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N POINT DUAL FUNCTIONS FOR MESONS WITH PHYSICAL TRAJECTORIES

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ABSTRACT

In this paper a graphical formalism is presented that allows to write any dual N point meson function. The technique allows to write the most general N point function of the type in which a polynomial modifies the Chan integrand.

We can then single out solutions that have the proper isospin structure and no ghosts along the leading and satisfy the bootstrap principle.

We have thoroughly analyzed the $G = -$ unnatural parity trajectories case for N pions. The novel features associated with physical amplitudes are :

a) leading trajectories have also multiplicities ;

b) factorization of daughter trajectories breaks down.

The rules for the other cases are explicitly given and the theoretical and phenomenological implications of the results are briefly discussed.

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INTRODUCTION

The generalized Veneziano model has been a fruitful laboratory for hadronic amplitudes [1]. Its interpretation has evolved rapidly. It started as a solution to analyticity and bootstrap constraints but lately an interpretation in terms of an infinite component field theory seems most promising.

Unfortunately, the models under study have serious shortcomings and, sometimes, it is legitimate to ask how much the simple amplitudes reflect the real world. The most outstanding problem along these lines is the introduction of spin $\frac{1}{2}$ particles into the theory. This problem in turn, when solved, might have profound influence over the integer spin case.

Our task here is to give a general solution in the tree approximation to the problem of N external physical pseudoscalar mesons coupled by means of physical trajectories.

We have been able to solve the problem for an arbitrary number of external particles and hence we can give a detailed account of the properties of the internal states. The procedure is simple but lengthy ; we have carried it out only for the case of the $\pi - A_1$ trajectory case for the N pion amplitude. The other cases can be solved as well.

The modifications we propose are most conservative [2]. As a whole the results of the simple model are the same with two exceptions. Due to the presence of different trajectories, cyclical symmetry is lost by individual terms. As a direct consequence the factorization properties are severely affected. The leading trajectory is not single anymore, though it is still not parity mixed. Daughter factorization is lost since the number of states is a function of the number of external particles.

We have been able to express the problem of ghosts and the isospin content of the leading trajectory in a concise, diagrammatic fashion. As a consequence, we have deduced the existence of a unique two parameter N point function for $G = -1$ trajectories. Any N point function can be written down without effort and hence application of these functions to phenomenology are possible immediately. However, the rules require detailed calculations to be justified. For the reader who is willing to accept our results the paper provides simple rules for writing amplitudes. The more enterprising one will find an Appendix with a detailed example of how to proceed.

The main features of the N π amplitude with G = - trajectories (πA_1) are :

a) the $\pi - A_1$ trajectory is simple, with $I = 1,0$ states except for the π ;

b) the $\rho - f^0$ trajectory is doubled ;

c) ghosts persist at the daughter level ;

d) factorization breaks down at the daughter level.

The second result is rather unpleasant at the ρ level and interesting at the 2^+ level because of the reported A_2 splitting. There is room experimentally for a doubled ρ but we feel that this doubling is embarrassing. The presence of ghosts has been explicitly checked and this result coupled to the lack of factorization poses a new challenge of how to make the theory consistent with causality.

Section 1 contains the full discussion of the $\pi - A_1$ case and the general diagrammatic rules. Section 2 generalizes the result for the odd number of legs and for the $(\omega - A_2)$ trajectory case. We present explicitly the seven point function that might be useful phenomenologically. We present a detailed example of how to check positivity and factorization in an Appendix.

1. UNNATURAL PARITY $G=-1$ TRAJECTORIES

Our method to construct N point functions is best described by means of graphical techniques. Using dual graphs, we will be able to establish a one-to-one correspondence between the factors composing the N point function and the geometrical structure of the graphs. These graphs allow also a simple understanding of the properties of the leading trajectories in all channels.

These rules were established by painful calculations but the results embodied in these rules are very simple and allow for the construction of the desired amplitude in a very simple fashion.

A typical term of the amplitude with N pions is assumed to have the structure

$$A(p_1 \cdots p_N) = \int \cdots \int \prod_{i,j} u_{ij}^{-\alpha_{ij}-1} P(\alpha_{12}, \cdots \alpha_{ij} \cdots \alpha_{N-2,N-1}; u_{12}, \cdots u_{N-2,N-1}) dV_N \quad (1)$$

where $\alpha_{ij} = a^{(K)} - b(p_i + p_{i+1} + \cdots p_j)^2$, $a^{(K)}$ $(K=1,2)$ being the intercept of the two possible types of trajectory, u_{ij} are the usual $N(N-3)/2$ Chan variables, dV_N is the volume element which includes δ functions to ensure duality. $P(\alpha_{ij}, u_{ij})$ is a polynomial in α_{ij} and u_{ij}. The main function of this polynomial is to eliminate the unwanted poles at $\alpha_{ij}=0$ if the intercept is positive. Other functions could be tried but the strict requirements of meromorphy and absence of ancestors make any other choice difficult, if not impossible. A natural choice like some simple denominator, for example, produces inevitably ancestors in some dual channel.

Let us now see the effect each "building block" has on the amplitude and associate to it a geometrical pattern that can be used in the dual graph :

1) $\alpha_{ij}(1-u_{ij}) \rightarrow$ ——— (continuous line in dual graph)

This factor clearly eliminates the $\alpha_{ij}=0$ pole in the amplitude and on all the trajectories dual to (i,j) because of the $(1-u_{ij})$ factor. Spin structure and Regge behaviour is not affected since both factors have compensating effects.

2) $(1-u_{ij})$ $\rightarrow$ - - - - - - (dotted line in dual graphs)

This term eliminates the $\alpha_{ij}=0$ on the trajectories dual to the channel (i,j) and shifts these trajectories to the first daughter level. The term has non-leading Regge behaviour for the variables dual to (i,j).

3) u_{ij} $\rightarrow$ + + + + + + + (lines with crosses in the dual diagrams)

These factors eliminate the $\alpha_{ij}=0$ and shift the trajectory to the first daughter level.

We then associate a graph to a given polynomial. To clarify how the procedure works, we describe a simple example. In a six-pion function a possible polynomial is

$$P = (1-u_{24})\,\alpha_{34}\,(1-u_{34})\,\alpha_{14}\,(1-u_{14}) \tag{2}$$

Using the afore-mentioned rules, one obtains the graph depicted in Fig.1. This graph then is associated to an amplitude that can be explicitly constructed and whose properties are, using the afore-mentioned rules : non-leading behaviour in the channels $(1,2)$, $(4,5)$, $(1,3)$, $(3,5)$ since they cross the dotted line and hence are dual to it; non-leading behaviour in $(3,4)$, $(2,4)$, etc., (they do not cross the dotted line). The $\alpha = 0$ pole is missing in all $G=+$ (two-body) channels.

Because of the factors we have introduced, cyclical symmetry is <u>not</u> preserved term-by-term. The total amplitude is then written

$$A = \sum_{\text{permutations}} Tr(\tau_1 \dots \tau_N)\, A(p_1 \dots p_N) \tag{3}$$

where $Tr(\tau_1, \dots, \tau_n)$ is the Chan-Paton isospin factor and whose presence ensures isospin invariance and absence of exotics. The sum is understood to be over all cyclical and non-cyclical permutations.

It is both amusing and rewarding that the properties of the amplitude, at the level of the leading trajectory, can be simply expressed in terms of geometrical properties of the graphs. These rules are true as one can verify by direct computation, but we omit here all these tedious calculations.

To illustrate the method, we go back to the example given by Eq. (2). We fix our attention on the (1,3) pole and consider the effect of all cyclical permutations when we study this particular singularity. As seen in Fig. 2, we get two graphs that contribute. The other four possible graphs have no leading trajectory in that channel since the (1,3) pole goes across a line of type 2. The contributions to the amplitude from these graphs will be further specified as follows :

$$\mathcal{R}\tilde{\mathcal{R}} + \mathcal{R}^T\tilde{\mathcal{R}}^T$$

where $\mathcal{R}$ is the half left figure, $\tilde{\mathcal{R}}$ is the reflected figure $\mathcal{R}$ with respect to the pole axis, $\mathcal{R}^T$ is the inverted figure with respect to the orthogonal direction. These definitions can be best understood by looking at Fig. 2. $\mathcal{R}$ can represent a sum of diagrams as well in the statements made below. The power of this technique stems from the following simple theorem :

a) if the sum of cyclically permuted graphs has the structure

$$\mathcal{R}\tilde{\mathcal{R}} + \mathcal{R}^T\tilde{\mathcal{R}}^T$$

then the leading trajectory is ghost free and has for each J both $I=0$ and $I=1$, except for $J=0$ where $I=1$ only obtains;

b) the sum reads

$$(\mathcal{R}+\mathcal{R}^T)(\tilde{\mathcal{R}}+\tilde{\mathcal{R}}^T)$$

In this case the leading trajectory is ghost free, for even J one gets isospin 1 (if $G=-$) or 0 (if $G=+$). The situation is reversed for odd J;

c) $$(\mathcal{R}-\mathcal{R}^T)(\tilde{\mathcal{R}}-\tilde{\mathcal{R}}^T)$$

Again, the leading trajectory is ghost free, for even J we get isospin 0 for $G=+$ and 1 for $G=-$ and the opposite for $J=\text{odd}$.

If the amplitude contribution <u>cannot</u> be expressed in any of these three forms, then ghosts on the leading trajectory are inevitable. We must emphasize once more that the study of ghosts is a lengthy unpleasant calculation, but these rules follow naturally once these expressions are written down. Once the amplitude has been brought to these forms, the spin-isospin part of the theorem is trivial. To have the full amplitude we must add the non-cyclical permutations given by the twisting operation to obtain

$$(\mathcal{R}\pm\mathcal{R}^T)(\tilde{\mathcal{R}}\pm\tilde{\mathcal{R}}^T) \qquad \text{(case a)}$$

$$[(\mathcal{R}+\mathcal{R}^T)\pm(\mathcal{R}+\mathcal{R}^T)](\tilde{\mathcal{R}}+\tilde{\mathcal{R}}^T) \qquad \text{(case b)}$$

$$[(\mathcal{R}-\mathcal{R}^T)\mp(\mathcal{R}-\mathcal{R}^T)](\tilde{\mathcal{R}}-\tilde{\mathcal{R}}^T) \qquad \text{(case c);}$$

since under twisting $\mathcal{R}\to\mathcal{R}^T$ or $\breve{\mathcal{R}}\to\tilde{\mathcal{R}}^T$ up to a sign. In fact, the signs depend on the spin-isospin structure. This comes as no surprise since we know that twisting is intimately connected to signature.

Each graph has a different analytic structure and factorizes by itself. However, if we accept a general solution with different diagrams contributing, we will increase the number of levels accordingly.

By means of these tools we analyze the possible solutions.

<u>The four-pion function</u>

This is the Lovelace formula which is obtained choosing

$$P(\alpha_{ij}\ldots, u_{ij}\ldots) = \alpha_{12}(1-u_{12}) \tag{4}$$

in the expression (3) :

$$A(P_1\ldots P_N) = \int_0^1\int_0^1 u_{12}^{-\alpha_{12}-1}\, u_{23}^{-\alpha_{23}-1}\, P(\alpha_{12},u_{12})\,\delta(1-u_{12}-u_{23})\,du_{12}\,du_{23} \tag{5}$$

The graph associated with the polynomial (4) is shown in Fig. 3, as well as the properties of the leading trajectory. We obtain two "b"-type graphs, so that the amplitude can be written as

$$\mathcal{A}\tilde{\mathcal{A}} + \mathcal{B}\tilde{\mathcal{B}}$$

where $\mathcal{A} = \mathcal{A}^T$; $\mathcal{B} = \mathcal{B}^T$. In this case $\mathcal{A}$ and $\mathcal{B}$ are analytically the same, even if the graphical structure is different (this is true only for the four-pion function) and thus we have a simple $G = +1$ trajectory of "b" - type, i.e., the ρ -f^o trajectory.

The six-pion functions

A systematic study considering the different possibilities to build the six-pion function is shown in Fig. 4.

Let us first consider those solutions which contain only continuous lines. They have maximum Regge behaviour term-by-term.

"A" has ghosts [this solution was in fact considered in Ref. 2), it cannot be cast in the forms a, b, or c];

"B" is of "a" type, has not the π pole, has no ghosts and has, as can be seen by direct calculation, pure d wave coupling for the $A_1 \rightarrow \rho\pi$ decay;

"F" has ghosts.

Thus the single solution which is leading term-by-term and has no ghosts would be "B". But "B" has no π pole and we have thus to consider amplitudes which have no leading Regge behaviour in all channels term-by-term.

"F'" has no leading behaviour in any $G = -1$ channel but has the ρ-ρ-ρ coupling;

"C" is a "maximum duality" type solution (maximum duality means maximum asymptotic behaviour compatible with the pole structure for each term) considered in Ref. 2), it has also ghosts;

"D" has no ghosts, is of "a" type and has s wave $A_1 \to \rho\pi$ coupling;

"E" if of "b" type and thus has no $J = 1$, A_1 pole. This is the solution considered in Ref. 3);

2"D"-"E" is a "c" type solution, has no π pole but has the $I = 1$, A_1 pole.

Thus the general solution for the six-pion amplitude should be

$$c_1 B + c_2 B' + c_3 D + c_4 (2D - E) + c_5 E \tag{6}$$

As we shall see, most of these solutions are ruled out by the bootstrap principle at the eight-pion level.

Eight-pion function

The "B" solution may be extended at the eight-pion level, as in Fig. 5. Unfortunately, it has ghosts and is thus ruled out. The same is valid for the (2D-E) solution. The "D" solution has no ghosts and gives two "b" type trajectories, as well as the "E" solution. Since, however, the "E" solution has not the A_1 trajectory and since the "2D-E" solution which gives the A_1 trajectory does not work for $G = +1$ poles, it follows that the eight-pion function should be written as

$$c_2 B' + c_3 D \tag{7}$$

the "B'" solution may be maintained if we want to have the $\rho - \rho - \rho$ coupling which does not exist either in the "D" solution or in "E".

N - pion function

This will be (7) where "B'" and "D" have to be defined for N pions. The "D" solution is defined in the general case through those graphs where the $G = +1$ lines are continuous and open (there are no $G = +1$ "loops") and where the $G = -1$ lines are dotted and have at most one extremity touching a $G=+1$ line. The total number of lines per graph being N-3. The "B'" solution is defined in the general case by the condition that it does not contain $G=-1$ lines. The total number of lines per graph being N-3.

The solution (7) has the $\pi-A_1$ trajectory singled and exchange degenerated and the $\rho-f^0$ trajectory doubled $(c_2=0)$ or tripled $(c_2\neq 0)$.

Leg dependence of the level structure

The degeneracy of the parent trajectory is easily seen to be leg independent. By looking at the diagrams, it is seen that only one configuration always contributes for the π trajectory and two for the $\rho-f^0$. Other cuts invariably cross a line that make the contribution non-leading.

For the daughters the situation is essentially different. At every level of the graph structure new possible cuts appear and these affect the degeneracy of some states. However, for a given spin, the number of lines that can be crossed is finite and so knowledge of these contributions completely determines the degeneracy of that level. However, the complete amplitude lacks factorization in the sense that some daughter states have an infinite degeneracy. This precludes their use for simple-minded loop calculations.

2. GENERALIZATION TO OTHER CASES

In this section we discuss briefly how one must generalize the rules of the previous section to discuss the following two cases :

a) odd number of pseudoscalar mesons as external particles;

b) natural parity trajectory in the even number case for three-body channels.

These two cases complete all possibilities for mesonic amplitudes.

The odd diagram case does not mix with the one discussed in Section 1 because of parity conservation. Hence, the bootstrap principle holds independently. There are two new "building blocks" to be added to the ones discussed in Section 1. These new elements are pentagons and hexagons.

Let us first consider the odd particle case. The equivalent of the Lovelace formula here is the Bardakçi-Ruegg one. It reads [4]

$$A(P_1 \cdot\cdot P_5) = Tr(\tau_1 \cdot\cdot \tau_5)\, \epsilon_{\mu\nu\theta\varphi}\, P_1^{\mu} P_2^{\nu} P_3^{\theta} P_4^{\varphi}\, B_5(1-\alpha_{12}, \cdot\cdot 1-\alpha_{ij}, 1-\alpha_{51}) \quad (9)$$

The contribution associated with these variables is now associated with a pentagon. More precisely, in a 2N+1 leg diagram consider any set of contiguous five-momenta. Take any four of them to saturate the epsilon tensor and write <u>always</u> dotted lines when any of the five momenta are internal lines. For all internal pentagon variables, write a u_{ij} factor. No lines should be drawn inside the pentagon. This completely defines the contribution at the five-point level. Consider now the seven-point function as depicted in Fig. 6. Clearly the bootstrap condition demands the function to break into the product of Lovelace's expression and formula (9). This completely defines the possible contribution as the one shown : a full line. Other configurations must be summed over. We have not investigated this problem in further detail, but a solution can obviously be built. Case b) can be analyzed likewise. The "Born" term here is the expression of Ref. [5]

$$\sum Tr(\tau_1 \cdots \tau_6)\, \epsilon_{\mu\nu\theta\varphi}\, P_1^{\nu} P_2^{\theta} P_3^{\varphi}\, \epsilon_{\mu\alpha\beta\gamma}\, P_4^{\alpha} P_5^{\beta} P_6^{\gamma} \times$$
$$B_6(1-\alpha_{12}, 1-\alpha_{23}, 1-\alpha_{45}, 1-\alpha_{56}, 2-\alpha_{34}, 2-\alpha_{25}, 1-\alpha_{13}, 3-\alpha_{24}, 3-\alpha_{35}) \qquad (10)$$

This will be associated with the contribution of the internal part of the hexagon. By choosing the appropriate number of u_{ij} factors, one reproduces the expression (10). The perimeter is treated like for the pentagon. Notice that the rules are local and hence no new phenomena appear. One can then proceed and construct the eight-point function as in the previous case. Adding this amplitude to the one of Section 1 gives the full solution of the N pseudoscalar function.

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APPENDIX

We illustrate the method of calculation of the pole structure at the level of the six-pion function. We shall do it for the "B" solution (see Fig.4) and look at the (1,3) poles. Following the rules used in the text, for a given order of the external legs, the sum over the cyclical permutations will give the following polynomials :

$$\alpha_{12}(1-u_{12})\alpha_{34}(1-u_{34})\alpha_{14}(1-u_{14})+\alpha_{23}(1-u_{23})\alpha_{45}(1-u_{45})\alpha_{25}(1-u_{25}) \tag{A.1}$$

and, using the duality relations $(1-u_{12}=u_{23}u_{24}u_{25}$, etc.$)$ we get :

$$\begin{aligned} A(p_1\ldots p_6) &= \alpha_{12}\alpha_{34}\alpha_{14}\, B(-\alpha_{12},1-\alpha_{13},-\alpha_{14},2-\alpha_{23},1-\alpha_{24},2-\alpha_{25},-\alpha_{34},1-\alpha_{35},2-\alpha_{45}) \\ &+\alpha_{23}\alpha_{25}\alpha_{45}\, B(2-\alpha_{12},1-\alpha_{13},2-\alpha_{14},-\alpha_{23},1-\alpha_{24},-\alpha_{25},2-\alpha_{34},1-\alpha_{35},-\alpha_{45}) \end{aligned} \tag{A.2}$$

In order to study the factorization properties, we shall consider for simplicity zero mass pions, with units such that the slope of the trajectories is unity and assume that the ρ $-f^0$ trajectory has the intercept one-half.

We first write the B function using Chan variables in the peripheral configuration :

$$\begin{aligned} &B(x_{12},x_{13},x_{14},x_{23},x_{24},x_{25},x_{34},x_{35},x_{45}) = \\ &\iiint du_{12}\,du_{13}\,du_{14}\; u_{12}^{x_{12}-1}(1-u_{12})^{x_{23}-1}u_{13}^{x_{13}-1}\times \\ &\times(1-u_{13})^{x_{34}-1}u_{14}^{x_{14}-1}(1-u_{14})^{x_{45}-1}(1-u_{12}u_{13})^{x_{24}-x_{23}-x_{34}}\times \\ &(1-u_{13}u_{14})^{x_{35}-x_{34}-x_{45}}(1-u_{12}u_{13}u_{14})^{x_{25}+x_{34}-x_{24}-x_{35}} \end{aligned} \tag{A.3}$$

where

$$x_{ij}=m_{ij}-\alpha_{ij}$$

We now make the change of variables

$$u_{12} = \frac{\rho_2'(1+f)}{1-(1-2\rho_2')f}\,;\; u_{13} = \frac{4f}{(1+f)^2}\,;\; u_{14} = \frac{\sigma_2'(1+f)}{1-(1-2\sigma_2')f}\,;\; \rho_1' = \sigma_1' = 1\,, \quad \rho_3' = \sigma_3' = 0 \tag{A.4}$$

$$\begin{cases} X_{12} = X_{\bar2\bar3}\,; X_{13} = X_{\bar1\bar2\bar3} = X_{\textcircled{1}\textcircled{2}\textcircled{3}}\,; X_{14} = X_{\textcircled{2}\textcircled{3}}\,; X_{23} = X_{\bar1\bar2}\,;\; X_{45} = X_{\textcircled{1}\textcircled{2}} \\ X_{24} = X_{\bar2\bar1\textcircled{1}} = X_{\bar3\textcircled{3}\textcircled{2}}\,; X_{25} = X_{\bar3\textcircled{3}}\,; X_{34} = X_{\bar1\textcircled{1}}\,; X_{35} = X_{\bar2\bar3\textcircled{3}} = X_{\bar1\textcircled{1}\textcircled{2}} \end{cases} \tag{A.5}$$

$$m_{12} = m_{\bar2\bar3}\,;\; m_{13} = m_{\bar1\bar2\bar3} = m_{\textcircled{1}\textcircled{2}\textcircled{3}}\,;\; \text{etc}\ldots \tag{A.6}$$

$$P_1 = \bar P_3\,,\; P_2 = \bar P_2\,,\; P_3 = \bar P_1\,,\; P_4 = q_1\,,\; P_5 = q_2\,,\; q_3 = P_6 \tag{A.7}$$

The change of variables corresponds to the configuration shown in Fig. 7. We also denote

$$\begin{aligned} a_{\bar2\textcircled{3}} &= 1 + m_{\bar2\bar3\textcircled{3}} - m_{\bar2\bar3} - m_{\bar3\textcircled{3}}\,; \quad a_{\bar3\textcircled{2}} = 1 - m_{\textcircled{2}\textcircled{3}} - m_{\bar3\textcircled{3}} + m_{\bar3\textcircled{2}\textcircled{3}} \\ a_{\bar2\textcircled{2}} &= -1 + m_{\bar1\textcircled{1}} + m_{\bar3\textcircled{3}} - m_{\bar3\textcircled{3}\textcircled{2}} - m_{\bar2\bar3\textcircled{3}} \\ a_{\bar2\textcircled{1}} &= 1 - m_{\bar1\bar2} - m_{\bar1\textcircled{1}} + m_{\bar2\bar1\textcircled{1}} \\ a_{\bar1\textcircled{2}} &= 1 - m_{\textcircled{1}\textcircled{2}} - m_{\bar1\textcircled{1}} + m_{\bar1\textcircled{1}\textcircled{2}} \\ a_{\bar1\textcircled{1}} &= -1 + m_{\bar1\textcircled{1}} \\ a_{\bar3\textcircled{3}} &= -1 + m_{\bar3\textcircled{3}} \\ a_{\bar3\textcircled{1}} &= m_{\bar2\bar3} + m_{\bar1\bar2} - m_{\bar1\bar2\bar3} - m_{\bar2\bar1\textcircled{1}} \\ a_{\bar1\textcircled{3}} &= m_{\textcircled{2}\textcircled{3}} + m_{\textcircled{1}\textcircled{2}} - m_{\textcircled{1}\textcircled{2}\textcircled{3}} - m_{\bar1\textcircled{1}\textcircled{2}} \end{aligned} \tag{A.8}$$

With the new variables (A.3) reads :

$$B = 4^{X_{\bar1\bar2\bar3}} \int d\rho_2'\, \varphi[\rho_2', X_{\bar1\bar2}, X_{\bar2\bar3}] \int d\sigma_2'\, \varphi[\sigma_2', X_{\textcircled{1}\textcircled{2}}, X_{\textcircled{2}\textcircled{3}}] \int df\, f^{X_{\bar1\bar2\bar3}-1} (1+f)^{-1} F(f, \rho_2', \sigma_2') \tag{A.9}$$

where $\varphi[\rho'_2, x_{\bar{1}\bar{2}}, x_{\bar{2}\bar{3}}] = (\rho'_1-\rho'_2)^{x_{\bar{1}\bar{2}}-1}(\rho'_2-\rho'_3)^{x_{\bar{2}\bar{3}}}$,

$$F = \prod_{i=1}^{3}\prod_{j=1}^{3}\left[1-(1-2\rho'_i)f(1-2\sigma'_j)\right]^{2\bar{p}_i q_j + a_{i(j)}} =$$

$$= \exp\left\{+\sum_{n=1}^{\infty}\frac{f^n}{n}\left[V^{(n)}W^{(n)} - \sum_{i,j} a_{i(j)}(1-2\rho'_i)(1-2\sigma'_j)\right]\right\}, \qquad \text{(A.10)}$$

$$V^{(n)} = \sum_{i=1}^{3}\sqrt{2}\,(1-2\rho'_i)^n\bar{p}_i \; ; \; W^{(n)} = -\sqrt{2}\sum_{j=1}^{3}(1-2\sigma'_j)\,q_j$$

Equations (A.9) and (A.10) may be used to study the whole factorization problem. Since we are interested here only in the leading trajectory, we shall take instead of (A.10) :

$$F'(\rho'_2, \sigma'_2, f) = \exp\left\{f\,V^{(1)}W^{(1)}\right\} \qquad \text{(A.10')}$$

since the other terms in (A.10) cannot contribute at the level of the leading trajectory.

We now go back to our example given by Eq. (A.2). The contribution to the leading trajectory is

$$A(p_1 \cdots p_6) = -2.4^{-\alpha_{\bar{1}\bar{2}\bar{3}}+1}\int d\rho'_2\,\varphi[\rho'_2, -\alpha_{\bar{1}\bar{2}}, -\alpha_{\bar{2}\bar{3}}]\int d\sigma'_2\,\varphi[\sigma'_2, -\alpha_{(1)(2)}, -\alpha_{(2)(3)}]\times$$

$$\int df\, f^{-\alpha_{\bar{1}\bar{2}\bar{3}}}\left[\alpha_{\bar{2}\bar{3}}(\rho'_1-\rho'_2)^2\alpha_{(2)(3)}(\sigma'_1-\sigma'_2)^2\bar{p}_1 q_1 + \alpha_{\bar{1}\bar{2}}(\rho'_2-\rho'_3)^2\alpha_{(1)(2)}(\sigma'_2-\sigma'_3)^2\bar{p}_3 q_3\right]\times \qquad \text{(A.11)}$$

$$\times\sum_n f^n (n!)^{-1}\left[V^{(1)}W^{(1)}\right]^n$$

We now take into account non-cyclical permutations and isospin invariance as given by Eq. (3) and thus consider the contribution of the twisted graphs.

Since under twisting the trace over τ matrices gives a factor $\pm$ for $I = 1/0$, $\rho'_1 \rightleftarrows \rho'_3$, $V^{(1)} \to -V^{(1)}$, $\alpha_{\bar{1}\bar{2}} \rightleftarrows \alpha_{\bar{2}\bar{3}}$, we get

$$A = -2.4^{-\alpha_{\bar1\bar2\bar3}+1} \int df \sum_{n=0}^{\infty} f^{-\alpha_{\bar1\bar2\bar3}+n} \int d\sigma_2' \int d\rho_2'$$

$$\varphi[\rho_2', -\alpha_{\bar1\bar2}, -\alpha_{\bar2\bar3}]\left[\bar{p}_1(\rho_1'-\rho_2')^2\alpha_{\bar2\bar3} \pm (-1)^n \bar{p}_3(\rho_2'-\rho_3')^2\alpha_{\bar1\bar2}\right] \times$$

$$(V^{(1)}W^{(1)})^n \varphi[\sigma_2', -\alpha_{12}, -\alpha_{23}]\left[q_1(\sigma_1'-\sigma_2')^2\alpha_{23} \pm (-1)^n q_3(\sigma_2'-\sigma_3')^2\alpha_{12}\right] \qquad (A.12)$$

and thus we have no ghosts and both isospins, i.e., we are in the "a" case, as discussed in the text.

Every term can be analyzed by this method. The connection between the diagrams and the symmetry of the expressions is now apparent. This is due, of course, to the simple transformation of V in this representation.

REFERENCES AND FOOTNOTES

1) See, for example : V. Alessandrini, D. Amati, M. Le Bellac and D. Olive, CERN Preprint TH. 1160 (1970), for a comprehensive review.

2) We follow the method proposed by V. Rittenberg and H.R. Rubinstein, Phys.Rev.Letters (to be published). Notice that the example fully discussed in that paper gives ghosts at the eight-point level and it cannot be considered satisfactory. In that case the 2^- recurrence of the π is doubled.

3) D. Olive and W.J. Zakrzewski, Nuclear Physics (to be published), have found, using less general propagator techniques, this solution. Notice that this solution also breaks down at the eight-point level a fact already noticed by these authors.

4) K. Bardakçi and H. Ruegg - Phys.Rev.Letters 28B, 671 (1969).

5) D.J. Dorren, V. Rittenberg, H.R. Rubinstein, M. Chaichian and E.J. Squires, Nuovo Cimento (to be published). See also J. Gabarro and L. Gonzalez Mestres, Orsay Preprint, May (1970).

FIGURE CAPTIONS

Figure 1 Graph illustrating the polynomial given by Eq. (2) for the six-pion function. Continuous line represents a $(1-u)\alpha$ factor ; dotted line represents a $(1-u)$ factor.

Figure 2 The contribution of the cyclically permuted graph shown in Fig. 1 to the (1,3) leading trajectory (wiggled line).

Figure 3 Graphical interpretation of the four-pion function.

Figure 4 Analysis of the $G = -1$ trajectory for different solutions of the six-pion functions. Continuous line represents $\alpha(1-u)$ factors, dotted lines represent $(1-u)$ factors, lines with crosses represent u factors, the wiggled lines represent the pole line.

Figure 5 Analysis of $G = +1$ trajectory for different solutions of the eight-pion functions. The different lines have the same meaning as in Fig. 4.

Figure 6 Graph used to write the $K\bar{K} + 5\pi$ function. The shaded pentagon represents the Bardakçi-Ruegg formulae for $K\bar{K} - 3\pi$, the dotted line and the continuous line have the same meaning as in Fig. 4.

Figure 7 The external legs ordering used in the Appendix for the six-pion functions.

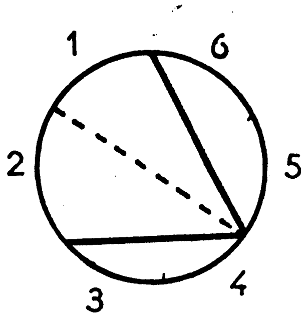


Fig. 1

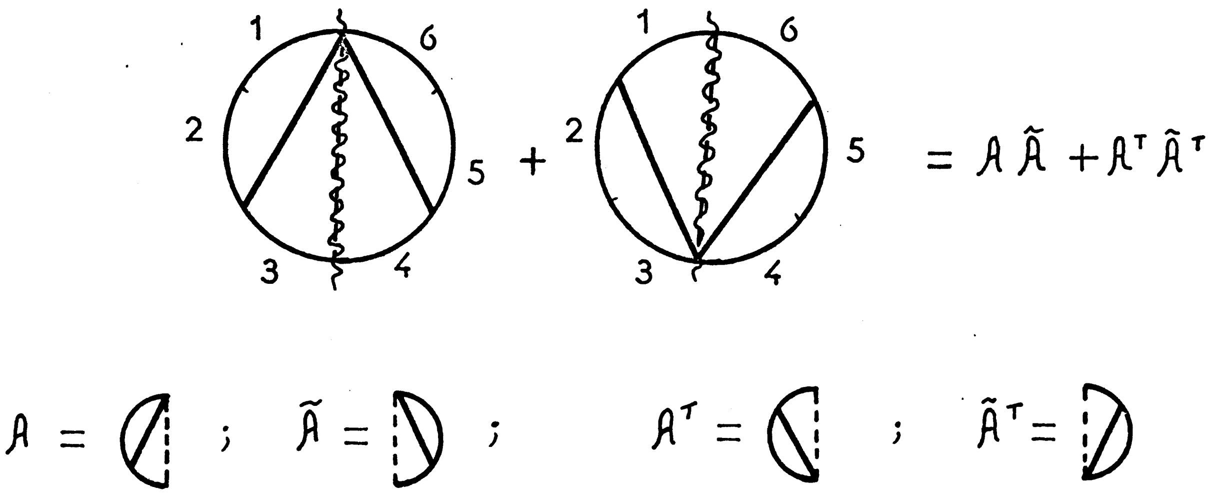


Fig. 2

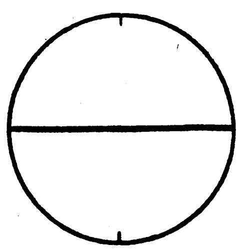
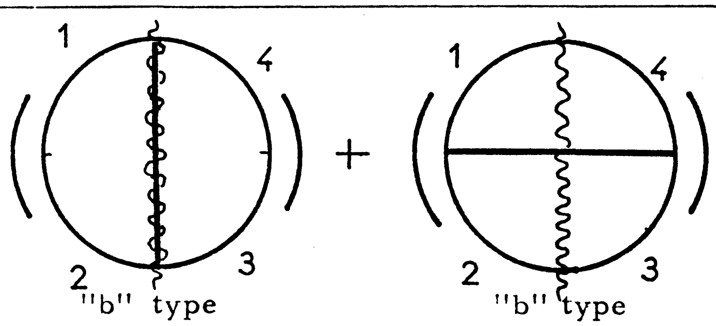
Symbol and representative figures	Pole decomposition of the leading trajectory
	

Fig. 3

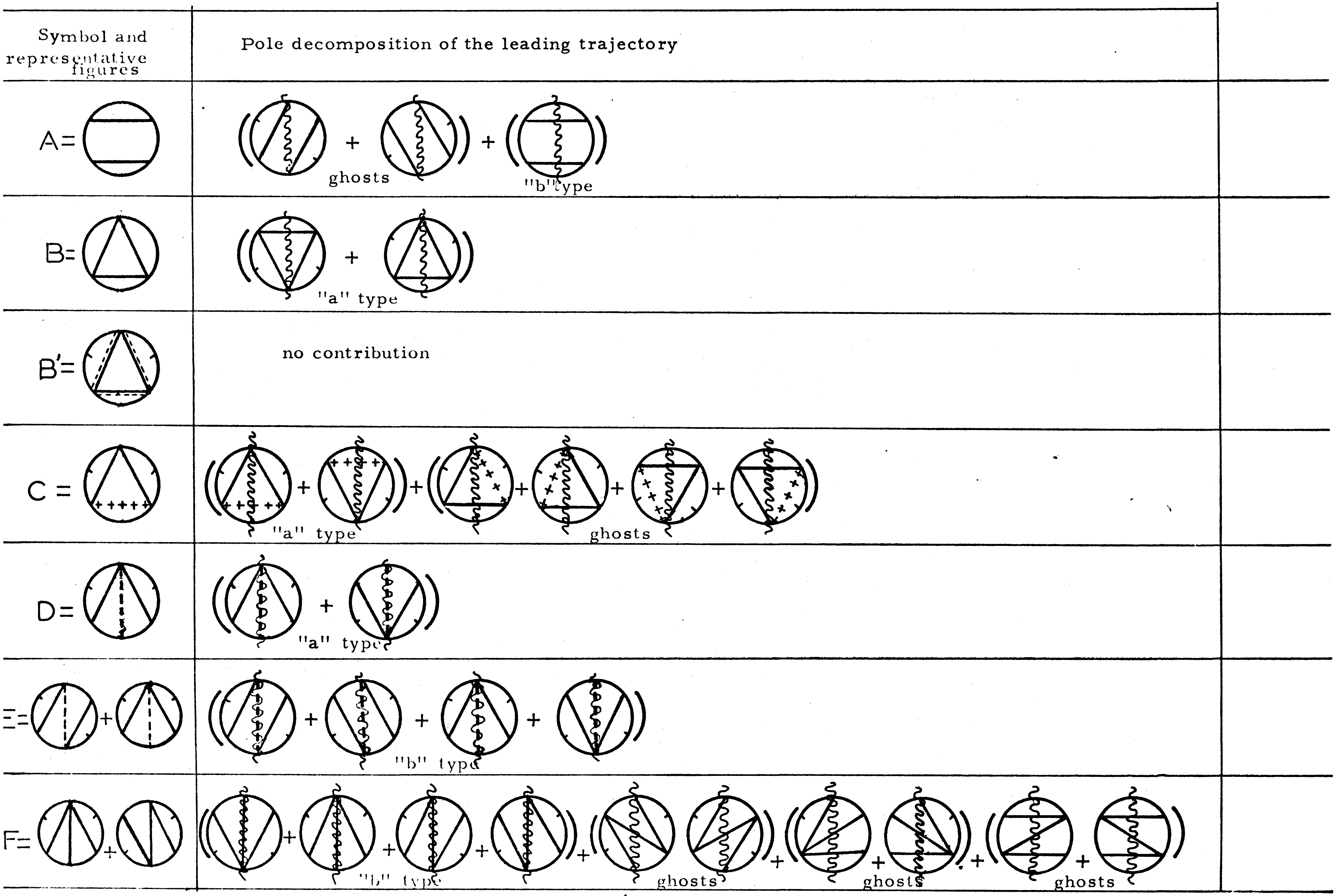

Symbol and representative figures
Pole decomposition of the leading trajectory
A=
ghosts
"b" type
B=
"a" type
B'=
no contribution
C =
"a" type
ghosts
D=
"a" type
E=
"b" type
F=
"b" type
ghosts
ghosts
ghosts

Fig. 4

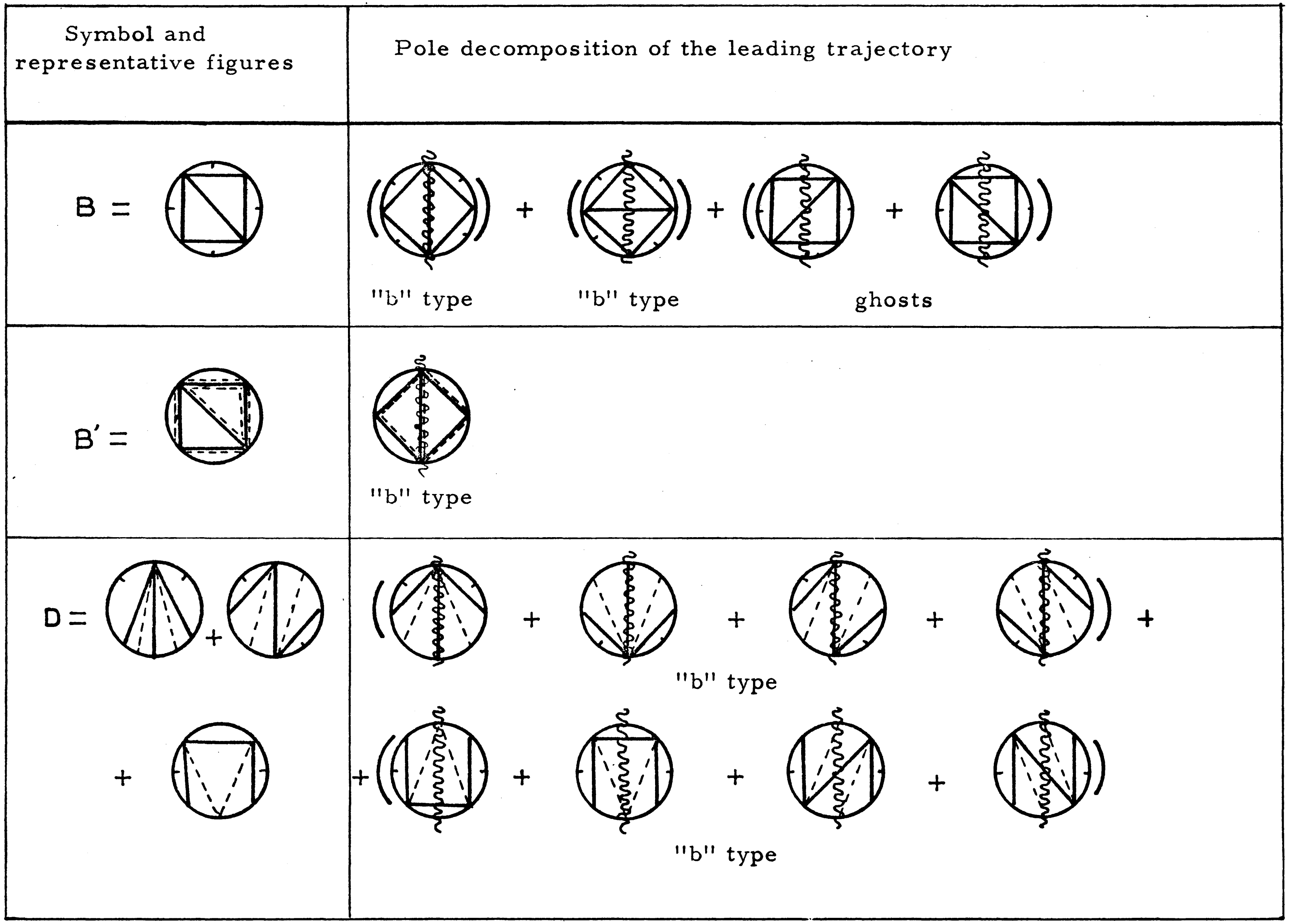

Symbol and representative figures
Pole decomposition of the leading trajectory
B =
"b" type
"b" type
ghosts
B′ =
"b" type
D =
"b" type
"b" type

Fig. 5

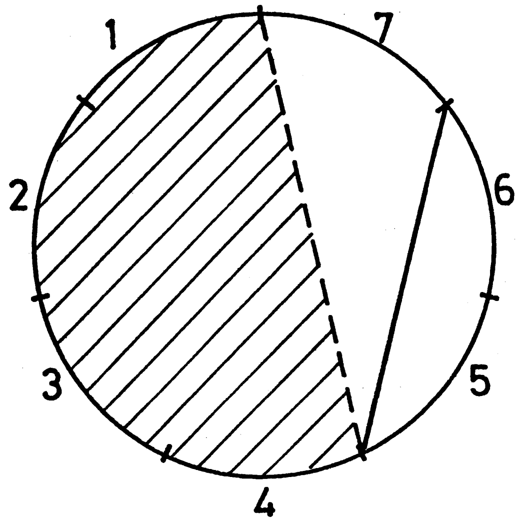


FIG. 6

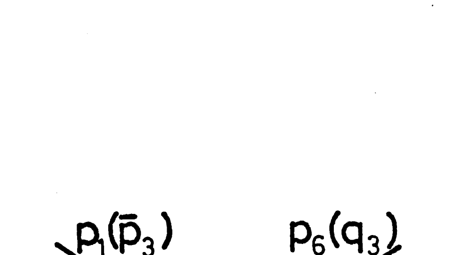


FIG. 7