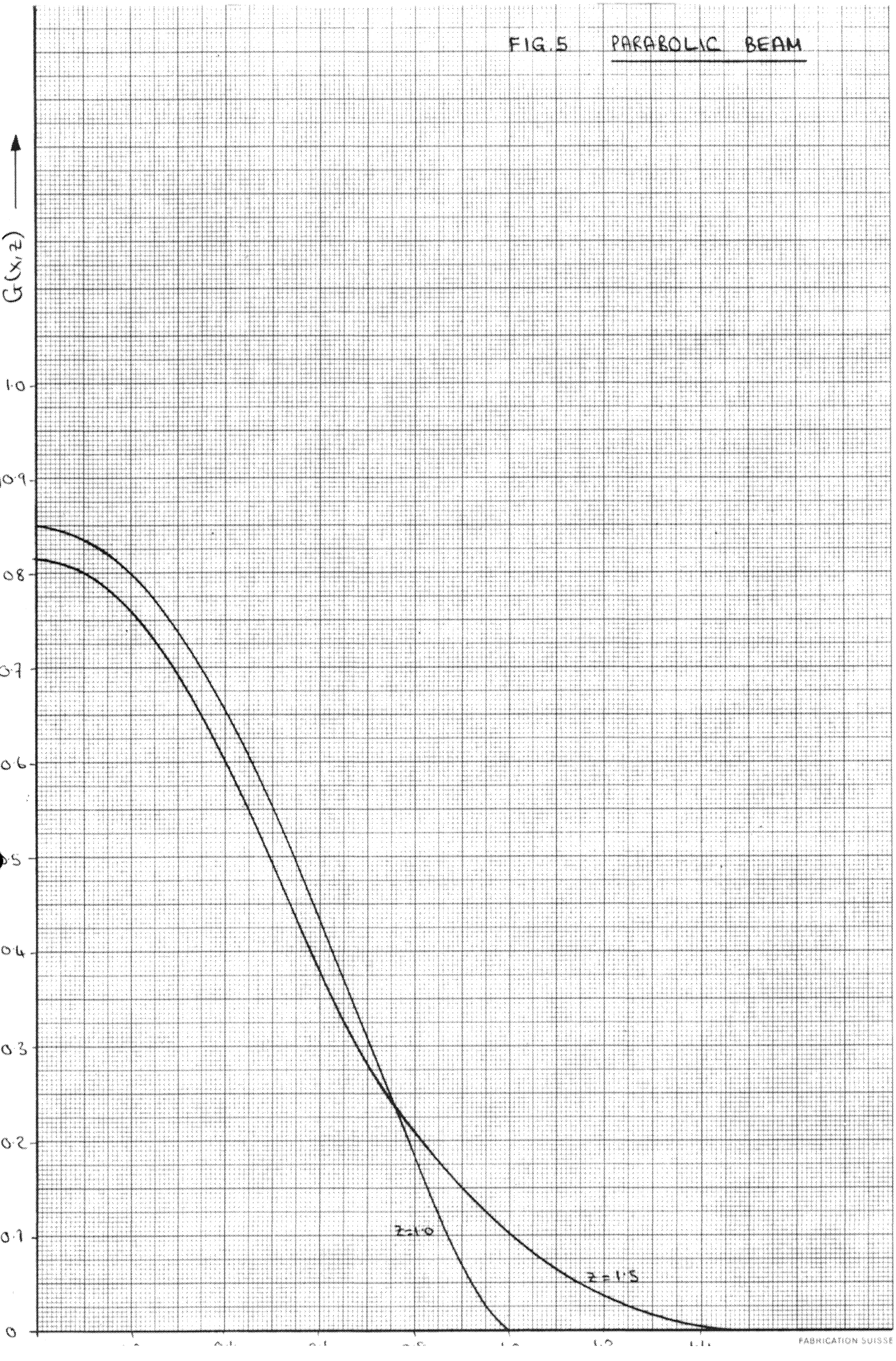


FIG. 5 PARABOLIC BEAM

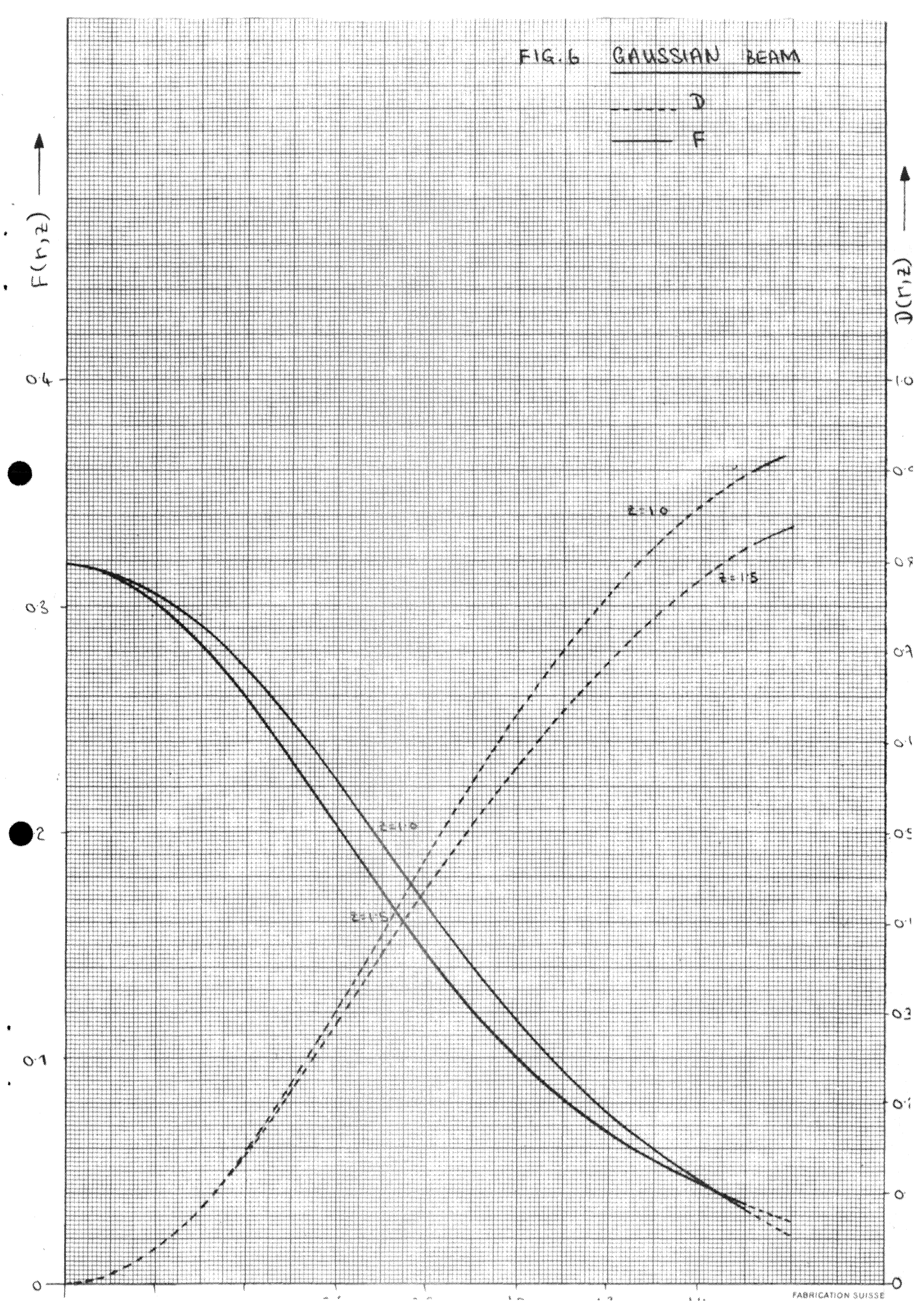
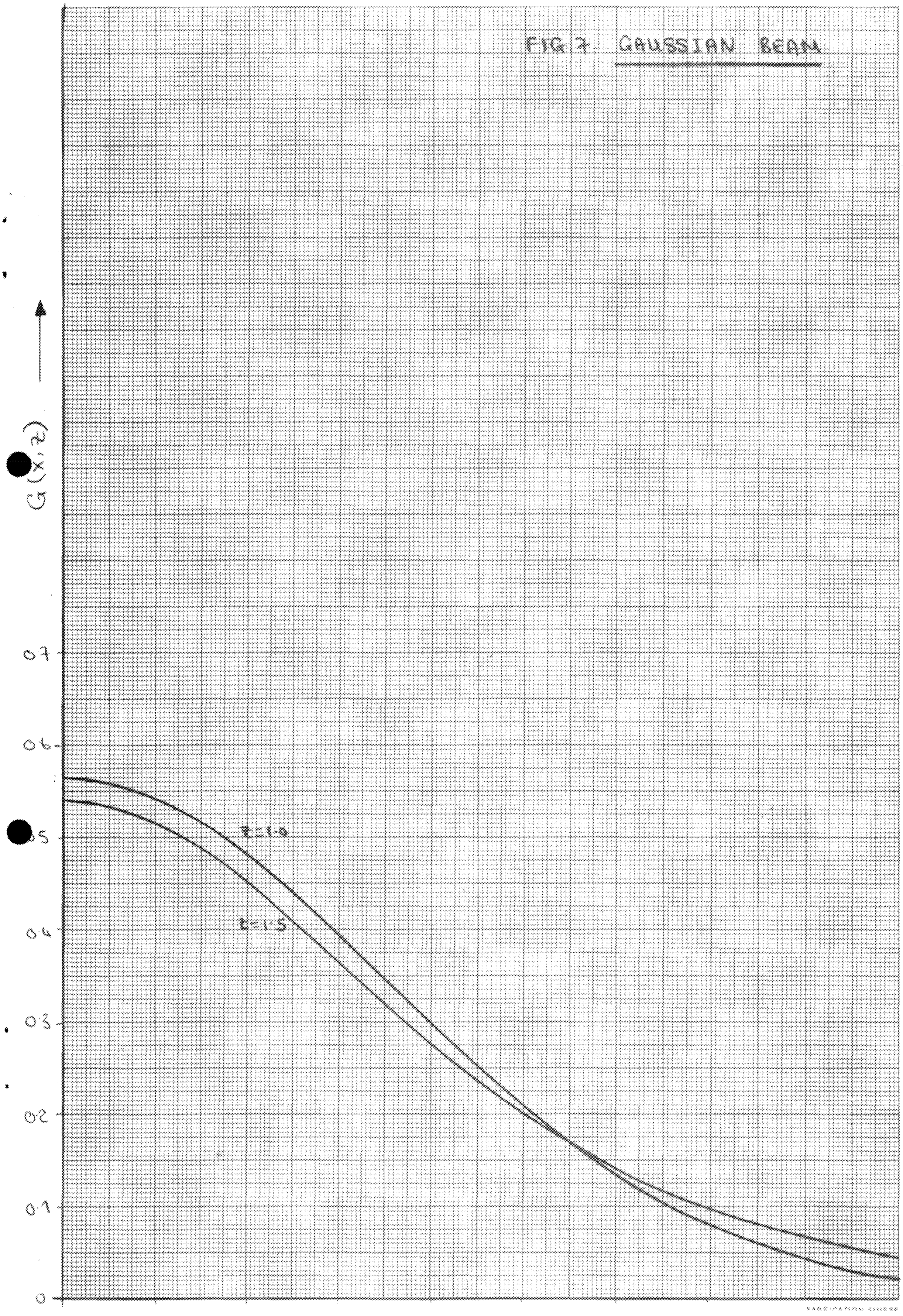


FIG. 6 GAUSSIAN BEAM
D
F
F(r, z)
D(r, z)
0.4
0.3
2
0.1
0
z = 1.0
z = 1.5
z = 1.0
z = 1.5
FABRICATION SUISSE

FIG. 7 GAUSSIAN BEAM



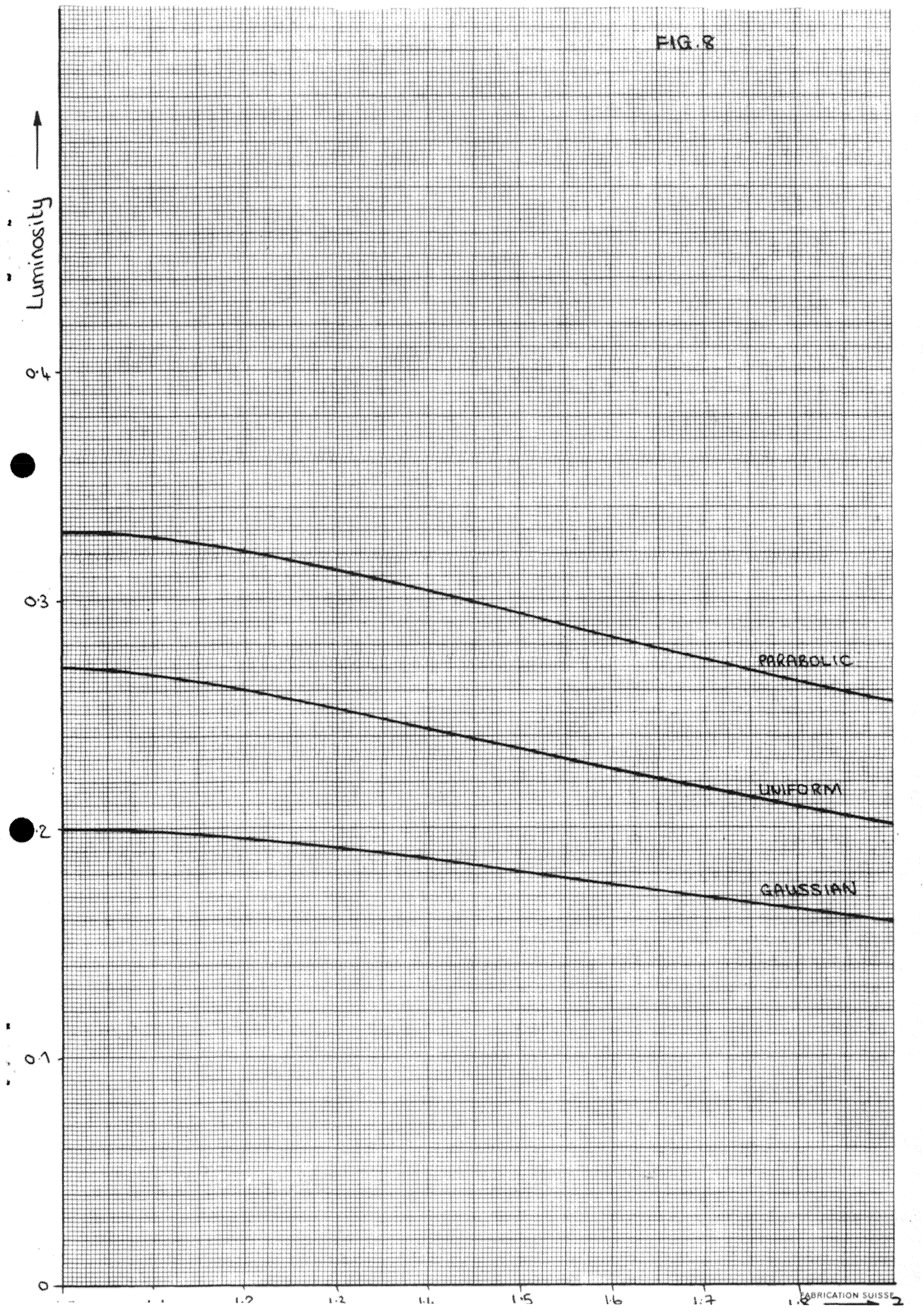


FIG. 8