Archives

Ref.TH.1259-CERN

QUASI-ELASTIC NEUTRINO-NUCLEUS INTERACTIONS

J.S. Bell
CERN - Geneva

and

C.H. Llewellyn Smith
SLAC - Stanford

A B S T R A C T

Calculations of the Pauli exclusion effect are made in a simple shell model. Shell structure effects, associated with spin-orbit splitting, appear, but are not large and decrease rapidly with increasing momentum transfer. The shell model remains less exclusive on the whole than the standard Fermi gas, but not sufficiently so to explain the apparent experimental absence of the exclusion effect.

Ref.TH.1259-CERN
30 November 1970

1. - INTRODUCTION

We have been led by two circumstances to look again at the theory of quasi-elastic neutrino reactions in nuclei. One is the apparent experimental absence [1)] of the expected Pauli exclusion effect. The other is the belief [2),3)] that this process must be an important background in the attempt [4)] to see the nuclear shadow effect [5),6)] for inelastic reactions.

The "quasi-elastic" reactions are those supposed to be initiated by an elastic collision with a single nucleon

$$\nu + n \longrightarrow \mu + p$$

Even to speak of such a class of events is already to picture the nucleus as a collection of almost free nucleons. The simplest version of this picture is the Fermi gas model, in which nucleon motion and the Pauli exclusion effect are allowed for [7),8)]. A good idea of the results of such calculations, for the differential cross-section with respect to angle integrated over secondary lepton energy, can be obtained by neglecting nucleon motion. Nucleon velocities are not very small $(v/c \lesssim 0.3)$ but there is some cancellation between nucleons moving in different directions, even when the exclusion effect distorts the region of integration. The result is then that the differential cross-section per neutron is that for a free neutron multiplied by an exclusion factor [*)]

$$1 - N^{-1} D$$

with

$$\begin{aligned} D &= Z && \text{for } 2x < u - v \\ &= \tfrac{1}{2} A \left\{ 1 - \tfrac{3x}{4}\left(u^2 + v^2\right) + \tfrac{x^3}{3} - \tfrac{3}{32x}\left(u^2 - v^2\right)^2 \right\} && \text{for } u - v < 2x < u + v \\ &= 0 && \text{for } 2x > u + v \qquad (1) \end{aligned}$$

---

*) For the symmetric case, $N = Z$, this formula was given by Gatto [9)]. The general formula was given by Berman [10)].

where

$$x = \left|\frac{\vec{q}}{2k_f}\right| \qquad u = \left|\left(\frac{2N}{A}\right)^{1/3}\right| \qquad v = \left|\left(\frac{2Z}{A}\right)^{1/3}\right|$$

(N,Z,A) ≡ (neutron, proton, nucleon) number

and $\vec{q}$ is the momentum transfer to a stationary free nucleon in secondary lepton production at the given angle.

Goulard and Primakoff [11] have gone some distance towards a more detailed account of finite nuclei in the "closure" approximation. However, in evaluating their formulae they assumed Wigner supermultiplet symmetry, and instead of using model wave functions made simple ansatzs for nuclear density distributions and correlation functions. Thus spin-orbit coupling and shell structure effects were excluded from the beginning. We therefore decided to investigate such effects by calculating with explicit shell model wave functions for some simple closed subshell, or almost closed subshell, configurations.

In Sections 2 and 3, the basic ideas, essentially those of Goulard and Primakoff, are set out. This leads us to consider three correlation functions (actually their Fourier transforms) which in this context have not usually been distinguished hitherto. Our shell model calculations of these quantities are described in Section 4. In Section 5 the bearing of data for the inverse reaction, nuclear muon capture, is considered. Some comparison with previous work is made in Section 6. Finally Section 7 gives some discussion and application to the two problems mentioned at the beginning.

It can be remarked here that similar considerations have a long history in connection with electron-nucleus interactions [*]. Calculations of the present kind might well have been made in that context, but as far as we know were in fact made [13] only for the particularly simple case of O^{16}. Experimental data of great relevance could also have been obtained in that context. But we know of very little [14)-19)], and of none at the energies and angles of interest in contemporary neutrino experiments.

---

*) See for example the review of de Forest and Walecka [12].

2. - PSEUDOPOTENTIAL

Consider the reaction

$$\nu(k) + A(p) \rightarrow \mu(k') + A'(p') \tag{2}$$

where A and A' are nuclear states ; four-momenta are indicated in brackets. The transition amplitude is given by the matrix element

$$\langle A'(p') \mid L_\mu J_\mu(0) \mid A(p) \rangle$$

where J_μ is the strangeness conserving hadronic weak current, and L_μ is a combination of lepton wave functions :

$$L_\mu = \ell_\mu (2k_0 2k_0')^{-1/2} \qquad \ell_\mu = (G_\beta/\sqrt{2})\,\bar{\mu}\, i\gamma_\mu (1+\gamma_5)\nu$$

where G_β is the strangeness conserving weak coupling :

$$G_\beta \approx 1.00 \; 10^{-5}$$

in units with

$$\hbar = c = M_N = 1$$

where M_N is nucleon mass.

The nuclear states will be described by the many-nucleon Schrödinger non-relativistic wave functions of conventional nuclear theory, and an appropriate form of J_μ must be given. A correct version would presumably include terms referring to more than one nucleon. If, however, only single nucleon terms are retained the general form can be written

$$L_\mu J_\mu(0) = \sum_n S(\vec{x}_n)\, \tau_n^+ T_n \tag{3}$$

where τ_n^+ increases by one unit the charge on the n'th nucleon, and T_n is some combination

$$T_n = T(\vec{\sigma}_n, \vec{K}'_n, \vec{K}_n) \tag{4}$$

of Pauli spin operators and differential operators, where $\vec{K}_n$ and $\vec{K}'_n$ denote the same differential operator

$$(1/i)\ \partial/\partial\vec{x}_n$$

envisaged, however, as standing to the right or left, respectively, of the δ function in (2). This inclusion of differential operators of arbitrary order allows the simulation of finite range and non-local effects.

For nuclear model calculations it is convenient to build momentum conservation into the matrix element ; in the usual way

$$(2\pi)^3\,\delta^3(\vec{q}+\vec{p}-\vec{p}\,')\,\langle A'(p')|\sum_n \delta(x_n)\,\tau_n^+ T_n\,|A(p)\rangle$$

$$= \langle A'(p')|\sum_n e^{i\vec{q}\cdot\vec{x}_n}\,\tau_n^+ T_n\,|A(p)\rangle$$

where the differential operators $\vec{K}'_n$ and $\vec{K}_n$ in T_n stand respectively to the left and right of the exponential. By partial integration one finds

$$\vec{K}' - \vec{K} \equiv \vec{q} \tag{5}$$

This can be used to eliminate $\vec{K}'$. We will regard initial nucleon motion as small, $\vec{K} \approx 0$, and then the operator character of T_n arises entirely from $\vec{\sigma}_n$.

The operator T can be expressed in terms of conventional form factors by considering the quasi-elastic reaction on a single neutron

$$\nu(k) + n(K) \rightarrow \mu(k') + p(K')$$

For this case

$$\langle K'|J_\mu(0)|K\rangle = \bar{p}\left(f_1 i\gamma_\mu - f_2 \sigma_{\mu\nu} i q_\nu + f_A i\gamma_\mu\gamma_5 - f_P q_\mu \gamma_5\right)n \qquad (6)$$

where in a standard representation the Dirac spinors p and n are related to Pauli spinors P and N by

$$n = \frac{1}{\sqrt{2K_0}}\begin{pmatrix}\sqrt{K_0+M_N}\;N \\ \dfrac{\vec{\sigma}\cdot\vec{K}}{\sqrt{K_0+M_N}}\;N\end{pmatrix} \qquad p = \frac{1}{\sqrt{2K_0'}}\begin{pmatrix}\sqrt{K_0'+M_N}\;P \\ \dfrac{\vec{\sigma}\cdot\vec{K}'}{\sqrt{K_0'+M_N}}\;P\end{pmatrix} \qquad (7)$$

In this way one finds

$$T(\vec{\sigma},\vec{K},\vec{K}') = \left(2k_0 2k_0' 2K_0 2K_0'\right)^{-1/2}\,\mathcal{J}(\vec{\sigma},\vec{K},\vec{K}') \qquad (8)$$

with

$$\mathcal{J}(\vec{\sigma},\vec{K},\vec{K}') = 2\left(1+\tfrac{1}{4}q^2\right)^{-1/2}\left[g_2\, i\vec{\sigma}\cdot\vec{q}\times\vec{\ell} - f_A\vec{\sigma}\cdot\vec{\ell}_t\left(1+\tfrac{1}{4}q^2\right) - g_1\ell_0\right] + O(\vec{K}) \qquad (9)$$

where terms which vanish with lepton mass have been ignored, terms which vanish with initial momentum $\vec{K}$ have not been written explicitly, and

$$(\vec{q}, iq_0) \equiv q_\mu = k_\mu - k'_\mu$$

$$(\vec{l}, il_0) \equiv l_\mu = \bar{\mu}(i\gamma_\mu)(1+\gamma_5)\nu \quad G_p/\sqrt{2}$$

$$\vec{l}_t = \vec{l} - |\vec{q}|^{-2}\, \vec{l}\cdot\vec{q}\; \vec{q}$$

$$g_1 = f_1 - \tfrac{1}{2}q^2 f_2 \qquad\qquad g_2 = f_2 + \tfrac{1}{2} f_1$$

The form factors f and g are functions only of the invariant q^2.

3. - <u>CLOSURE APPROXIMATION</u>

The differential cross-section for secondary lepton production in a direction specified by the unit vector $\hat{k}'$ and in an element of solid angle $d\Omega$ is given by

$$(2\pi)^3 \delta^3(\vec{P}'' - \vec{P}) \frac{d\sigma}{d\Omega} = \left(\frac{1}{2\pi}\right)^2 \Sigma \overline{\Sigma}$$

$$\Theta(k_0') \, k_0'^2 \langle A(P'') | O^+(\vec{q}) | A'(P') \rangle \langle A'(P') | O(\vec{q}) | A(P) \rangle \qquad (10)$$

where Θ is the step function, $\Sigma \overline{\Sigma}$ denotes summation over final states and averaging over initial nuclear spins, and

$$O(\vec{q}) = \sum_n e^{i\vec{q}\cdot x_n} \tau_n^+ T_n \qquad (11)$$

$$k_0' = k_0 + P_0 - P_0' \qquad (12)$$

$$\vec{q} = \vec{k} - k_0' \hat{k}' \qquad (13)$$

If we could neglect $P_0 - P_0'$ in (12) and (13), then $\vec{q}$ and k_0' in (10) would not depend on the final excited state A', and the summation over excited states could be performed immediately by closure, yielding a matrix element of

$$O^+(\vec{q}) \; O(\vec{q})$$

Instead of neglecting $P_0 - P_0'$ entirely we will calculate it regarding the nucleus as a collection of almost free slowly moving nucleons. For small $\vec{q}$ this does not make much difference ; however, it is a more plausible (although not necessarily correct) procedure for large $\vec{q}$. For such a system of free particles the summation over final states would include an integral over momentum

$$\int d^3 \vec{K}' \, \delta^3(\vec{K}' - \vec{K} - \vec{q})$$

Some care must be taken because $\vec{q}$ depends on $\vec{P}'$ through (13) and (12). Taking Cartesian co-ordinates with $\hat{k}'$ as z axis

$$\begin{aligned} \int d^3 \vec{K}' \, \delta^3(\vec{K}' - \vec{K} - \vec{q}) &= (1 - \partial K_0'/\partial K_3')^{-1} \\ &\simeq (1 - K_3'/K_0')^{-1} = - k_0' K_0'/k' \cdot K' = - k_0' K_0'/k \cdot K \\ &= k_0 K_0 \, k_0' K_0' \, / (M^2 k_0^2) \end{aligned} \tag{14}$$

when we neglect initial nucleon momentum $\vec{K}$. Because of the appearance of these factors the result is more neatly expressed in terms of $\mathcal{J}$ rather than T [see Eq. (8)]. In this way we arrive at

$$(2\pi)^3 \delta^3(\vec{P} - \vec{P}') \frac{d\sigma}{d\Omega} = \frac{1}{64} \frac{1}{\pi^2} \left(\frac{k_0'}{k_0}\right)^2 \Sigma \bar{\Sigma}$$

$$\langle A(P'') | \sum_n \tau_n \mathcal{J}_n^+ e^{-i\vec{q}\cdot\vec{x}_n} \sum_m e^{i\vec{q}\cdot\vec{x}_m} \tau_m^+ \mathcal{J}_m | A(P) \rangle \tag{15}$$

in which the first summation Σ is only over final lepton spin.

For a state of momentum $\vec{P}$ the non-relativistic many-body wave function has the form

$$e^{i\vec{P}\cdot\vec{R}} \quad \Phi(\vec{x}_1 - \vec{x}_2, \; \cdots)$$

where $\vec{R}$ is the centre-of-mass co-ordinate and ϕ does not depend on $\vec{R}$ or $\vec{P}$. It is the integration over $\vec{R}$ which gives the momentum conserving δ function on the left-hand side of (15). It is convenient in nuclear model calculations to use instead wave functions in which the centre-of-mass is assigned a fictitious bounded motion

$$\Psi = \phi(\vec{R}) \; \Phi(\vec{x}_1 - \vec{x}_2, \; ---)$$

where ϕ is some $\vec{P}$ independent normalized wave function. The model wave functions that we will use are of this form, exactly [20),21)]. Since the operator in (15) involves only relative co-ordinates, $x_n - x_m$, the only effect of using these model wave functions, rather than the momentum eigenstates, is the disappearance of the δ function.

So finally

$$\frac{d\sigma}{d\Omega} = \frac{1}{64} \frac{1}{\pi^2} \frac{1}{M_N^2} \left(\frac{k_0'}{k_0}\right)^2 \bar{\sum}\sum \left(\Psi, \left[\sum_{nm} \tau_n \mathcal{T}_n^+ e^{-i\vec{q}\cdot(\vec{x}_n - \vec{x}_m)} \mathcal{T}_m \tau_m^+\right] \Psi\right)$$

with

$$(k_0'/k_0) = \left(1 + 2 k_0 \sin^2 \tfrac{1}{2}\theta\right)^{-1} \tag{16}$$

as calculated for free stationary nucleons.

The operator $\mathcal{T}$ in (16) has the form [Eq. (9)]

$$\mathcal{T} = \alpha + \beta_x \sigma_x + \beta_y \sigma_y + \beta_z \sigma_z \tag{17}$$

taking the z axis along $\vec{q}$, where the coefficients α and β (neglecting initial nucleon momenta) depend only on q_μ and ℓ_μ. With nuclear spins averaged over it follows from rotation and reflection symmetry about the axis $\vec{q}$ that there are no cross contributions between the four terms in (17). So we require only the squares of these coefficients ; from (9)

$$|\beta_z|^2 = 0 \qquad |\alpha|^2 = 4\left(1+\tfrac{1}{4}q^2\right)^{-1} g_1^2\, \ell_0^* \ell_0$$
$$|\beta_x|^2 + |\beta_y|^2 = 4\left(1+\tfrac{1}{4}q^2\right)^{-1} \left| g_2 \vec{\ell} \times i\vec{q} + f_A\left(1+\tfrac{1}{4}q^2\right)\vec{\ell}_t\right|^2 \tag{18}$$

The lepton wave function factors can be obtained for example from the covariant trace

$$\bar{\ell}_\mu \ell_\nu = 8\left(k_\mu k'_\nu + k'_\mu k_\nu - \delta_{\mu\nu} k'\cdot k + \epsilon_{\mu\nu\rho\sigma} k'_\rho k_\sigma\right)\left(G_\beta^2/2\right) \tag{19}$$

whence

$$\left.\begin{aligned} \ell_0^* \ell_0 &= 2G_\beta^2\left\{(k_0+k_0')^2 - q^2\left(1+\tfrac{1}{4}q^2\right)\right\} \\ \vec{\ell}_t^{\,*}\cdot\vec{\ell}_t &= 2G_\beta^2\left\{(k_0+k_0')^2/\left(1+\tfrac{1}{4}q^2\right) + q^2\right\} \\ \mathrm{Re}\,\vec{\ell}^{\,*}\cdot\vec{\ell}\times i\vec{q} &= 4G_\beta^2(k_0+k_0')\,q^2 \end{aligned}\right\} \tag{20}$$

Then

$$|\alpha|^2 = 8G_\beta^2\left\{\frac{(k_0+k_0')^2}{1+\frac{1}{4}q^2} - q^2\right\} g_1^2$$
$$|\beta_x|^2+|\beta_y|^2 = 8G_\beta^2\left\{\left(\frac{(k_0+k_0')^2}{1+\frac{1}{4}q^2}+q^2\right)\left(g_2^2q^2 + f_A^2\left(1+\tfrac{1}{4}q^2\right)\right) + 4g_2 f_A q^2(k_0+k_0')\right\} \tag{21}$$

Returning to (16), note that the $n=m$ terms are trivial, because

$$\tau_n^-\tau_n^+ = \tfrac{1}{2}(1-\tau_3)_n\,, \qquad \sigma_{xn}\sigma_{xn} = 1\,, \quad \text{etc.}$$

The $n \neq m$ terms can be expressed in terms of three functions of $\vec{q}^{\,2}$

$$D_{S,T,L} = -\left(\Psi, \left\{\sum_{n\neq m} e^{i\vec{q}\cdot(\vec{x}_n-\vec{x}_m)}\,\tau_n^-\tau_m^+\left(1, \sigma_{xn}\sigma_{xm}, \sigma_{zn}\sigma_{zm}\right)\right\}\Psi\right) \tag{22}$$

The result is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64}\frac{1}{\pi^2}\frac{1}{M^2}\left(\frac{k_0'}{k_0}\right)^2 \left\{ |\alpha|^2 (N - D_s) + \left(|\beta_x|^2 + |\beta_y|^2\right)\left(N - D_T\right) \right\} \tag{23}$$

For the antineutrino reaction

$$\bar{\nu} + A \rightarrow \bar{\mu} + A'$$

the $g_2 f_A$ term in (21) changes sign and in (23) N is replaced by Z.

4. - SHELL MODEL CALCULATIONS

We have calculated the correlation functions D, for a number of cases, in the nuclear shell model with oscillator wave functions.

In the simplest cases the nuclear ground state is represented by a single determinantal wave function. Then the two-body operator in (22) can give rise to the usual direct and exchange contributions. Here the direct contributions are zero, because of the charge-exchange operators $\Upsilon^{\pm}$. From the exchange terms

$$D_{S,T,L} = \sum_{\phi,\psi} (\psi | e^{-i\vec{q}\cdot\vec{x}} (1, \sigma_x, \sigma_z) | \phi)(\phi | e^{i\vec{q}\cdot\vec{x}} (1, \sigma_x, \sigma_z) | \psi) \qquad (24)$$

where the single particle matrix elements involve the spin-orbit states ϕ of protons and ψ of neutrons, and the summation is over all pairs of such occupied states.

In more complicated cases the wave function is a superposition of several determinants. Then as well as several contributions of the above type there can be cross terms between different members of the superposition. The only case of this kind that we will consider is that of Fe^{56}. The simplest shell model description involves two neutrons outside the closed neutron subshells, and two protons missing from the top proton subshell. We will assume the simplest coupling scheme, in which the two neutrons are separately coupled to zero angular momentum, and so have always opposite magnetic quantum numbers, and likewise for the two proton holes. Then a transition from one member of the superposition to another would require changing the states of two neutrons, or of two protons, or of two neutrons and two protons. But the operator in (22) can change the state of only one neutron and one proton. So here there are no cross terms. We have only to average over a number of sums of the type (24), which amounts to giving appropriate fractional weights in a single such sum to those terms arising from incomplete subshells.

With oscillator wave functions it is a mechanical matter to evaluate the matrix elements and perform the summations, and this has been done by computer. One version of the programme uses directly the single particle states labelled by quantum numbers (n, ℓ, j, m), where n specifies oscillator energy $(n = 1, 2, \ldots)$, ℓ specifies orbital angular momentum $(\ell = n-1, n-3, -1 \text{ or } 0)$, j specifies total angular momentum $(j = \ell + \frac{1}{2}, \text{ or } [\text{if } \ell > 0] \ \ell - \frac{1}{2})$, and m is the projection of total angular momentum on the z axis. The matrix elements are then dependent on Clebsch-Gordan coefficients, and the programme uses an available subroutine (written by H. Yoshiki) for evaluating these coefficients. A second version of the programme uses states classified by three Cartesian oscillator energies (n_x, n_y, n_z) and uses projection operators to project onto (n, ℓ, j) subshells. There are obvious sum rules (used to check and then to speed up the programme) which indicate that these projections can be omitted when summing over a complete shell.

The oscillator ground state wave function having the form

$$\psi \propto e^{-\frac{1}{2} r^2/b^2} \tag{25}$$

(which defines the conventional size parameter b) the quantities $D_{S,T,L}$ are each of the form

$$e^{-\frac{1}{2} b^2 |\vec{q}|^2} \left\{ C(0) + C(1)\left(\frac{|\vec{q}|^2 b^2}{2}\right) + C(2)\left(\frac{|\vec{q}|^2 b^2}{2}\right)^2 + \cdots \right\} \tag{26}$$

The numerical values of the C's are listed in Table I for a number of cases. They are quoted to four figures, but of course are available in the machine to much higher accuracy. They are in fact rational numbers, but we have not taken the extra trouble required to obtain them in this form. The configurations adopted are just the simplest possible. For the $N = Z$ nuclei both neutrons and protons are assigned the following closed subshells :

He^4	$2(1s_{1/2})$						
C^{12}	"	$4(2p_{3/2})$					
O^{16}	"	"	$2(2p_{1/2})$				
Si^{28}	"	"	"	$6(3d_{5/2})$			
Ca^{40}	"	"	"	"	$4(3d_{3/2})$	$2(2s_{1/2})$	
28^{56}	"	"	"	"	"	"	$8(4f_{7/2})$

The configuration used for Fe^{56} comes from that of the hypothetical doubly magic 28^{56} (for which also results are quoted) by adding two neutrons (coupled to zero angular momentum) in the next $p_{\frac{3}{2}}$ subshell and removing two protons (coupled to zero angular momentum) from the $f_{\frac{7}{2}}$ subshell.

For He^4, O^{16} and Ca^{40}, the T and L coefficients do not differ from the S. This is the case [22] when all the (n, ℓ) subshells of either neutrons or protons are closed. The summation over either ψ or ϕ in (4) is then complete with respect to spin, and one of the σ's can be commuted through and multiplied directly with the other, giving unity. In general the three sets of coefficients are different. It will be noted that in all cases

$$C_s(0) = Z$$

This is because we have used the same spin-orbitals for protons and neutrons ; it is clear from (4) that D_S then reduces for $q = 0$ to the number of common states. The values of $C_T(0)$ and $C_L(0)$ are also readily calculated. They must be equal by rotational symmetry (i.e., $D_T = D_L$ at $q = 0$). For $q = 0$, ψ and ϕ in (4) can be restricted to have the same (n, ℓ) values. When both subshells, $j = \ell \pm \frac{1}{2}$, are fully occupied for either protons or neutrons (or both empty for either) the σ's in (4) can be eliminated, as already remarked. When the $j = \ell - \frac{1}{2}$ subshell is empty and the $j = \ell + \frac{1}{2}$ is full, the difference from the complete (n, ℓ) subshell is readily incorporated by using projection operators. Thus the contribution

$$2(2\ell+1)$$

of the full (n, ℓ) subshell is reduced by a factor which is the expectation value in the orbital state of

$$\frac{1}{2}\,\mathrm{Trace}\;\frac{\vec{\sigma}\cdot\vec{L}+\ell+1}{2\ell+1}\,(1, \sigma_x)\,\frac{\vec{\sigma}\cdot\vec{L}+\ell+1}{2\ell+1}\,(1, \sigma_x)$$

where the trace is with respect to spin, $\vec{L}$ is the orbital angular momentum operator, the 1 is for S and the σ_x for T or L. This reduces to

$$\frac{\ell(\ell+1)+(\ell+1)^2}{(2\ell+1)^2}\,, \quad \frac{-\frac{1}{3}(\ell+1)\ell+(\ell+1)^2}{(2\ell+1)^2}$$

on using

$$L_x^2 \equiv L_y^2 \equiv L_z^2 \equiv \frac{1}{3}\vec{L}^2 = \frac{1}{3}\ell(\ell+1)$$

So

$$C_S(0) - C_{L,T}(0) = \frac{4}{3}\,\frac{\ell(\ell+1)}{(2\ell+1)^2} \times 2(2\ell+1) \tag{27}$$

where ℓ is the value for the incomplete (n, ℓ) subshell. With $\ell = 1,2,3$ for Carbon, Silicon and 28^{56}, this reproduces the results quoted in the Table. Going from 28^{56} to Fe^{56} we have to reduce the difference (27) by a factor (6/8) because two of the relevant eight protons are removed.

Another regularity in the Table is the systematic vanishing when $N = Z$ of the coefficient $C(1)$ in the scalar case. This coefficient is simply related to the "unretarded dipole" term given especial attention by Foldy and Walecka [23]. It can be computed separately as follows. The definition

$$Z - D_s = \langle A | \sum \tau^- e^{-i\vec{q}\cdot\vec{x}} \sum \tau^+ e^{i\vec{q}\cdot\vec{x}} | A \rangle$$

is equivalent [24], in the isosymmetric case, to

$$Z - D_s = \tfrac{1}{2} \langle A | \sum \tau_3 e^{-i\vec{q}\cdot\vec{x}} \sum \tau_3 e^{i\vec{q}\cdot\vec{x}} | A \rangle$$

We can then write for the term in D_s of second order in q

$$\begin{aligned} & -\tfrac{1}{2} \langle A | \sum \tau_3 \vec{q}\cdot\vec{x} \sum \tau_3 \vec{q}\cdot\vec{x} | A \rangle \\ & = -\tfrac{1}{2} Z^2 \langle A | (\vec{q}\cdot\vec{X}_n - \vec{q}\cdot\vec{X}_p)^2 | A \rangle \end{aligned} \qquad (28)$$

where X_n and X_p are centre-of-mass co-ordinates of protons and neutrons. Now for these simple wave functions the relative centre-of-mass motion has an oscillator ground state wave function proportional to

$$\exp\left(- \frac{Z}{4b^2} (\vec{X}_n - \vec{X}_p)^2 \right)$$

so that (28) is equal to

$$Z \left(- \tfrac{1}{2} b^2 |\vec{q}|^2 \right)$$

Since $C(0) = Z$, this is completely accounted for by the expansion of the exponential in (26), so that indeed $C(1) = 0$.

The results are displayed graphically in the figures, where the functions

$$1 - N^{-1} D_{S,T,L}$$

- i.e., the reduction factors for neutrons in nuclei as compared with free neutrons - are plotted against momentum transfer squared $\vec{q}^2$ in $(\mathrm{GeV})^2$. The scale parameters b are chosen to reproduce empirical root mean square charge radii through the relation

$$\langle r^2 \rangle_{charge} = \left[\cdot 64 + \tfrac{1}{2} b^2 \left\{ Z^{-1} (3Z_1 + 5Z_2 + 7Z_3 + \cdots) - 3A^{-1} \right\} \right] 10^{-26} \, \mathrm{cm}^2$$

This allows for finite proton (but not neutron) mean square charge radius (0.64) and for centre-of-mass motion $(3A^{-1})$; Z_1, Z_2, Z_3, etc., are the numbers of protons in the 1st, 2nd, 3rd, etc., oscillator shells [i.e., summing over quantum numbers (ℓ, j, m) for given oscillator energy]. The empirical values used are displayed in Table II.

In Fig. 1 is shown for the closed shell nuclei He^4, O^{16}, Ca^{40}, (for which S, T and L factors are equal) the corresponding exclusion factors, and for comparison the Fermi gas factor (1). In Figs. 2, 3 and 4, the three different factors are shown for each of the spin non-saturated nuclei C^{12}, Si^{28} and Fe^{56}, and again for comparison the corresponding Fermi gas factors (1).

5. - MUON CAPTURE

For muon capture

$$\mu^- + A \rightarrow \nu + A'$$

the roles of proton and neutron are interchanged as compared with the neutrino induced reaction (1). Then in (23) N is replaced by Z. The theory of total capture rates in nuclei, in closure approximation, has been developed especially by Primakoff [26] and by Luyten, Rood and Tolhoek [22]. It is convenient for us to refer to a short summary given elsewhere [24]. Translating formula (1) of that paper into the present notation, the total capture rate is

$$\Lambda = \frac{\nu^2}{2\pi} |\phi|^2_{av} \left\{ G_V^2 + 2 G_A^2 + (G_A - G_P)^2 \right\}$$
$$\times \left\{ a[Z - D_S] + b(Z - D_T) + c(Z - D_L) \right\} \qquad (29)$$

where ν is some suitable average neutrino energy, the D's are to be evaluated for momentum transfer $|\vec{q}| = \nu$, $|\phi|^2_{av}$ is the muon probability density averaged over the nucleus, and

$$G_V = 1.01\, G_\beta \qquad G_A = -1.38\, G_\beta \qquad G_P = -0.58\, G_\beta$$

$$a = G_V^2 / (G_V^2 + 2 G_A^2 + (G_A - G_P)^2) = .19$$
$$b = 2 G_A^2 / (G_V^2 + 2 G_A^2 + (G_A - G_P)^2) = .69$$
$$c = (G_P - G_A)^2 / (G_V^2 + 2 G_A^2 + (G_A - G_P)^2) = .12$$

and

$$G_\beta = 1.00 \times 10^{-5} \; M_N^{-2}$$

(where M_N is nucleon mass) is the $\Delta S = 0$ weak coupling constant.

The approximations involved in (29) are :

(a) neglect of other than single body terms in the effective interaction ;

(b) the closure approximation ; unfortunately the result is very sensitive to the choice of average neutrino energy ν, varying with a power between the third and fourth ; we will quote results for

$$\nu = 80 \text{ MeV}$$

as originally proposed by Primakoff ;

(c) neglect of initial nucleon velocity ; estimates [22),27)] of the relevant correction terms suggest that (1) should be increased by some 10% ; this is not included in the theoretical values of Table III ;

(d) replacement of the muon wave function by a value constant over the nucleus ; this seems [28)] to be quite a good approximation ; it is convenient to introduce a quantity Z_{eff} such that

$$\pi Z |\phi|^2_{av} = m_\mu^3 \alpha^3 Z^4_{eff}$$

where m_μ is muon mass and α the fine structure constant. We will use values of Z_{eff} computed by Ford and Wills [29)].

It is convenient to quote reduced capture rates Λ_r defined by

$$\Lambda = \{G_V^2 + 2G_A^2 + (G_A - G_P)^2\}\{m_\mu^2 (m_\mu \alpha)^3/(2\pi)^2\} Z^4_{eff} \Lambda_r$$

$$= (278 \text{ sec}^{-1}) Z^4_{eff} \Lambda_r$$

Some experimental values are quoted in Table III. They are computed from Λ's listed by Eckhause et al. [30]. The error given in the Table arises only from the quoted experimental error for Λ and does not reflect, for example, any uncertainty in the calculation of Z_{eff} [28].

According to the theory

$$\Lambda_r = \left(\frac{\nu}{m_\mu}\right)^2 \left\{ a(1-Z^{-1}D_S) + b(1-Z^{-1}D_T) + c(1-Z^{-1}D_L) \right\} \tag{30}$$

The values resulting from our shell model calculations are listed in Table III.

Comparing the first and second columns in Table III reasonable agreement is found for the really closed shell nuclei He^4, O^{16} and Ca^{40}, but poor agreement for the intermediate cases C^{12}, Si^{28} and Fe^{56}. In the third column therefore we give the results of discarding our functions D_T and D_L and using everywhere the shell model D_S. This is clearly an improvement. There is here a clear suggestion that the very simplest shell model wave functions exaggerate greatly the effect of the spin orbit splitting in destroying the Wigner supermultiplet symmetry.

Even for the three closed shell nuclei it is likely that the degree of agreement with experiment is more than the theory deserves. It is known [22] that without the closure approximation the oscillator shell model gives much too large capture rates. The situation has been improved [31)-34] by invoking collective motion notions which lead to important modifications of both excitation energies and matrix elements. These notions are not immediately applicable for the higher momentum transfers of interest in neutrino reactions. Even for the smaller momentum transfers different authors disagree [35] about just how the ingredients should be combined in obtaining the over-all agreement with experiment.

6. - COMPARISON WITH OTHER WORK

B. Goulard and H. Primakoff [11)]

Our work is very close to that of Goulard and Primakoff. There are two main differences. The first is that they assume from the outset on the basis of Wigner supermultiplet symmetry in which spin dependent forces are ignored that to a sufficient approximation

$$D_S = D_T = D_L \tag{31}$$

We have tested this in our model, but have not been able to improve on it. The second difference is that instead of calculating with model wave functions they directly assume simple forms for nuclear density distributions and correlation functions. In the case of Fe^{56} we show in Fig. 5, as well as our shell model D_s curve and the Fermi gas curve, the exclusion factor calculated from the prescriptions of Primakoff and Goulard. The differences between the three curves are not very significant. The Primakoff-Goulard curve has more structure than the others ; this is presumably because their construction of the relevant correlation function produces a long-range [36),37)] as well as a short-range part, and because they have given (for simplicity) sharp boundaries to both parts.

When the relation (31) is assumed, the quasi-elastic scattering by a nucleus is in the case of forward scattering proportional to $N - D_s(0)$. In all our examples this had the value $N-Z$, so that

$$\left[\frac{d\sigma}{d\Omega}(\text{nucleus})\Big/\frac{d\sigma}{d\Omega}(\text{nucleon})\right]_{\theta=0} = N - Z \tag{32}$$

Goulard and Primakoff give a general argument for this result. It suffices to assume that the weak charge is identical with the isobaric spin. Then in closure approximation we are concerned with the matrix element

$$\langle A | \int d\vec{x}\, J_0^-(\vec{x}) \int d\vec{y}\, J_0^+(\vec{y}) | A \rangle = \langle A | T^- T^+ | A \rangle$$

where T^+ and T^- are isobaric raising and lowering operators. With the usual assumption that the nuclear ground state is the least charged member of isomultiplet,

$$T^- | A \rangle = 0$$

Then

$$\langle A | T^- T^+ | A \rangle = \langle A | [T^-, T^+] | A \rangle = -\langle A | T_3 | A \rangle = N - Z$$

whence

$$N - D_S(0) = N - Z \tag{33}$$

So the result (32), while dependent on the closure approximation, does not depend on the neglect of correlations in the wave function and many-body terms in the pseudopotential.

J. Frazier, C.W. Kim and M. Ram [38]

These authors had the idea of inferring an exclusion effect in forward quasi-elastic neutrino reactions from the observed effect in muon capture not just qualitatively but in a precise and rather model independent way - always in the context of the closure approximation. There are two essential difficulties in relating the two processes :

(a) they involve different combinations of the functions $D_{S,T,L}$;

(b) they involve different momentum transfers.

Thus some theory must be invoked in the comparison, and for this the authors turn to Primakoff's original treatment of muon capture [26]. Here the relation (31) is assumed, which disposes of difficulty (a). Then it can be argued that in a certain approximation [26),24)]

$$Z - D_s(\nu) = Z\left(1 - \frac{N}{2A}\delta\right) \tag{34}$$

where δ is a constant and ν the momentum transfer of muon capture. Together with (33), this gives

$$N - D_s(0) = \frac{N-Z}{Z\left(1-\frac{N}{2A}\delta\right)}\left(Z - D_s(\nu)\right) \tag{35}$$

Using empirical values of $Z - D_S(\nu)$ from muon capture one can then predict values of $N - D_S(0)$ and so of forward neutrino reactions. In so far as (34) fits the muon capture data (and it does so quite surprisingly well [36),24),37)]) one comes back essentially to (32), and this is what Frazier, Kim and Ram find. Actually their argument is a little more complicated than this, but we believe that it is essentially in this way that they overcome the difficulties (a) and (b). It seems to us that (32) can then be regarded as more securely based than before only if (35) is regarded as more securely based than (33), which gives (32) directly without appeal to capture data. We do not take this view, having seen (33) to follow from rather general arguments.

A. Bogan [28)]

Extensive calculations have been done by Bogan [28)] with shell model wave functions based on an infinitely deep square well potential. He assumed the relation (31) at the outset and calculated only D_S.

We would certainly expect the square well potential to be more relevant than the parabolic for heavy nuclei. Bogan finds (his Fig. 1) that for Pb the square well exclusion curve is rather close to the asymmetric Fermi gas curve. We have used the latter in some estimates for Pb referred to in the next Section.

Bogan gives numerical results only for the momentum transfer (he takes 80 MeV/c) of muon capture. In this connection a comparison of oscillator and square well potentials was already made by Luyten, Rood and Tolhoek [22)] for O^{16} and Ca^{40} ; they found no great difference. This remains so for the other light nuclei that we have considered, as set out in Table IV. The last column gives the square well results of Bogan (from his Table 2) and the penultimate column gives our oscillator values. The latter, for this purpose, were recalculated with Bogan's size prescription : he neglected centre-of-mass motion and proton size, and took for the mean square radius of the proton distribution

$$\langle r^2 \rangle^{1/2} = (3/5)^{1/2} \left(1.123\, A^{1/3} + 2.352\, A^{-1/3} - 2.070\, A^{-1} \right)$$

This is rather rough for He^4. In column 3 of the Table we show the values consequent on the b's of our Table II, as used in Section 5. Also, for general interest, we quote the Fermi gas values ($k_f = 268$ MeV) with and without the Primakoff linear approximation.

Bogan gave particular attention to the value, for N = Z nuclei, of the expansion parameter a in

$$1 - N^{-1} D_S(|\vec{q}|) = a|\vec{q}|^2 + O(|\vec{q}|^4) + \cdots \qquad (36)$$

It was observed by Foldy and Walecka [23] that a can be related to the bremsstrahlung weighted photonuclear reaction cross-section, and that an empirical formula of Levinger [39] then implies

$$a = .416\ A^{1/3}\ (\text{fermi})^2 \tag{37}$$

Bogan found that his wave functions gave a value close to this, at least for O^{16}. With the parabolic well, as remarked in Section 4,

$$a = b^2/2$$

We compare in Table IV the values obtained with the b's of Table II, and from the Foldy-Walecka formula (37). The agreement is not good. To whatever extent formula (37) represents the data even for these light nuclei we would have here another failure of the simple parabolic well shell model. In view of the agreement between columns 3, 4 and 5 of Table IV (setting aside He^4) we would be surprised if the square well version were very different in this respect. We believe that Bogan made a slip in this connection, using different values of k_f in arriving at his values (12) and (13), which are then not directly comparable.

C.A. Piketty [40),41)] ; J. Løvseth [42] ; T.W. Donnelly [43]

These authors have presented calculations, for finite nuclei, in which the closure approximation is not made. Piketty (see also Piketty and Orkin-Lecourtois [41]), allowing the recoil nucleon to move in a complex potential well, was interested in the direct ejection of fast protons - which is only a subset of the processes with which we are concerned. Donnelly made the recoil nucleon move in a real potential well, i.e., ignored absorption ;

it would be interesting to have the relevant integrals of his results for comparison with the closure approximation. Løvseth followed his former Fermi gas calculation [7] but with a momentum distribution more appropriate to finite nuclei ; but then, as he remarks, it is ambiguous what to do about the exclusion effect, which was our main concern here.

7. - DISCUSSION AND APPLICATION

We have seen that nuclear shell structure effects do appear. But they are not large even for small $|\vec{q}|$, and muon capture data suggest that the effects will be smaller in reality than in our model. Moreover these effects disappear quite rapidly with increasing $\vec{q}$, and so even if real will not be easy to see.

One place where such effects might play a role, and it was one of the motivations of this work, is in the small angle experiment of Borer et al. [4]. This experiment was designed to look for a possible nuclear shadow effect in inelastic reactions, by comparing nuclei of different sizes as inelastic scatterers, and the quasi-elastic contribution is an unwanted background. In Table VI are given calculated quasi-elastic cross-sections per nucleon, averaged over the CERN neutrino spectrum with the experimental cuts (muon angle $< 5^{\circ}$, muon energy < 1.2 GeV) of the small angle sample. For carbon is quoted the result of using the exclusion factor of the Fermi gas, of our shell model functions and of using the shell model D_S everywhere. For Pb, our oscillator wave functions would not be relevant. However, Bogan [28] found with infinite square well wave functions an exclusion curve quite close to that of the appropriate asymmetric $(N/Z = 126/82)$ Fermi gas. For Pb we quote only the result with this latter. Despite the appreciable variation between the three versions of the exclusion effect exhibited for carbon, carbon is in each case substantially less efficient per nucleon than Pb as a quasi-elastic scatterer. This goes in the opposite direction to the hypothetical nuclear shadow effect, as has been discuted elsewhere [2),3)].

Despite the varying structure of the nuclei considered, the shell model exclusion curves of Figs. 1 - 4 are all rather similar, except for very small $|\vec{q}|^2$. They lie rather higher, except again for small $|\vec{q}|^2$, than the Fermi gas curves ; this is perhaps to be expected simply because average nuclear densities are smaller than central nuclear densities, on which the standard Fermi gas is based. The curves are not high enough, however, to account for the experimental results of Kustom et al. [1]. This is illustrated in Figs. 6 and 7, which apart from the broken lines that we have added are Figs. 3c and 3b respectively of Kustom et al., presenting the same data on different scales. In each case the higher solid curve is their fit without nuclear effects, i.e., for free neutrons, and in the lower solid curve they have allowed for exclusion in a Fermi gas model. In Fig. 6 we have added what we get by multiplying the free neutron curve by various exclusion factors : (1) the symmetric Fermi gas factor of Figs. 1, 2, 3 ; (2) the more appropriate asymmetric Fermi gas curve of Fig. 4 ; (3) the shell model curve (on this scale it does not matter which) of Fig. 4. This shell model curve is repeated on Fig. 7, and also what we get on increasing it by 30 % - this is interesting because the experimenters regard their absolute flux as possibly uncertain to that degree.

If the finding of Kustom et al. is confirmed, despite the strong exclusion effect in muon capture and the indications from electron scattering, then the assumptions and approximations of the theory will come under renewed scrutiny. We have in mind especially the closure approximation, and the neglect of many-body terms in the pseudo-potential and of correlations in the wave function. We will close with a general remark on these last two questions. It is not always realized that there is no clear separation between them. Given a picture involving wave functions Ψ and operators O, an equally valid picture is obtained by unitary transformation *)

$$\Psi' = e^{iS}\,\Psi \qquad O' = e^{iS}\,O\,e^{-iS}$$

In general single and many-body terms in operators will be changed and mixed up by the transformation, and the correlations in the wave functions will be altered. Loosely speaking, the same effect can appear in one picture as due to a correlation and in another as due to a many-body term in the operator. Thus effective interactions and correlation functions are not separately well defined concepts. Yet it is possible

---

*) For example, one of us has used such a transformation in connection with the hard core problem [44].

to find in the literature estimates of many-body terms in interactions made without any specification of the particular picture in which they are appropriate. This is not very important when the effects are small and the calculations are regarded as only exploratory. For serious calculations of large effects the question would be a vital one.

In connection with this work we have profited from discussions with M. Rho and O. Kofoed-Hansen, and from correspondance with A. Bogan, C.W. Kim, C.A. Piketty and H. Primakoff.

	N	Z		C(0)	C(1)	C(2)	C(3)	C(4)	C(5)	C(6)
			S							
He^4	2	2	T	2.000×10^0						
			L							
			S	6.000×10^0	0	8.889×10^{-1}				
C^{12}	6	6	T	4.222×10^0	8.889×10^{-1}	8.889×10^{-1}				
			L	4.222×10^0	1.778×10^0	8.889×10^{-1}				
			S							
O^{16}	8	8	T	8.000×10^0	0	2.000×10^0				
			L							
			S	1.400×10^1	0	4.560×10^0	-1.600×10^{-1}	8.000×10^{-2}		
Si^{28}	14	14	T	1.080×10^1	3.520×10^0	2.640×10^0	1.600×10^{-1}	8.000×10^{-2}		
			L	1.080×10^1	5.760×10^0	1.360×10^0	4.800×10^{-1}	8.000×10^{-2}		
			S							
Ca^{40}	20	20	T	2.000×10^1	0	1.000×10^1	-2.000×10^0	5.000×10^{-1}		
			L							
			S	2.800×10^1	0	1.503×10^1	-2.588×10^0	6.916×10^{-1}	-2.177×10^{-3}	2.902×10^{-3}
28^{56}	28	28	T	2.343×10^1	7.771×10^0	8.393×10^0	-3.412×10^{-1}	3.259×10^{-1}	2.393×10^{-2}	2.902×10^{-3}
			L	2.343×10^1	1.189×10^1	4.162×10^0	1.487×10^0	1.238×10^{-2}	5.007×10^{-2}	2.902×10^{-3}
			S	2.600×10^1	4.000×10^0	1.133×10^1	-1.342×10^0	5.151×10^{-1}	1.224×10^{-3}	4.082×10^{-3}
Fe^{56}	30	26	T	2.257×10^1	9.829×10^0	5.989×10^0	6.631×10^{-1}	1.608×10^{-1}	2.653×10^{-2}	4.082×10^{-3}
			L	2.257×10^1	1.291×10^1	2.633×10^0	2.263×10^0	-1.429×10^{-1}	5.184×10^{-2}	4.082×10^{-3}

TABLE I Non-zero expansion coefficients for the exclusion factors of Eq. (24) :

$$D_{S,T,L} = \left\{ \sum C_{S,T,L}(n) \left(\frac{\vec{q}^2 b^2}{2} \right)^n \right\} \exp \left(- \frac{\vec{q}^2 b^2}{2} \right)$$

	He^4	C^{12}	O^{16}	Si^{28}	Ca^{40}	Fe^{56}	
$\sqrt{\langle r^2 \rangle}$ charge	1.68	2.50	2.70	3.04	3.52	3.80	10^{-13}cm
b	1.39	1.66	1.76	1.77	1.99	2.04	10^{-13}cm

TABLE II Experimental root mean square charge radii and corresponding oscillator size parameters. The experimental root mean square radii are representative values from the compilation of Hofstadter and Collard [25)]. The range of values quoted by these authors is such that the last figure given in the Table is not really significant.

	Λ_r(exp)	Λ_r (shell model)	$\Lambda_r(D_L, D_T \rightarrow D_S)$
He^4	0.086 ± 0.007	0.085	0.085
C^{12}	0.125 ± 0.004	0.209	0.113
O^{16}	0.111 ± 0.004	0.122	0.122
Si^{28}	0.137 ± 0.005	0.181	0.122
Ca^{40}	0.130 ± 0.001	0.139	0.139
Fe^{56}	0.108 ± 0.001	0.151	0.137

TABLE III Experimental and theoretical results for reduced capture rates. The second column gives the results directly implied by the simplest shell model wave functions, and the third column results from using the shell model D_S also for D_T and D_L.

	C^{12}	Pb^{208}
no exclusion	1.82 $10^{-40} cm^2$	2.20 $10^{-40} cm^2$
Fermi gas	0.84 "	1.28 "
shell model	1.05 "	
shell model (D_T, $D_L \to D_S$)	0.94 "	

TABLE VI Quasi-elastic cross-sections per nucleon for production of muons, within 5^o of the forward direction and with energy greater than 1.2 GeV, averaged over the CERN neutrino spectrum [2)]. Conventional assumptions about form factors were made [2)]. The first row is the result for a collection of free stationary neutrons and protons, in the ratio N/Z, without nuclear effects. The remaining rows allow for the exclusion principle in various ways.

	He^4	C^{12}	O^{16}	Si^{28}	Ca^{40}	
a (Levinger-Foldy-Walecka)	0.66	0.95	1.05	1.26	1.42	$10^{-26} cm^2$
$a = b^2/2$	0.96	1.38	1.55	1.57	1.98	"

TABLE V The expansion parameter a of Eq. (36), as calculated from the empirical formula (37) and from the parabolic well shell model with the size parameters of Table II.

	Fermi gas	Linear approx. to Fermi gas	Parabolic well with b of Table II	Parabolic well with Bogan size	Infinite square well of Bogan
He^4	0.222	0.222	0.147	0.216	0.220
C^{12}	0.222	0.222	0.196	0.209	0.212
O^{16}	0.222	0.222	0.211	0.235	0.218
Si^{28}	0.222	0.222	0.211	0.236	0.212
Ca^{40}	0.222	0.222	0.242	0.242	0.221
Fe^{56}	0.170	0.165	0.218	0.222	0.192

TABLE IV Exclusion factor $[1 - Z^{-1} D_S(80 \text{ MeV/c})]$. The first column is the Fermi gas value from Eq. (1). The next is the approximation of Primakoff [26),24)] to this value, $[1 - 4(N/A^2)D_S]$ with D_S evaluated at $N = Z$. The third column gives our parabolic well values, and the fourth gives parabolic well results with radii fixed according to the prescription of Bogan [28)]. Finally the results of Bogan are quoted from his Table 2, except for Fe^{56}, where the quotation is from his Table 3 ; in the latter case his calculation involves an interpolation between nuclei with closed subshells.

REFERENCES

1) R.L. Kustom, D.E. Lundquist, T.B. Novey, A. Yokosawa and F. Chilton - Phys.Rev.Letters 22, 1014 (1969).

2) J.S. Bell and C.H. Llewellyn Smith - CERN Preprint TH.1198 (1970), to appear in Nuclear Phys.

3) J.S. Bell - CERN Preprint TH.1222 (1970), to appear in the Proceedings of the 11th Scottish Universities Summer School in Physics.

4) K. Borer, B. Hahn, H. Hofer, F. Krienen and P.G. Seiler - Phys. Letters 30B, 572 (1969).

5) J.S. Bell - Phys.Rev.Letters 13, 57 (1964).

6) J.S. Bell - CERN Preprint TH.887 (1968).

7) J. Løvseth - Phys.Letters 5, 199 (1963).

8) York-Peng Yao - Phys.Rev. 176, 1680 (1968).

9) R. Gatto - Nuovo Cimento 10, 1559 (1953) ; 2, 670 (1955).

10) S.M. Berman - Proc.High Energy Theor.Conference, CERN 61-22, 9 (1961).

11) B. Goulard and H. Primakoff - Phys.Rev. 125, B1139 (1964).

12) T. de Forest and J.D. Walecka - Adv.in Physics 15, 1 (1966).

13) K.W. McVoy and L. Van Hove - Phys.Rev. 125, 1034 (1962).

14) J. Chollet, P. Bounin and G.R. Bishop - "Diffusion Inélastique d'Electrons de Haute Energie sur $A\ell^{27}$", unpublished.

15) H.F. Ehrenfest and R. Hofstadter - Phys.Rev. 110, 544 (1958).

16) W. Csyz - Phys.Rev. 131, 2141 (1963).

17) P. Bounin and G.R. Bishop - Journal de Physique 24, 974 (1963).

18) G.R. Bishop, D.B. Isabelle and C. Betourne - Nuclear Phys. 54, 97 (1964).

19) E.J. Moniz - Phys.Rev. 184, 1154 (1969).

20) H.A. Bethe and M.E. Rose - Phys.Rev. 51, 283 (1936).

21) J.P. Elliott and T.H.R. Skyrme - Proc.Roy.Soc. A232, 561 (1955).

22) R.J. Luyten, H.P.C. Rood and H.A. Tolhoek - Nuclear Phys. 41, 236 (1963).

23) L.L. Foldy and J.D. Walecka - Nuovo Cimento 36, 1257 (1965).

25) H.R. Collard and R. Hofstadter - Landolt-Börnstein, Vol. 2, Springer Verlag, Berlin (1967).

26) H. Primakoff - Revs.Modern Phys. 31, 802 (1959).

27) R. Klein, T. Neal and L. Wolfenstein - Phys.Rev. 138, 886 (1965).

28) A. Bogan - Nuclear Phys. B5, 431 (1968).

29) K.W. Ford and J.G. Wills - Nuclear Phys. 35, 295 (1962).

30) M. Eckhause, R.T. Siegel, R.E. Welsh and T.A. Filippas - Nuclear Phys. 81, 575 (1966).

31) L.L. Foldy and J.D. Walecka - Nuovo Cimento 34, 1026 (1964).

32) M. Rho - Phys.Rev. 161, 955 (1967).

33) V.M. Novikov and M.G. Urin - Soviet J.Nuclear Phys.(USA) 3, 302 (1966).

34) G.G. Bunatyan - Soviet J.Nuclear Phys.(USA) 3, 613 (1966).

35) L.L. Foldy and R.H. Klein - Phys.Letters 24B, 540 (1967).

36) R.H. Klein - Thesis, Carnegie Tech. (1963).

37) R.H. Klein - Phys.Rev. 146, 756 (1966).

38) J. Frazier, C.W. Kim and M. Ram - Phys.Rev. D1, 3162 (1970).

39) J.S. Levinger - "Nuclear Photo Disintegration", Oxford University Press (1960).

40) C.A. Piketty - Nuovo Cimento 44A, 1163 (1966).

41) C.A. Piketty and A. Orkin-Lecourtois - Nuovo Cimento 50A, 927 (1967).

42) J. Løvseth - Nuovo Cimento 57A, 382 (1968).

43) T.W. Donnelly - Nuclear Phys. A150, 393 (1970).

44) J.S. Bell - "The Many Body Problem", Lectures Notes for the 1st Bergen International School of Physics (Ed. C. Fronsdal) Benjamin (1962).

FIGURE CAPTIONS

Figure 1 Shell model and Fermi gas exclusion factors for closed shell nuclei.

Figure 2 Shell model and Fermi gas exclusion factors for C^{12}.

Figure 3 Shell model and Fermi gas exclusion factors for Si^{28}.

Figure 4 Shell model and Fermi gas exclusion factors for Fe^{56}.

Figure 5 Comparison with Goulard and Primakoff. The curve is from their Eqs. (38) and (41) with the parameters of the central curve in their Fig. 1 : $(d/r_o) = 1.5$, $r_o = 1.25$ fm. This gives a root mean square radius $3/5\ r_o\ A^{1/3} =$ $= 3.17$ fm.

Figure 6 Figure 3c from Kustom et al. [1]. The upper and lower solid curves are their free neutron and Fermi gas theoretical curves. The three dashed curves we obtained from the free neutron curve by incorporating various exclusion factors : (1) symmetric Fermi gas ; (2) asymmetric Fermi gas ; (3) shell model.

Figure 7 Figure 3b from Kustom et al. [1]. The solid curves are as in Fig. 6, and (3) is again the shell model curve ; (4) is the result of increasing (3) by 30 %. The experimenters regard the absolute flux as possibly uncertain to this degree.

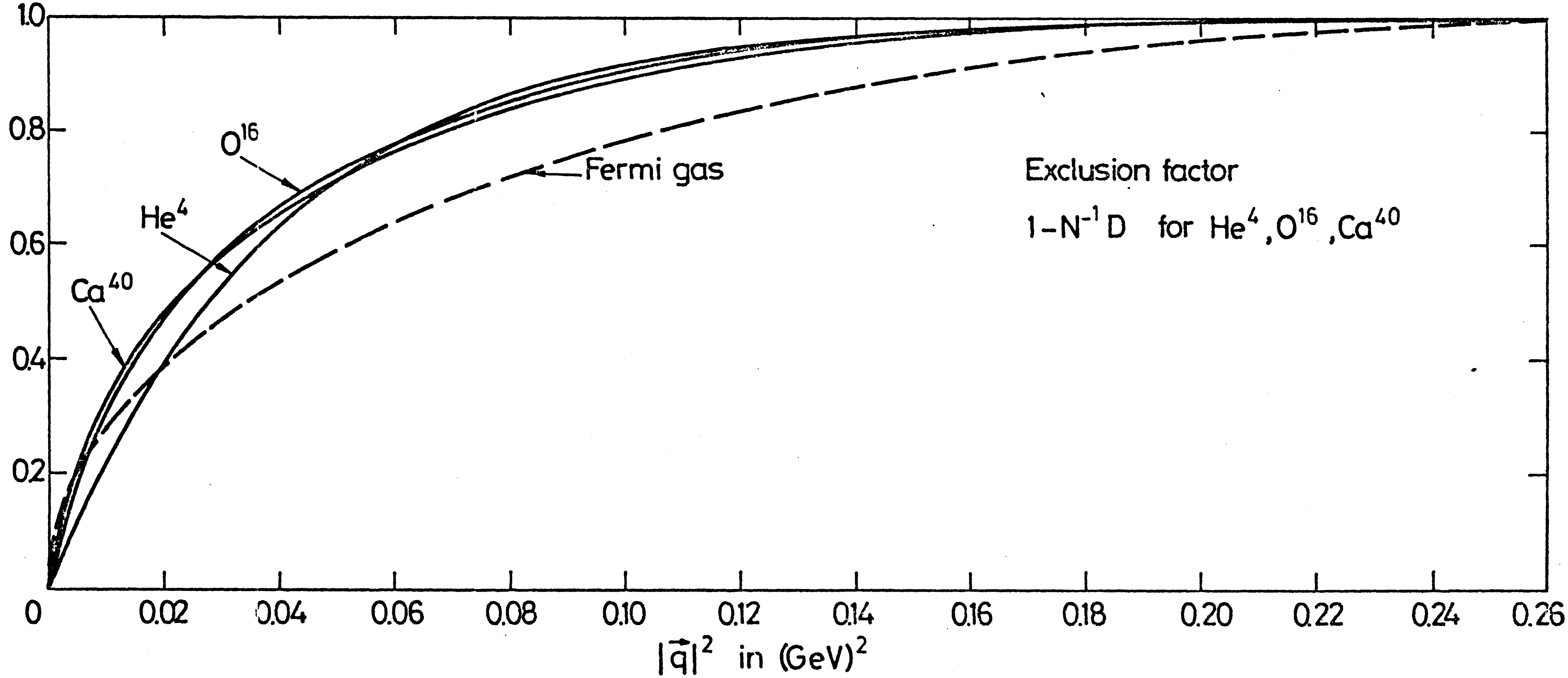
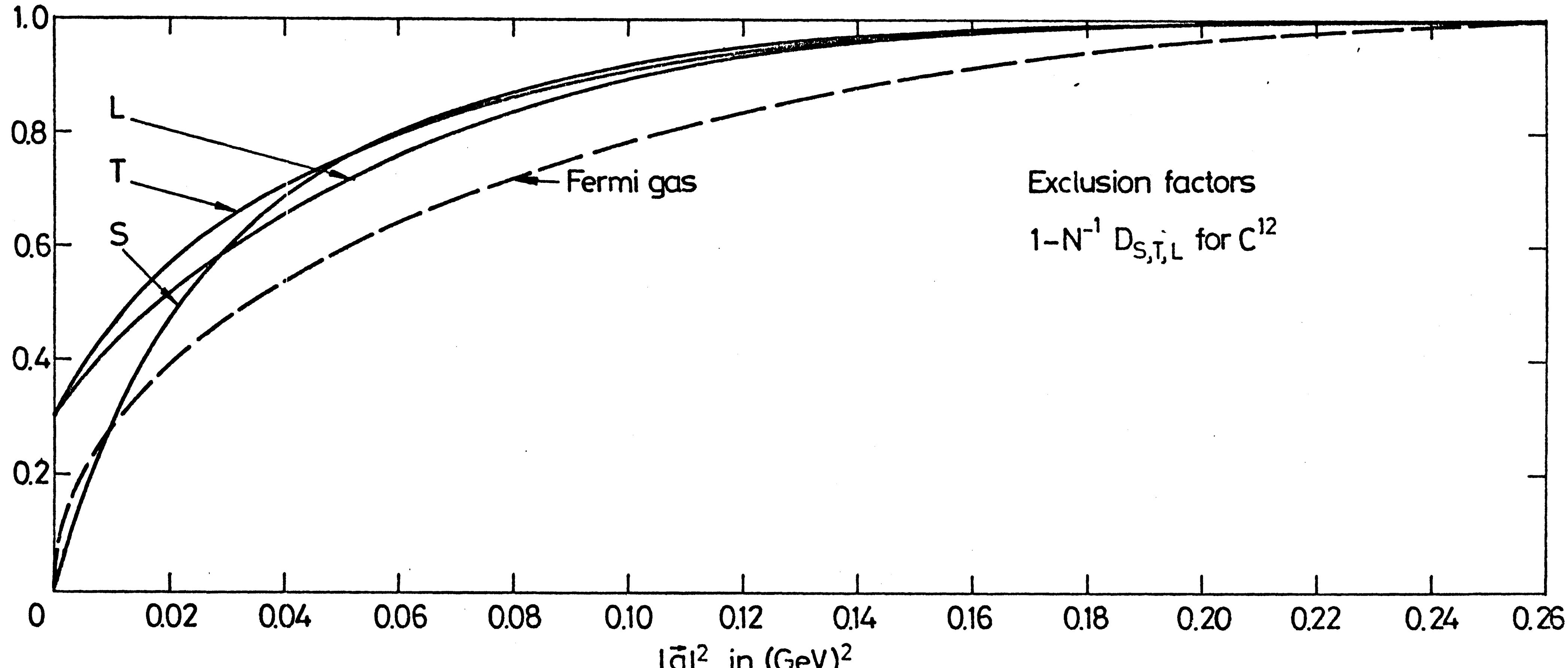

1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26
$|\vec{q}|^2$ in $(GeV)^2$
O^{16}
He^4
Ca^{40}
Fermi gas
Exclusion factor
$1-N^{-1}D$ for He^4, O^{16}, Ca^{40}

FIG.1



1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26
L
T
S
Fermi gas
Exclusion factors
$1-N^{-1}\,D_{S,T,L}$ for C^{12}
$|\vec{q}|^2$ in $(GeV)^2$

FIG. 2

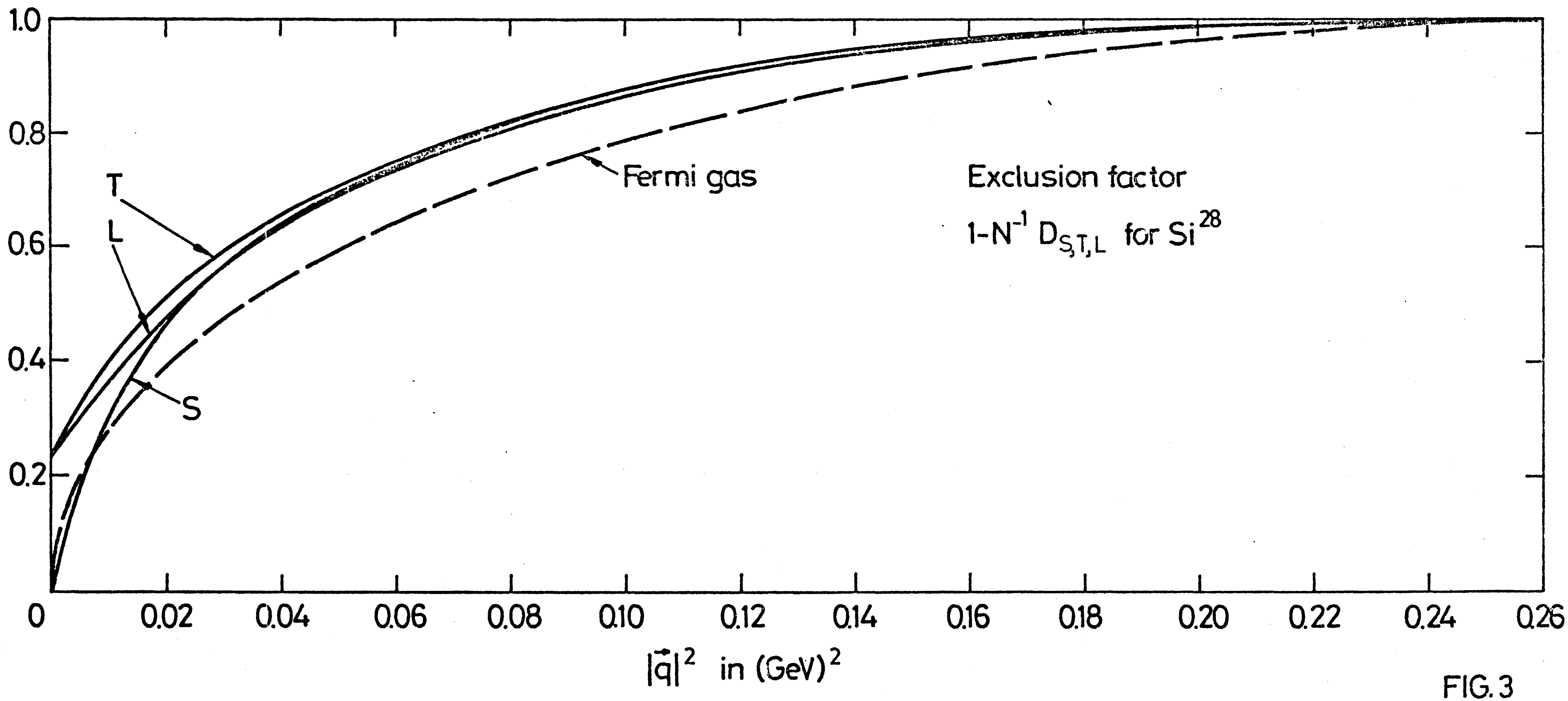
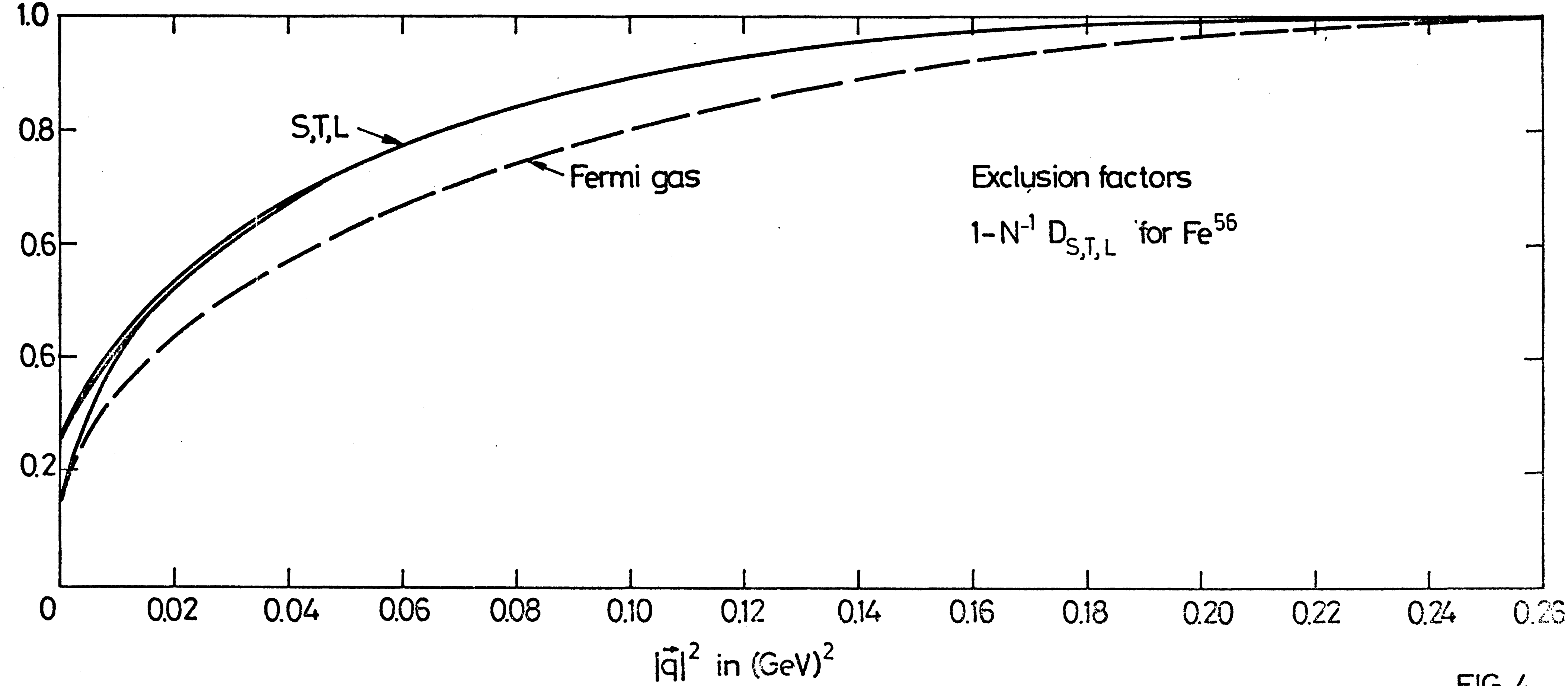

1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26
$|\vec{q}|^2$ in $(GeV)^2$
T
L
S
Fermi gas
Exclusion factor
$1-N^{-1} D_{S,T,L}$ for Si^{28}

FIG. 3



1.0
0.8
0.6
0.6
0.2
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26
S,T,L
Fermi gas
Exclusion factors
$1-N^{-1}\,D_{S,T,L}$ for Fe^{56}
$|\vec{q}|^2$ in $(GeV)^2$

FIG. 4

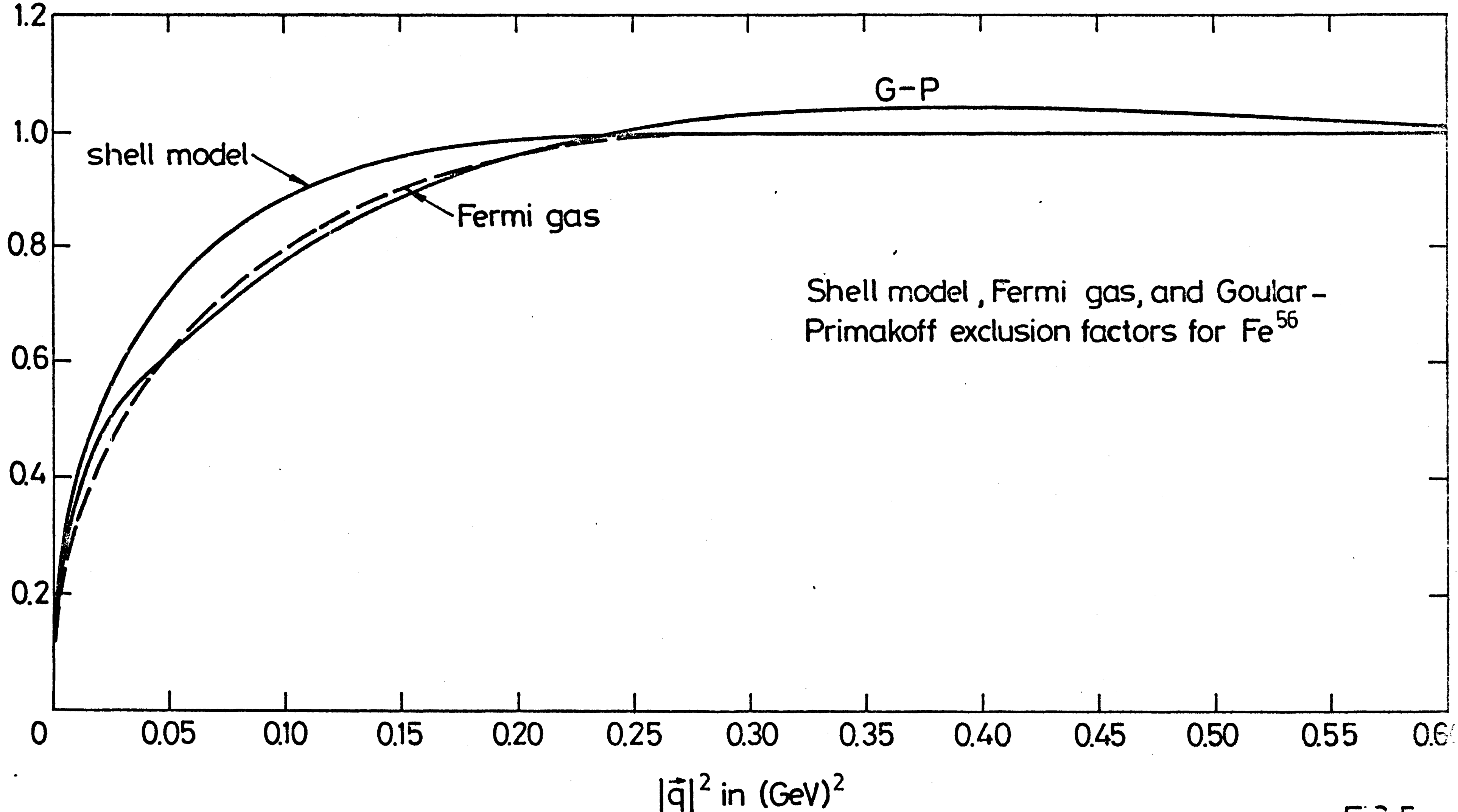
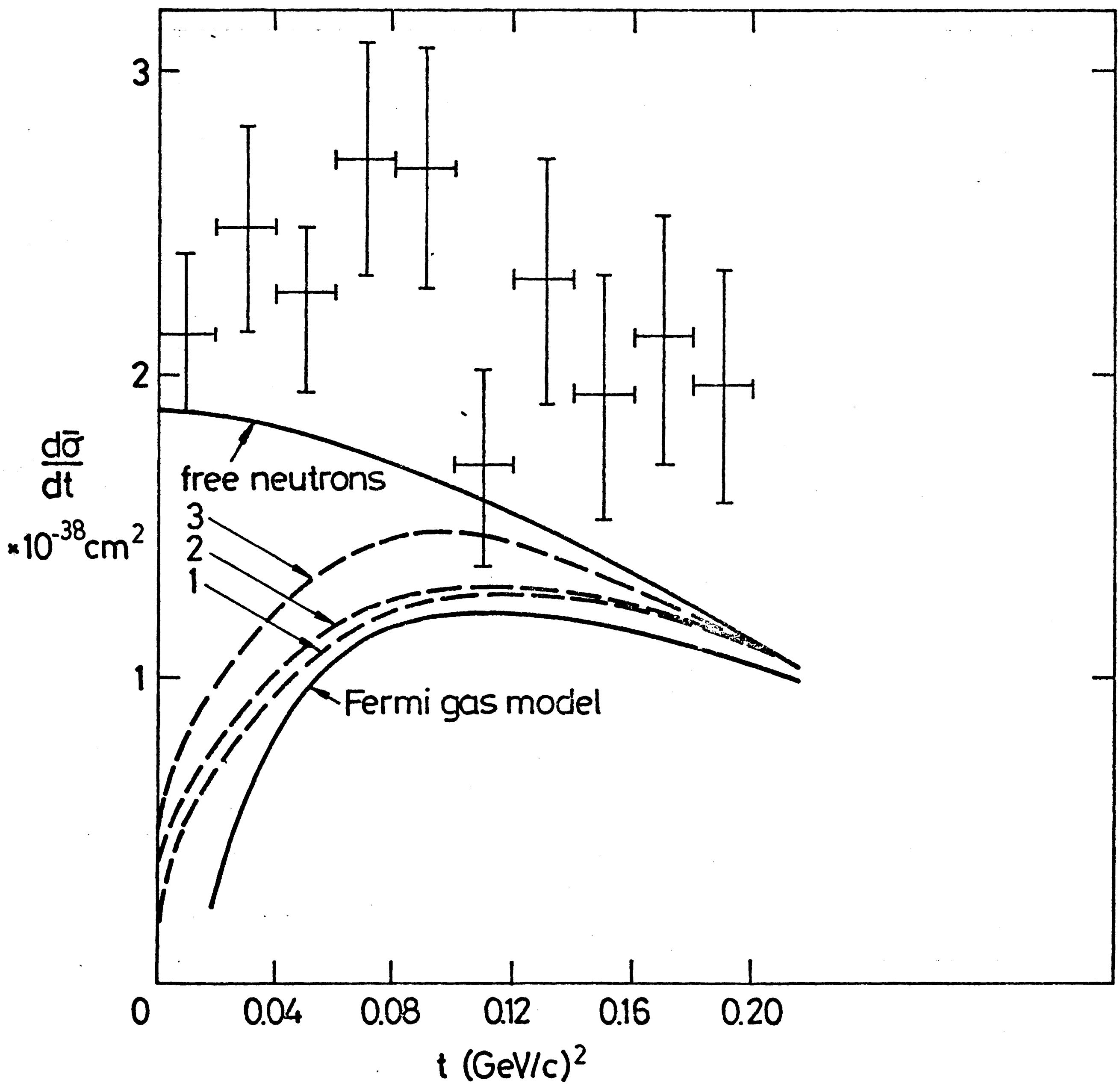
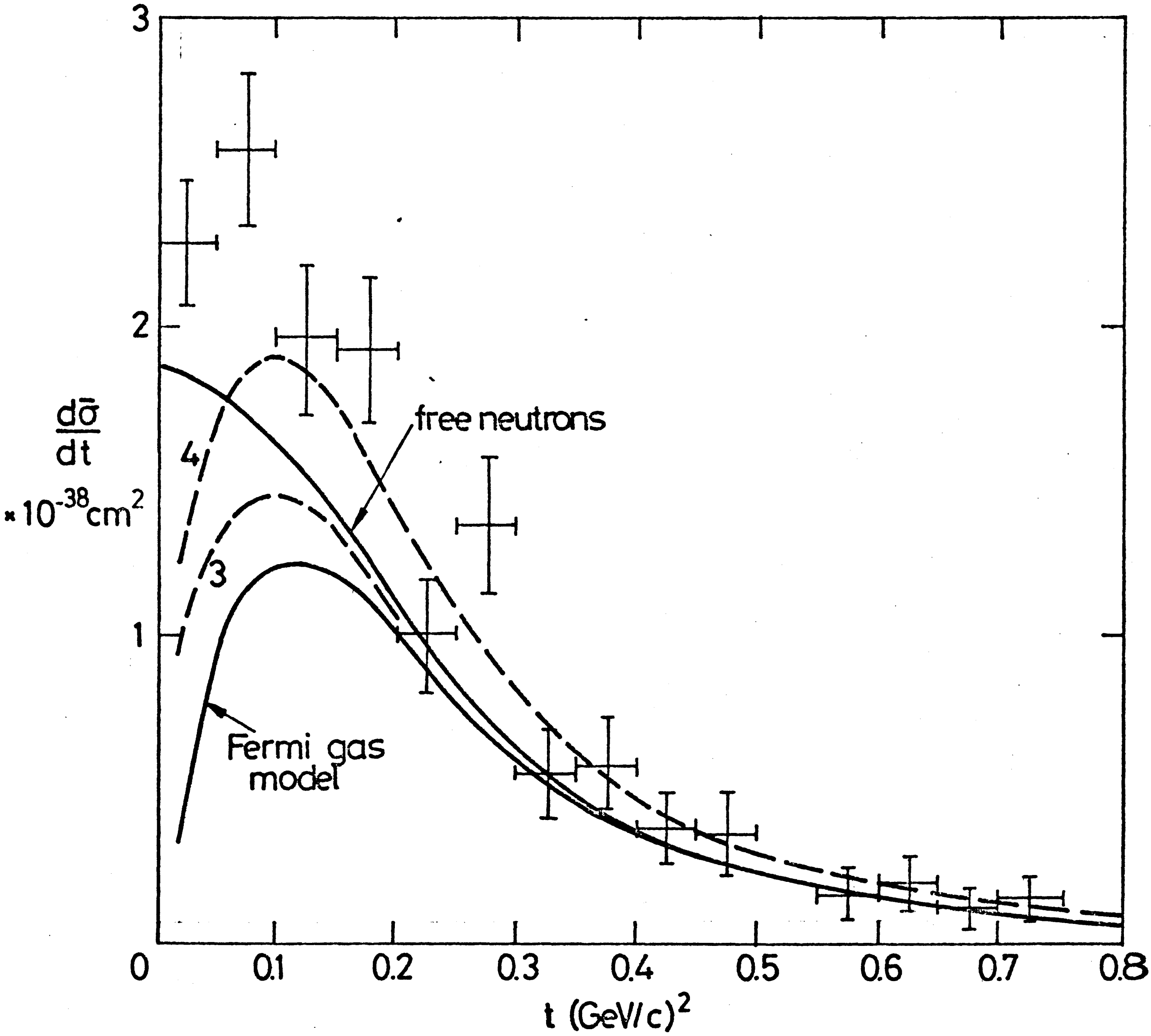

1.2
1.0
0.8
0.6
0.4
0.2
0
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
G–P
shell model
Fermi gas
Shell model, Fermi gas, and Goular-
Primakoff exclusion factors for Fe^{56}
$|\vec{q}|^2$ in $(GeV)^2$

FIG. 5



3
2
1
$\frac{d\bar{\sigma}}{dt}$
$\times 10^{-38} cm^2$
free neutrons
3
2
1
Fermi gas model
0
0.04
0.08
0.12
0.16
0.20
t $(GeV/c)^2$

FIG.6



3
2
1
0
$\frac{d\bar{\sigma}}{dt}$
$\times 10^{-38} cm^2$
free neutrons
4
3
Fermi gas model
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
t $(GeV/c)^2$

FIG. 7