CERN/EF/RF 81-5
7 July 1981

TUNING OF A SUPERCONDUCTING ACCELERATING CAVITY UNDER OPERATING CONDITIONS

PART 1: THEORY

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ABSTRACT

Recently a 4-cell s.c. accelerating 500 MHz cavity has been tested at CERN [1]. The mechanical support of this cavity allows to change the lengths ℓ_i of individual cavity cells in a reversible way even with the cavity mounted within the test cryostat and cooled down to 4.2 K. In the following we present a cavity model and describe a perturbation method which uses only this property of the support (fig. 1) to monitor the excitation of different cells and to tune the cavity to a flat π-mode.

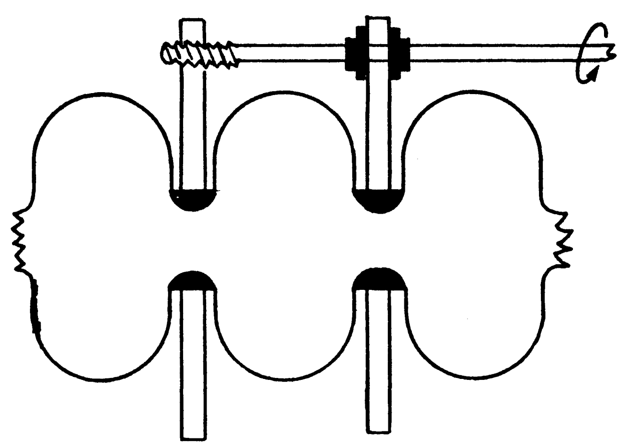

Fig. 1
Sketch of cavity and support

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1. INTRODUCTION

Superconducting RF cavities in the frequency-range of a few hundred MHz are limited in field by a rapidly increasing field-emission. Therefore for multicell-cavities operated in the π-mode the field flatness is even more essential than for copper-structures.

On the other hand, it is technically much more complicated to measure the field with a perturbing object [2], if the evacuated cavity is in a bath of liquid helium especially since the perturbing object and its support have to be removed from the cavity for the final high field and high Q tests of the cavity.

Therefore we have developed a method of measuring the field-distribution and tuning the cavity accordingly without introducing an object into the cavity, but simply using the tunable support [3].

In part 1 we present a lumped-circuit model for cavity half-shells of aperture-coupled cavities and from these building blocks we construct the whole multi-cell cavity model. It is shown that this model gives the correct eigenvalues and eigenvectors for any type of end-cells and especially the correct behaviour of the mode-frequencies for different couplings.

In the second part we show - based on this model - how one can measure the field excitation of the different cells in squeezing the cells by a fixed amount and measuring the corresponding change of the mode-frequency.

Finally we give a method of how to approach the field-flatness and desired mode-frequency in a few tuning-steps.

This method was used and found to be correct during three runs of the CERN 4-cell cavity (F_π = 500 MHz) between December 1980 and April 1981 [1,3].

The results are given in part II of this paper.

2. THE MODEL

Monitoring and tuning will involve first order perturbation calculations i.e. we will take up an approach used already earlier [4,5,6] Our model, however, eliminates two drawbacks of these attempts:

or full-end-cell cavities the models used so far have no flat π mode in their unperturbed state.

(b) Their $\pi/2$-mode frequency is independent of coupling which is not true for aperture coupled TM_{01} cavities.

The basic building bloc of a model with correct behaviour is, in analogy to a cavity half-cell, a half-section having a coupling condenser $C_k/2$ as shunt and a series resonator composed of L/2 and 2C as series element (fig. 2).

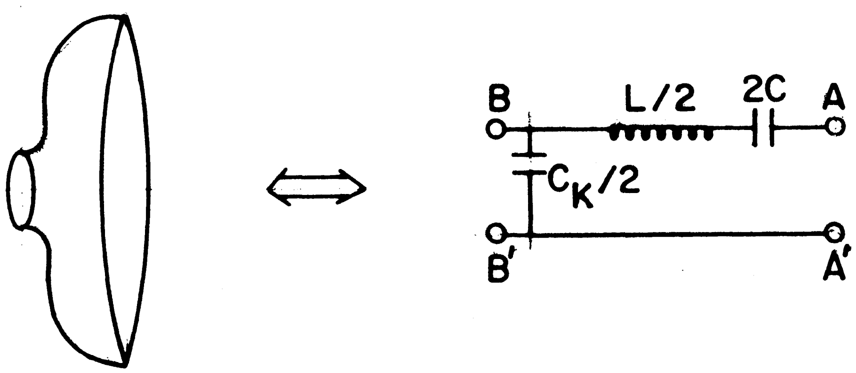

Fig. 2
Cavity half-shell and its equivalent model

Half-sections are then connected in the same way as half-cells would have been to form a cavity. Fig. 3 shows for example a section of 3-cells out of a multicell cavity:

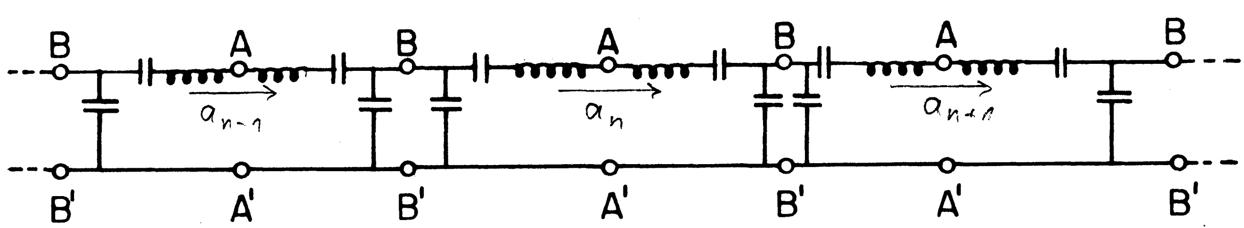

Fig. 3
Model of 3 full cells out of a multicell cavity

2.1 The chain of infinite length

To obtain the dispersion relation of such models we interprete fig. 3 as three meshes of an infinite chain with a travelling wave excited on it travelling from left to right. Then the current amplitudes in adjacent cells (losses here neglected!) may only differ by a phase angle θ: (θ is in cavity language the mode angle)

$$\frac{a_n}{a_{n-1}} = \frac{a_{n+1}}{a_n} = e^{-j\theta} \qquad (1)$$

On the other hand, if Y is the coupling admittance

$$Y = j\Omega\, C_k \qquad (2)$$

(Ω = ang. freq. = $2\,\pi\,F$; F are mode-frequencies, f cell-frequencies) and Z the resonator impedance

$$Z = j\,\Omega\,L + \frac{1}{j\,\Omega\,C} \qquad (3)$$

we find from Kirchhoffs laws

$$\frac{1}{Y}\,(a_n - a_{n-1}) + Z\,a_n - \frac{1}{Y}\,(a_{n+1} - a_n) = 0 \quad \rightarrow \qquad (4)$$

$$\frac{a_{n-1}}{a_n} + (2 + Z\,Y) - \frac{a_{n+1}}{a_n} = 0 \qquad (5)$$

Combining this equation with eq. 1 we obtain

$$2 + Z\,Y = e^{j\theta} + e^{-j\theta} = 2\cos\theta \qquad (6)$$

which expresses the dispersion relation for any ladder network with shunt and series elements Y and Z. Using now eqs 2 and 3 we find for the special case here the dispersion formula

$$\frac{\Omega^2}{\omega_0^2} = 1 + 2\,K(1 - \cos\theta) \qquad (7)$$

with

$$\omega_0^2 = \frac{1}{L\,C} \quad \text{and} \quad K = \frac{C}{C_k}$$

In the cavity language ω_0 is the resonance-frequency of one cell terminated by electric mirrors (metal plates) in the iris plane and K is the coupling coefficient between cells. Now the lowest possible frequency Ω corresponds to $\theta = \emptyset$ (zero mode frequency) and is independent of coupling as approximately found in aperture coupled TM_{01} mode cavities

$$\Omega_z^2 = \omega_0^2 = \frac{1}{L\ C}$$

The highest possible frequency corresponds to $\theta = \pi$ (π-mode frequency)

$$\Omega_\pi^2 = \omega_0^2\ (1 + 4\ K)$$

For small K we get as relative height of the dispersion step (relative band width)

$$\frac{\Omega_\pi - \Omega_z}{\Omega_z} \sim \frac{1}{2}\ \frac{\Omega_\pi^2 - \Omega_z^2}{\Omega_z^2} = 2\ K = \frac{C}{C_k/2}$$

2.2 Chains of finite length

According to eq. 1 all currents a_i are equal in the zero mode. Hence the coupling condensors are not charged, there is no voltage between points B and B' and connecting them does not perturb the zero mode.

In contrast for the π-mode no current passes through points B and B', and cutting the wire there will not perturb the π-mode. This offers two possibilities to delimit "full end cell" chains of finite length.

The first, terminated by short circuits between end points B and B' corresponds to a cavity delimited by electric mirros in the iris plan and may resonate in the zero mode but not in the π-mode. A cavity delimited by metal plates in the iris is evidently not suitable for particle acceleration.

The second, with open circuits at both ends, resonates in the π-but not in the zero mode. Finally we note, that connecting points A to A' perturbs neither π-nor zero mode and corresponds to cavities with half end cells.

Open circuits correspond to magnetic mirrors in the iris plane. Such objects can be realized in computer codes like SUPERFISH or LALA but in practice only be approximated by cut-off tubes. If used to terminate a TM_{01} cavity such tubes store predominantly electric energy and are consequently in first approximation modelled by a small condenser C_{tube} connected between terminating points B and B' as indicated in fig. 4

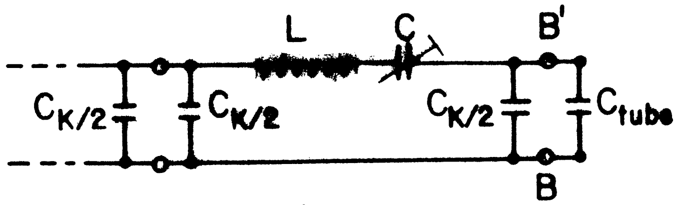

Fig. 4

Model for an end-cell with cut-off tube

The resulting series L-C circuits at the end of the chain are now tuned to slightly lower frequencies than in the ideal open circuited case, but evidently a small reajustment of the resonator condenser C (end cell correction) can re-establish the ideal conditions i.e. we treat a cavity with beam tubes and corrected end cells like a cavity with all cells equal and delimited by magnetic mirrors.

If we now apply Kirchhoffs laws to an open ended, N-mesh chain, fig. 3 being an example of a 3-mesh model, we obtain by adding the voltages around the first mesh

$$\left(\frac{3}{Y} + Z\right) a_1 - \frac{1}{Y} a_2 = 0 \tag{9}$$

In the same way for the last mesh

$$-\frac{1}{Y} a_{N-1} + \left(\frac{3}{Y} + Z\right) a_N = 0 \tag{10}$$

The remaining mesh equations being those of the infinite chain (eq. 4).

Multiplying all these N equations by $j\,\Omega\,C$ and introducing matrix notation we obtain the eigenvalue equation

$$\underline{\overset{0}{H}}\,|a\rangle = \overset{0}{\Omega}{}^2\,|a\rangle \tag{11}$$

Here is $|a\rangle$ a column vector having the resonator current amplitudes a_n as components. (We adopt the following notation: capital underlined letters are matrixes, small letters with brackets are either column vectors: $|p\rangle$ or row vectors: $\langle q|$; finally we write a scalar product as: $\langle q|p\rangle$)

$\underline{\overset{0}{H}}$ has the components:

$$\underline{\overset{0}{H}} = \begin{pmatrix} \omega_0^2(3\,K+1) & -\omega_0^2\,K & 0 \ldots\ldots & & \\ -\omega_0^2\,K & \omega_0^2(2\,K+1) & -\omega_0^2\,K & 0 \ldots\ldots & \\ 0 & -\omega_0^2\,K & \omega_0^2(2\,K+1) & -\omega_0\,K & 0 \ldots\ldots \\ \vdots & & & & \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots 0 & -\omega_0^2\,K & \omega_0^2(3\,K+1) \end{pmatrix}$$

Solving eq. 11 for its eigenvalues $\overset{0}{\Omega}{}^2_m$ we find the same functional relationship between Ω_m and θ_m as for the infinite chain but now with a constraint on the permitted values of the mode angles θ_m

$$\frac{\overset{0}{\Omega}{}^2_m}{\omega_0^2} = 1 + 2\,K\,(1-\cos\theta_m) \qquad (12)$$

$$\theta_m = m\,\frac{\pi}{N}\,; \quad m = 1,\ldots,N$$

and solving further for the N components of the N eigenvectors of eq. 11

$$\overset{0}{a}{}^m_n = \sin\,(2n-1)m\,\frac{\pi}{2N} \qquad (13)$$

$$n,m = 1,\ldots,N.$$

The N eigenvectors are orthogonal due to the fact that for this ideal case the system matrix $\underline{\overset{0}{H}}$ is symmetric.

For the real cavity in its untuned state different cells will have in general different frequencies ω_n.

The corresponding model will then have a matrix $\underline{H}$ where the ω_0 of $\underline{\overset{0}{H}}$ are in each line replaced by the appropriate ω_n with the result that symmetry of $\underline{H}$ and orthogonality of eigenvectors are no longer guaranteed.

Nevertheless, after numerical checks assuming a natural relative variation of 10^{-3} between cell frequencies ω_n we are confident, that first order perturbation theory, which preassumes orthogonality, can be applied with adequate precision.

3. FIRST ORDER PERTURBATION IN ω (*)

Let us assume that we apply a small perturbation to the untuned cavity changing the individual cell frequencies from ω_n^2 to ω'^2_n with

$$\omega'^2_n = \omega_n^2 \left(1 + \frac{\delta(\omega_n^2)}{\omega_n^2} \right) .$$

The cavities system matrix in this perturbed state is then to first approximation

$$\underline{H}' = \underline{H} + \delta\underline{H} = (\underline{1} + \underline{P})\, \underline{H}$$

with the diagonal perturbation matrix $\underline{P}$

$$\underline{P} = \begin{pmatrix} \ddots & & & 0 \\ & \frac{\delta(\omega_n^2)}{\omega_n^2} & & \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & & & 0 \\ & P_n & & \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix}$$

(*) In our case variations in K between different cells can be neglected as also verified experimentally. However, even with different individual K_n the formulas derived here remain correct, if one uses the corresponding unperturbed eigenvalues Ω^2 and eigenvectors $|a\rangle$. Nevertheless as communicated by [8] field-flatness problems may arise due to variations in K for equally pre-tuned cells.

Assuming now that the Ω^2_m and $|a^m\rangle$ of $\underline{H}$ are known, we can find the Ω'^2_m and $|a'_m\rangle$ of $\underline{H}'$ in using well established perturbation methods (see for example [7])

$$\frac{\Omega'^2_m}{\Omega^2_m} = 1 + \frac{\delta(\Omega^2_m)}{\Omega^2_m} = 1 + \frac{\langle a^m|\delta H|a^m\rangle}{\Omega^2_m \langle a^m|a^m\rangle} = 1 + \frac{\langle a^m|PH|a^m\rangle}{\Omega^2_m \langle a^m|a^m\rangle}$$

and using $\underline{H}|a^m\rangle = \Omega^2_m|a^m\rangle$ we obtain

$$\frac{\Omega'^2_m}{\Omega^2_m} = 1 + \frac{\langle a^m|P|a^m\rangle}{\langle a^m|a^m\rangle} \tag{14}$$

and

$$|a'^m\rangle = |a^m\rangle + |\delta a^m\rangle \text{ with}$$

$$|\delta a^m\rangle = \sum_{k\neq m} \frac{\langle a^k|P|a^m\rangle}{\langle a^k|a^k\rangle} \frac{\Omega^2_m}{\Omega^2_m-\Omega^2_k} |a^k\rangle \tag{15}$$

We assume in the following, that all eigenvectors have been normalized

$$\langle a^m|a^m\rangle = 1.$$

Written in components eqs 14 and 15 have then the form

$$\frac{\delta(\Omega^2_m)}{\Omega^2_m} = \sum_{n=1}^{N} (a^m_n)^2 \cdot \frac{\delta(\omega_n)^2}{\omega^2_n} = \sum_{n=1}^{N} S_{mn} \frac{\delta(\omega_n)^2}{\omega^2_n} \tag{14a}$$

$$\delta a^m_j = \sum_{n=1}^{N} T^{(m)}_{jn} \cdot \frac{\delta(\omega_n)^2}{\omega^2_n} \tag{15a}$$

$$T^{(m)}_{jn} = \sum_{k\neq m}^{N} a^k_n a^m_n a^k_j \frac{\Omega^2_m}{\Omega^2_m-\Omega^2_k} . \tag{16}$$

The differences of mode frequencies in the denominator of eq. 16 imply the well known fact that the sensitivity of a mode against perturbations increases with 1/K and especially for the π-mode with N^2.

3.1 Field-measurement

The system of eq. (14a) is evidently a special formulation of Slaters perturbation theorem stating that the reaction of a mode frequency to a perturbation is proportional to the relative change of the modes stored energy, which for each cell is in turn proportional to the square of the local amplitude a_n^m. We can now make use of this to measure in fact the $(a_n^m)^2$ of different cavity cells by squeezing them by equal amounts $\delta\ell/\ell$ i.e. applying equal $\delta\omega/\omega$ and noting the corresponding $\delta\Omega_m/\Omega_m$. Since we measured only the squares of the local amplitudes we have then to guess the right sign which for not too badly tuned cavities can be done by comparing with the ideal eigenvectors of eq. (13). We then calculate, after having normalised all vectors, the δa_j^m with respect to the ideally tuned case.

3.2 Tuning

The remaining task is now to calculate corrections to the cell lengths appropriate to improve the cavity tune. This can be done in making use of eqs. (15a) and (16) after elimination of the following difficulty: If we detune all cells of a cavity by the same small amount we obviously shift the whole dispersion curve without changing neither its shape nor the field distribution (eigenvectors) of the cavity modes: The matrix $\underline{T}^{(m)}$ transforms any vector $|\delta\omega/\omega\rangle$ with equal components into the zerovector and is therefore singular (proof in Appendix 1). We exclude this case in demanding that one cell, for example the n-th one, is never detuned ($\delta\omega_n = \emptyset$) i.e. we imagine that the present detuned state has been arrived at by touching only the remaining (N-1) cells. We calculate now the reverse corrections to these cells from their δa_n^m and the non singular (N-1)x(N-1) matrix obtained from $\underline{T}^{(m)}$ by leaving away the n-th line and column. This procedure will lead to an ideally tuned state but with a mode frequency Ω_m in general different from the desired one. Therefore we have to follow up by a homogenous deformation of all cells (all $\delta\omega$ equal) calculable from the m-th line of eq. (14a) to correct also the mode frequency.

In practice both steps can be finally merged in one by replacing one line in matrix $\underline{T}^{(m)}$ by the m-th line of eq. (14a). We get a nonsingular N x N - Matrix which allows, when inverted, to calculate from $\delta(\Omega_m^2)/\Omega_m^2$ and (N-1) measured δa_n^m, corrections to all cells, which both improve the tune, and shift Ω_m in the right direction.

Numerical examples calculated for a 4- and 5-cell cavity with a 1% dispersion step are given in Appendix 2.

Acknowledgement

We want to thank all our colleagues who have helped us to clarify the ideas represented here in many critical discussions.

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APPENDIX 1

The matrix $\underline{T}$ of eq. (16) is singular since

$$\begin{pmatrix} T^{(m)} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

which is equivalent to

$$\sum_{n=1}^{N} T^{(m)}_{jn} = 0 \text{ for all } j$$

Using the definition (16) of $\underline{T}$ we obtain

$$\sum_{n=1}^{N} T^{(m)}_{jn} = \sum_{n=1}^{N} \sum_{K \neq m}^{N} a^k_n \, a^m_n \, a^k_j \, \frac{\Omega^2_m}{\Omega^2_m - \Omega^2_k}$$

$$= \sum_{k \neq m}^{N} a^k_j \, \frac{\Omega^2_m}{\Omega^2_m - \Omega^2_k} \cdot \sum_{n=1}^{N} a^k_n \, a^m_n$$

The eigenvectors are orthogonal, therefore

$$\sum_{n=1}^{N} a^k_n \, a^m_n = \langle a^k | a^m \rangle = \delta_{km}$$

In the definition eq. (16) of $\underline{T}^{(m)}$ we have to sum over k with the exception of m, therefore δ_{km} is always equal to zero hence also the total sum.

APPENDIX 2

NUMERICAL EXAMPLES

The eigenvalues in all equations have been the squares of the angular frequencies. We measure, however, the frequencies themselves. Therefore we use the relation $\delta(\omega^2)/\omega^2 = \delta(f^2)/f^2 \sim 2\,\delta f/f$ valid for $\delta\omega \ll \omega$, which is always true for our linear approximation ($\omega \leftrightarrow f$; $\Omega \leftrightarrow F$).

We use the following definitions

$\lvert\delta a\rangle$	=	$\lvert a$ (measured)$\rangle$ - $\lvert a$ (ideal)$\rangle$
δF_m	=	F_m (measured) - F_m (ideal) (mode-frequencies)
$\delta f_n^{(*)}$	=	frequency-shift to be executed for cell n to approach the ideal situation.

For a 4- resp. 5-cell cavity with equivalent cells (end-cells corrected) with a total dispersion step of 1% (K = 0.005) we get in first order approximation the following relations (note that $\underline{S}$ of eq. (14a) is independent of K).

(*) The mechanical movement $\delta\ell/\ell$ to obtain a given $\delta f/f$ has to be calibrated separately, f.e. by stretching all cells by the same amount $\delta\ell$ and measuring $\delta F_\pi/F_\pi$, equal to $\delta f/f$ in that case.

4-cell

Eigenvectors (normalised)

$$\pi: \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{pmatrix} \quad 3/4\,\pi: \begin{pmatrix} -0.653 \\ 0.271 \\ 0.271 \\ -0.653 \end{pmatrix} \quad 1/2\,\pi: \begin{pmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix} \quad 1/4\,\pi: \begin{pmatrix} 0.271 \\ 0.653 \\ 0.653 \\ 0.271 \end{pmatrix}$$

We define $\underline{\tilde{T}}$ by $\underline{\tilde{T}} = -2\,\underline{T}^{(\pi)}$ so that

$$\begin{pmatrix} & & \\ & \tilde{T} & \\ & & \end{pmatrix} \begin{pmatrix} \delta f_1/f_1 \\ \vdots \\ \delta f_4/f_4 \end{pmatrix} = \begin{pmatrix} \delta a_1^{\pi} \\ \vdots \\ \delta a_4^{\pi} \end{pmatrix}$$

$$\underline{\tilde{T}} = \begin{pmatrix} -178.5 & -22.5 & 76.5 & 127.5 \\ 25.5 & 76.5 & -25.5 & -76.5 \\ 76.5 & 25.5 & -76.5 & -25.5 \\ -127.5 & -76.5 & 25.5 & 178.5 \end{pmatrix}$$

We define $\underline{\tilde{S}}$ by $\underline{\tilde{S}} = -\underline{S}$ which yields

$$\begin{pmatrix} & & \\ & \tilde{S} & \\ & & \end{pmatrix} \begin{pmatrix} \delta f_1/f_1 \\ \vdots \\ \delta f_4/f_4 \end{pmatrix} = \begin{pmatrix} \delta F_{\pi}/F_{\pi} \\ \vdots \\ \delta F_{\pi/4}/F_{\pi/4} \end{pmatrix}$$

$$\underline{\tilde{S}} = \begin{pmatrix} -.250 & -.250 & -.250 & -.250 \\ -.427 & -.073 & -.073 & -.427 \\ -.250 & -.250 & -.250 & -.250 \\ -.073 & -.427 & -.427 & -.073 \end{pmatrix}$$

$$\begin{pmatrix} & & \\ & (\tilde{S},\tilde{T})^{-1} & \\ & & \end{pmatrix} \begin{pmatrix} \delta F_{\pi}/F_{\pi} \\ \delta a_2 \\ \delta a_3 \\ \delta a_4 \end{pmatrix} = \begin{pmatrix} \delta f_1/f_1 \\ \vdots \\ \delta f_4/f_4 \end{pmatrix}$$

$$(\tilde{S},\tilde{T})^{-1} = \begin{pmatrix} -1. & -.9804\ 10^{-2} & .4902\ 10^{-2} & -.4902\ 10^{-2} \\ -1. & 1.4706\ 10^{-2} & 0 & +.4902\ 10^{-2} \\ -1. & -.4902\ 10^{-2} & -.9804\ 10^{-2} & -.4902\ 10^{-2} \\ -1. & 0 & .4902\ 10^{-2} & +.4902\ 10^{-2} \end{pmatrix}$$

5-cell

Eigenvectors (normalised)

$$
\begin{matrix} \pi & 4/5\,\pi & 3/5\,\pi & 2/5\,\pi & 1/5\,\pi \\
\begin{pmatrix} .447 \\ -.447 \\ .447 \\ -.447 \\ .447 \end{pmatrix} &
\begin{pmatrix} .602 \\ -.372 \\ 0 \\ +.372 \\ -.602 \end{pmatrix} &
\begin{pmatrix} -.512 \\ -.195 \\ +.632 \\ -.195 \\ -.512 \end{pmatrix} &
\begin{pmatrix} -.372 \\ -.602 \\ 0 \\ +.602 \\ +.372 \end{pmatrix} &
\begin{pmatrix} .195 \\ .512 \\ .632 \\ .512 \\ .195 \end{pmatrix}
\end{matrix}
$$

We define $\underline{\tilde{T}}$ by $\underline{\tilde{T}} = -\,2\,\underline{T}^{(\pi)}$ so that

$$
\begin{pmatrix} & & \\ & \tilde{T} & \\ & & \end{pmatrix}
\begin{pmatrix} \delta f_1/f_1 \\ \vdots \\ \delta f_5/f_5 \end{pmatrix}
=
\begin{pmatrix} \delta a_1^{\pi} \\ \vdots \\ \delta a_5^{\pi} \end{pmatrix}
$$

$$
\underline{\tilde{T}} = \begin{pmatrix}
-219.0 & -73.0 & +36.5 & +109.5 & +146.0 \\
+73.0 & +109.5 & 0 & -73.0 & -109.5 \\
+36.5 & 0 & -73.0 & 0 & +36.5 \\
-109.5 & -73.0 & 0 & +109.5 & +73.0 \\
+146.0 & +109.5 & +36.5 & -73.0 & -219.0
\end{pmatrix}
$$

We define $\underline{\tilde{S}}$ by $\underline{\tilde{S}} = -\,\underline{S}$ which yields

$$
\begin{pmatrix} & & \\ & \tilde{S} & \\ & & \end{pmatrix}
\begin{pmatrix} \delta f_1/f_1 \\ \vdots \\ \delta f_5/f_5 \end{pmatrix}
=
\begin{pmatrix} \delta F_{\pi}/F_{\pi} \\ \vdots \\ \delta F_{\pi/5}/F_{\pi/5} \end{pmatrix}
$$

$$
\underline{\tilde{S}} = \begin{pmatrix}
-.200 & -.200 & -.200 & -.200 & -.200 \\
-.362 & -.138 & 0 & -.138 & -.362 \\
-.262 & -.038 & -.40 & -.038 & -.262 \\
-.138 & -.362 & 0 & -.362 & -.138 \\
-.038 & -.262 & -.40 & -.262 & -.038
\end{pmatrix}
$$

$$
\begin{pmatrix} & & \\ & (\tilde{S},\tilde{T})^{-1} & \\ & & \end{pmatrix}
\begin{pmatrix} \delta F_{\pi}/F_{\pi} \\ \delta a_2 \\ \vdots \\ \delta a_5 \end{pmatrix}
=
\begin{pmatrix} \delta f_1/f_1 \\ \vdots \\ \delta f_5/f_5 \end{pmatrix}
$$

$$
(\tilde{S},\tilde{T})^{-1} = \begin{pmatrix}
-1. & -1.096\ 10^{-2} & +.548\ 10^{-2} & -.548\ 10^{-2} & +.548\ 10^{-2} \\
-1. & +1.644\ 10^{-2} & 0 & +.548\ 10^{-2} & -.548\ 10^{-2} \\
-1. & -.548\ 10^{-2} & -1.096\ 10^{-2} & -.548\ 10^{-2} & 0 \\
-1. & 0 & +.548\ 10^{-2} & +1.096\ 10^{-2} & +.548\ 10^{-2} \\
-1. & 0 & 0 & -.548\ 10^{-2} & -.548\ 10^{-2}
\end{pmatrix}
$$