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NON-LINEAR PLASMA OSCILLATIONS IN ONE-DIMENSIONAL FLUX

АРУС

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НЕЛИНЕЙНЫЕ ПЛАЗМЕННЫЕ КОЛЕБАНИЯ В ОДНОМЕРНОМ ПОТОКЕ

В гидродинамическом приближении рассмотрены нестационарные колебания конечной амплитуды в одномерном потоке плазмы или электронном пучке. Получено точное решение задачи с начальными условиями для самосогласованной системы уравнений, описывающих нелинейные явления в потоке. Как примеры применения общих формул рассмотрены некоторые конкретные задачи, иллюстрирующие роль нелинейности.

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NON-LINEAR PLASMA OSCILLATIONS IN ONE-DIMENSIONAL FLUX

The non-stationary non-linear oscillations in one dimensional plasma flux or electron beam are considered in the hydrodynamical approximation. The exact solution of the problem with initial conditions for the relevant self-consistent system of equations is obtained. As application of the obtained general formulae some physical processes as the density relaxation, the evolution of a flux with density and velocity modulation are considered illustrating the role of the non-linearity.

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In the linear approximation the waves arising in plasma fluxes or electron beams have been investigated by a number of authors in order to understand the processes of electron bun - ching in various electronic UHF devices (travelling wave tubes, klystrons, etc.)

The non-stationary non-linear processes have been consi - dered by means of non-linear interactions of harmonic waves which is, however, justified only in the case of weakly non - linear processes when the flux density and velocity perturba tion is sufficiently small [1] . For the study of the strongly non-linear processes in electron beams one uses the methods of "large particles" or the method of strongly non-linear (over- turned) wave expansion of the unknown magnitudes [2].

In present work the non-stationary plasma oscillations of finite amplitude in one-dimensional flux or electron beam are considered in the hydrodynamical approximation. It is obtained the exact solution of the problem with initial conditions for the selfconsistent system of equations describing the non - linear phenomena in the flux. Some concrete problems illus - trating the role of non-linearity are considered as examples of approximation of the general formulae.

1. Let us consider a homogene plasma with compensating background of ions in rest the density of which is n_0. The electrons move in the direction of z axis with a constant velocity u_0 $(u_0 \ll c)$. If one takes into account the elec- tron-ion collisions then in order to maintain a constant current corresponding to the velocity u_0 it is necessary the presence of an electric field E_0 in the system.

To study the process of the development of the space charge longitudinal oscillations in the plasma flux one can use the following equations : The equation of motion for the average electron hydrodynamical velocity $u(z,t)$ phenomenologically taking into account the collisions:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \nu u = -\frac{e}{m}E \,, \qquad (1)$$

where ν is the effective frequency of the collisions, e and m are the electron absolute charge and mass, respectively; The continuity equation:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0 \,, \qquad (2)$$

where $n(z,t)$ is the electron density;
The Poisson equation:

$$\frac{\partial E}{\partial z} = 4\pi e(n_0 - n) \qquad (3)$$

Here E is the electric field directed along axis z.
Further we shall study the system of equations for velocity $u(z,t)$ and field $E(z,t)$. By eliminating the density $n(z,t)$ from (2) and (3) we obtain:

$$\frac{\partial}{\partial z}\left(\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial z} - 4\pi e n_0 u\right) = 0$$

or

$$\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial z} - 4\pi e n_0 u = 4\pi I(t) \qquad (4)$$

where $I(t)$ is an arbitrary function of time. Since it has been assumed that for the unperturbated state (at $t<0$) $E=E_0$ and $u=u_0$, from (4) we obtain $I(t)=-en_0u_0$. Equations (1) and (4) form a close system for determination of the field E and velocity u. Let us rewrite these equations in a form suitable for our further purposes:

$$\mathcal{D}u+2\alpha u+v=0,$$
$$\mathcal{D}v-u+u_0=0, \qquad (5)$$

where the following notations are used:

$$\mathcal{D}=\frac{\partial}{\partial\tau}+u\frac{\partial}{\partial x},\quad \tau=\omega_p t,\quad x=\omega_p z, \qquad (6)$$
$$v=\frac{e}{m}\frac{E}{\omega_p},\quad 2\alpha=\frac{\nu}{\omega_p},\quad \omega_p^2=\frac{4\pi n_0 e^2}{m}.$$

Further, for the generality of the consideration, we shall assume that the parameter α in (5) may take positive as well as negative values. The case $\alpha>0$ describes damping due to collisions. The negative values of this parameter may be used for the model description of a non-equilibrium medium [3].

Let us replace in eqs.(5) the functions u and v by new functions u' and v' using the relations:

$$u=u_0+e^{-\alpha\tau}(u'-v'),$$
$$v=-2\alpha u_0+e^{-\alpha\tau}(u'+v'). \qquad (7)$$

As a result we have

$$\mathcal{D}u'=-(1-\alpha)v',$$
$$\mathcal{D}v'=(1+\alpha)u'. \qquad (8)$$

The solutions of this system have qualitively different behaviour for various values of α. Further, for simplicity let us limit ourselves by the case when $|\alpha| < 1$ (the case $|\alpha| > 1$ requires evident modifications).

From eqs. (8) it follows that the non-linear oscillation frequency is equal to $\omega_p\sqrt{1-\alpha^2}$ and that the magnitude $\mathcal{D}\left((1+\alpha)u'^2+(1-\alpha)v'^2\right)$ becomes equal to zero. Therefore, it is convenient to use a coordinate frame rotating with a frequency $\omega_p\sqrt{1-\alpha^2}$ by means of following transformations:

$$\sqrt{1+\alpha}\,u' = u_1 \cos\tau' - v_1 \sin\tau', \qquad (9)$$
$$\sqrt{1-\alpha}\,v' = u_1 \sin\tau' + v_1 \cos\tau',$$

where $\tau' = \tau\sqrt{1-\alpha^2}$, while u_1 and v_1 are new functions of x and τ.

Substituding these expressions for u' and v' into (8) one obtains:

$$\mathcal{D}u_1 = 0, \quad \mathcal{D}v_1 = 0. \qquad (10)$$

Thus, u_1 and v_1 remain constant along a characteristic curve defined by the equation:

$$\frac{dx}{d\tau} = u = u_0 + e^{-\alpha\tau}(u' - v') =$$

$$u_0 + e^{-\alpha\tau}\left(\frac{u_1\cos\tau' - v_1\sin\tau'}{\sqrt{1+\alpha}} - \frac{u_1\sin\tau' + v_1\cos\tau'}{\sqrt{1-\alpha}}\right). \qquad (11)$$

Integrating this equation for constant and and choosing the integration constant in such a way that the characteristic would pass through the origin of the coordinate frame

$\left(x(\tau=0)=0 \right)$ we obtain:

$$x-u_0\tau-\left(\frac{u_1}{\sqrt{1+\alpha}}-\frac{v_1}{\sqrt{1-\alpha}}\right)\left[e^{-\alpha\tau}\left(\sqrt{1-\alpha^2}\sin\tau'-\alpha\cos\tau'\right)+\alpha\right]- \tag{12}$$

$$-\left(\frac{u_1}{\sqrt{1-\alpha}}+\frac{v_1}{\sqrt{1+\alpha}}\right)\left[e^{-\alpha\tau}\left(\alpha\sin\tau'+\sqrt{1-\alpha^2}\cos\tau'\right)-\sqrt{1-\alpha^2}\right]\equiv y=\text{const}$$

Returning to the functions u' and v' we have:

$$y= x-u_0\tau-e^{-\alpha\tau}\left(u'+v'\right)+ \tag{13}$$

$$+\left(u'+v'\right)\left(\cos\tau'-\frac{\alpha}{\sqrt{1-\alpha^2}}\sin\tau'\right)-\left(u'-v'\right)\frac{\sin\tau'}{\sqrt{1-\alpha^2}}.$$

The solutions of eqs. (10) can be expressed by the characteristic (12) or (13) in the following form:

$$u_1=f_1(y),\quad v_1=f_2(y), \tag{14}$$

where $f_1(x)$ and $f_2(x)$ are the initial values of the functions u_1 and v_1 $\left(f_1(x)=u_1(x,0),\ f_2(x)=v_1(x,0)\right)$.
According to (7) and (9) the functions $f_1(x)$ and $f_2(x)$ are connected with the initial values of u and v by the relations

$$\begin{aligned} 2\frac{f_1(x)}{\sqrt{1+\alpha}} &= u(x,0)-u_0+v(x,0)+2\alpha u_0, \\ 2\frac{f_2(x)}{\sqrt{1-\alpha}} &= v(x,0)+2\alpha u_0-u(x,0)+u_0. \end{aligned} \tag{15}$$

Again according to (7) and (9) for any moment we have the following expressions for u and v :

$$u - u_0 = e^{-\alpha\tau}\left[\left(\frac{u_1}{\sqrt{1+\alpha}} - \frac{v_1}{\sqrt{1-\alpha}}\right)\cos\tau' - \left(\frac{u_1}{\sqrt{1-\alpha}} + \frac{v_1}{\sqrt{1+\alpha}}\right)\sin\tau'\right],$$

(16)

$$v + 2\alpha u_0 = e^{-\alpha\tau}\left[\left(\frac{u_1}{\sqrt{1-\alpha}} - \frac{v_1}{\sqrt{1+\alpha}}\right)\sin\tau' + \left(\frac{u_1}{\sqrt{1+\alpha}} + \frac{v_1}{\sqrt{1-\alpha}}\right)\cos\tau'\right]$$

For the given initial conditions these formulae determine completely the electric field and the velocity of the plasma particles at any moment. The electron density $n(x,\tau)$ is determined by the Poisson equation:

$$n(x,\tau) = n_0\left(1 - \frac{\partial v}{\partial x}\right). \qquad (17)$$

2. As an example consider the following simple relaxation problem: Let for some reason at the moment $t=0$ it appears a surplus or deficiency of electrons with constant density n in the region $x_1 < x < x_2$. Assume also that $u(x,0)=0$, $u_0 = 0$. Since the density is constant from (17) we obtain for the initial field: $v(x,0) = \left(1 - \frac{n}{n_0}\right)x \equiv \alpha x$.
From eqs (15) we obtain for $f_1(x)$ and $f_2(x)$: $f_1 = \sqrt{1+\alpha}\,\frac{\alpha x}{2}$, $f_2(x) = \sqrt{1-\alpha}\,\frac{\alpha x}{2}$. Meanwhile from (14) it follows

$$\frac{u_1}{\sqrt{1+\alpha}} = \frac{v_1}{\sqrt{1-\alpha}}.$$

Taking into account this equation the expression for y takes the form

$$y = x - \frac{2v_1}{\sqrt{1-\alpha}\sqrt{1-\alpha^2}}\left[e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right) - \sqrt{1-\alpha^2}\right].$$

From (14) one obtains

$$v_1 = \frac{\sqrt{1-\alpha}\; ax/2}{1 + \frac{a}{\sqrt{1-\alpha^2}}\left[e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right) - \sqrt{1-\alpha^2}\right]} . \tag{18}$$

The substitution of expressions for u_1 and v_1 into (16) gives:

$$u = -\frac{ax\, e^{-\alpha\tau}\sin\tau'}{\sqrt{1-\alpha^2}(1-a) + a\, e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right)},$$

$$v = \frac{ax\, e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right)}{\sqrt{1-\alpha^2}(1-a) + a\, e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right)} \tag{19}$$

Now from (17) we have:

$$\frac{n(\tau)}{n_0} = \frac{\sqrt{1-\alpha^2}(1-a)}{\sqrt{1-\alpha^2}(1-a) + a\, e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right)} \tag{20}$$

These solutions are determined in the region:

$$x_1 < \frac{x\sqrt{1-\alpha^2}}{\sqrt{1-\alpha^2}(1-a) + a\, e^{-\alpha\tau}\left(\alpha \sin\tau' + \sqrt{1-\alpha^2}\cos\tau'\right)} < x_2. \tag{21}$$

The relaxation front velocity on the boundary x_i is given by:

$$u_F = -\frac{a x_i}{\sqrt{1-\alpha^2}} e^{-\alpha\tau}\sin\tau'. \tag{22}$$

The small values of the parameter $a = 1 - \frac{n}{n_0}$ correspond to the linear case. In this case one can use the power series with respect to a. In the case when there is no damping $\alpha = 0$

maintaining only the first expansion terms we obtain density oscillations with plasma frequency ω_p . If one takes into account the non-linearity as it is seen from (19) and (20) for $\alpha = 0$, it will be certainly changed the frequency spectrum and the feature of the oscillations the amplitude of which has a complicated dependence on a. As it is seen from (20) during the relaxation process the plasma electron density remains uniform. The density will be a certain positive magnitude only if the denominator in (19) - (21) is positive for all the $\tau > 0$. This takes place if $a < 0$ (electron surplus). If $a > 0$ (electron deficiency) and it takes place the inequality $a < \left[1 + \exp\left(-\frac{\pi\alpha}{\sqrt{1-\alpha^2}}\right)\right]^{-1}$ (when $\alpha > 0$) then the denominator is positive too. If the last inequality is violated then the denominator has zeros and the solution of the problem under consideration is non-unique. This means that the system (5) has unambiguous smooth solutions only for a limited class of initial conditions. As it is mentioned in [4] the absence of unique solutions of the Cauchy problem for system even simplier than (5) is rather a rule than an exclusion.

In conclusion of this section it is worthy to note that there is no a smooth and unambiguous for all $\tau > 0$ solution of the problem under consideration for the case $\alpha < 0$ (non-equilibrium medium).

3. Now let us consider the evolution of a flux having a density and velocity modulation at the initial moment:

$$u(x,0) = u_0 + \tilde{u} \cos k_0 z = u_0 + \tilde{u} \cos kx,$$
$$n(x,0) = n_0 \left(1 - kV_0 \cos kx\right). \qquad (23)$$

According to (17) the initial value of the field v is given by the expression (the integration constant is assumed to be zero)

$$v(x,0) = v_0 \sin kx \,. \tag{24}$$

From the positivity condition of the density $n(x,0)$ for all x it follows a condition for the modulation parameters v_0 and k : $kv_0 \leq 1$. For simplicity we shall assume that there is no damping, i.e. $\alpha = 0$.

Using these initial conditions and formulae (15) and (16) we obtain

$$u = u_0 + v_0 \sin\tau \sin\psi + \tilde{u} \cos\tau \cos\psi = \\ = -\frac{v_0-\tilde{u}}{2}\cos(\psi+\tau) + \frac{v_0+\tilde{u}}{2}\cos(\psi-\tau), \tag{25}$$

$$v = v_0 \cos\tau \sin\psi - \tilde{u} \sin\tau \cos\psi = \\ = \frac{v_0-\tilde{u}}{2}\sin(\psi+\tau) + \frac{v_0+\tilde{u}}{2}\sin(\psi-\tau),$$

where

$$\psi = kx - ku_0\tau + k\left[v(1-\cos\tau) - (u-u_0)\sin\tau\right] \equiv \\ = kx - ku_0\tau + Z(x,\tau). \tag{26}$$

Considering the space charge waves in a plasma flux or electron beam one usually limits himself by the linear approximation when the perturbation amplitudes v_0 and $\tilde{u}$ are small, i.e. $v_0 \ll u_0$, $\tilde{u} \ll u_0$ and $kv_0 \ll 1$. As it is seen from (25) and (26) this fact corresponds to the condition $|Z| \ll 1$ and as a result the solution is the sum of slow and fast waves with phase velocities $u_0 - \frac{\omega_p}{k_0}$ and $u_0 + \frac{\omega_p}{k_0}$, respectively, which is anticipated [5,6].

In order to study the non-linear phenomena in electron beams it is necessary to investigate the functions $Z(x,\tau)$ for non-small values of the amplitudes v_0 and $\tilde{u}$. Since $Z(x,\tau) = K\left[v(1-\cos\tau) - (u-u_0)\sin\tau\right]$, substituding the values of u and v from (25) we have the following equation for determination of $Z(x,\tau)$

$$Z = \mathcal{E}(\tau)\sin\left(Z + Kx - Ku_0\tau - \varphi\right), \qquad (27)$$

where

$$\mathcal{E}(\tau) = K\left[\tilde{u}^2\sin^2\tau + v_0^2(1-\cos\tau)^2\right]^{1/2}, \quad tg\varphi = \frac{\tilde{u}\sin\tau}{v_0(\cos\tau - 1)}. \qquad (28)$$

The number of the possible solutions of eq. (27) is determined mainly by the function $\mathcal{E}(\tau)$, which is periodic with a period 2π (when $v_0 \neq 0$). When $0 < \tau < 2\pi$ this function has a single maximum at $\tau = \pi$, equal to $\mathcal{E}(\pi) = 2Kv_0$ if the condition $\tilde{u}^2 \leq 2v_0^2$ is fulfilled. If $\tilde{u}^2 > 2v_0^2$ this function has two maxima at the points $\tau_\pm = \pi \pm \arccos\left(v_0^2/(\tilde{u}^2 - v_0^2)\right)$ and $\mathcal{E}(\tau_\pm) = \frac{K\tilde{u}^2}{\sqrt{\tilde{u}^2 - v_0^2}}$ and $\mathcal{E}(\tau_\pm) > \mathcal{E}(\pi)$. Therefore, independently from the fact, is the condition $\tilde{u}^2 > 2v_0^2$ fulfilled or not, with the growth of τ the values of the function $\mathcal{E}(\tau)$ will exceed unity if $2Kv_0 > 1$. As it follows from the well known theorem on the unexplicit functions, eq. (27) determines an one-valued and differentiable function $Z(x,\tau)$ for all the values of x and τ if only $\mathcal{E}_{max} < 1$. If this condition is fulfilled the eq. (27) within an accuracy of notations ~~coincides with the classic Kepler equation defining the~~

excentric anomaly. In this case the function $Z(x,\tau)$ may be presented in a form of a known series [7]

$$Z(x,\tau)=2\sum_{n=1}^{\infty}\frac{J_n(n\xi)}{n}\sin\left(kx-ku_0\tau-\varphi\right), \qquad (29)$$

where $J_n(n\xi)$ are the Bessel functions.

Thus, when $\xi_{max}< 1$ the expressions (25), (26) and (29) determine two strongly non-linear waves the front slope of which increases gradually up to overturning when τ grows. When $2KV_0\gg 1$ the solution $Z(x,\tau)$ of theequation (27) is a many-valued function. This means that with the growth of τ the slope of the front of the non-linear wave (27) increases and when $\tau=\tau_1$, where τ_1 is the smallest root of the equation $\xi(\tau)=1$, some singularities appear on the front which result in overturning of the wave and arising of many-beam states. The condition $2KV_0=2\frac{K_0V_0}{\omega_p}\gg 1$ means that the charcteristic half period of the perturbations is less than the period of plasma oscillations. Therefore, the plasma "has no time to follow" the fast beam perturbation changes. This fact results in arising of small-scale processes as a result of which many beam states are formed the description of which is given by expressions (25), (26) and (27).

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