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CONSTRUCTION OF THE $\pi\pi$ SCATTERING AMPLITUDE FROM SUPERCONVERGENCE

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A B S T R A C T

Assuming Regge trajectories fall linearly for negative t, we formulate superconvergence relations at infinitely many discrete t values. We saturate with an infinite number of resonances ; but only a finite number is involved at each finite t. Assuming the spacing in the grid $\Delta s = \Delta t = (\alpha')^{-1}$ we construct a unique solution, which turns out to be the Veneziano formula. Duality is not used as an input, it comes out as a consequence.

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Our central assumption is that the t channel Regge trajectories continue to drop linearly for negative t. We assume that the Regge expansion is dominated by these Regge poles, i.e., that there are no higher lying cuts, $\alpha_{cut}(t) > \alpha_{pole}(t)$. This requires that more and more superconvergence relations (SCR) *) are satisfied as we go to larger values of $(-t)$ **).

We saturate the SCR's by direct-channel poles, i.e., poles in the s and u channels. We neglect the non-resonating background in the direct channels. Within the framework of the narrow resonance approximation, $\mathrm{Im}\,\alpha(s) = 0$, $\mathrm{Im}\,\beta(s) = 0$, there are no double spectral functions. Therefore, the problem of fixed poles in the signatured amplitude at wrong-signature, unphysical integer values of j does not arise. Resonance saturation in the direct channels forces us to ignore the Pomeranchuk pole in the crossed channel (t channel) ***). Therefore, we can formulate SCR's even for $I_t = 0$, where a fixed Pomeranchuk singularity would otherwise destroy superconvergence ****).

In order to simplify our expressions we consider the s channel process $\pi^+\pi^- \to \pi^+\pi^-$. The u channel has $I_u = 2$ and contains no resonances. This induces exchange degeneracy in the s

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*) We consider here only superconvergence relations [1] in the proper sense, but not finite energy sum rules, which are sometimes called generalized superconvergence relations.

**) This is quite different from the usual SCR approach [1] where one stays at $t=0$, but enforces superconvergence by taking $I_t = 2$ and/or helicity flip $\neq 0$.

***) The "Harari ansatz" [2] is an inescapable consequence of the following two assumptions : (a) resonance saturation for the imaginary part ; (b) no resonances in channels like K^+p, pp.

****) These assumptions about the Pomeranchuk play no essential role in the following. E.g., we could consider the s channel process $K^+K^- \to K^0\bar{K}^0$.

and t channels for the trajectory functions α and the reduced residue functions β *) :

$$\alpha_\rho(t) = \alpha_f(t) = \tfrac{1}{2} + t \qquad (1)$$

$$\beta_{\rho^0, \pi^+\pi^-}(t) = \beta_{f^0, \pi^+\pi^-}(t) \qquad (2)$$

By eliminating signature we remove the qualitative difference between even and odd momentum SCR's. We shall talk directly about the poles in $\pi^+\pi^- \to \pi^+\pi^-$. We can forget about their isotopic spin assignments (which can always be reconstructed by using Bose statistics) and the I- spin Clebsch-Gordan coefficients. The s channel is identical to the t channel. However in this article the s-t symmetry will never be used, because we work at $t < 0$ (SCR's) and at $s = m_{res}^2 > 0$ (resonance saturation).

It is convenient to introduce a discrete energy scale, i.e., to combine all direct-channel resonances with masses within a bin of $\Delta s = 1$ GeV^2 into one "direct-channel pole". Its residue function $R_i(t)$ is the sum of a finite number of Legendre polynomials. - The choice of 1 GeV^2 for Δs is motivated (a) by the experimental spacing, (b) by the fact that two resonances with the same j are indistinguishable if $\Delta s \lesssim 2m\Gamma \sim 0.5$ GeV^2.

Where should we cut off the superconvergence integral ? Because t is negative the SC integral, which runs from $s = 0$ to $s = \infty$, covers the unphysical region $z_s < -1$ for low s. In this region the direct-channel resonances generally give very large contributions, because $z_s \ll -1$ in $P_\ell(z)$. As we increase the energy s (integration variable) at fixed t, we come to the point $\vartheta_s = 180^\circ$. We know experimentally (from K^-p backward elastic scattering) that the backward amplitude is very small if one cannot exchange a u channel Regge pole. As we increase s even more (at fixed t) we come into the region $\vartheta_s < 180^\circ$, and the experimental cross-sections become even smaller. Therefore we cut off the SC integral just before we reach

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*) All our units are in GeV^2. We set $m_\pi^2 = 0$, $m_\rho^2 = \frac{1}{2}$, $d\alpha/dt = 1$, $\alpha(0) = \frac{1}{2}$.

180°, i.e., at $s < s(t, \vartheta_s = 180^\circ)$. This means that at each finite t we saturate with the finite number of those direct-channel poles which are in the unphysical region. As $(-t)$ becomes larger, this number of direct-channel poles becomes larger.

That this is very reasonable is illustrated in Fig. 1, where we show the analogous s channel process $\bar{K}N \to \bar{K}N$ with $I_s = 0$. We choose this example because it contains a string of five experimentally known resonances on one (exchange degenerate) trajectory. We have plotted their contributions $C[\mathrm{Im}\ \bar{B}]$ to the SCR $\int \mathrm{Im}\ B\, d\nu = 0$ at $t = -3.34\ \mathrm{GeV}^2$. The first four poles lie in the unphysical region beyond 180°, $m^2 < s(t, \vartheta_s = 180^\circ)$. They give large contributions since the large $|z_s|$ values together with the large $\alpha(s)$ cause the $P_\ell(z_s)$ to blow up. On the other hand the fifth resonance, the $Y_0^*(2350)$ sits at $s = s(\vartheta_s = 180^\circ)$ for the t value used in Fig. 1. It gives a very small contribution, only 1.4% of the $5/2^+$ contribution. The experimental amplitude at 180° is even smaller, because the various partial waves cancel. Note that $(d\sigma/du)_{exp}$ at 180° [3] is about f... times smaller than the contribution by the $Y^*(2350)$. It is therefore an excellent approximation to cut off the SCR at $s \lesssim s(t, \vartheta_s = 180^\circ)$.

Is the saturation by the leading resonances a good approximation for $z_s \ll -1$? The fear that resonance saturation becomes worse and worse as $|z_s|$ becomes larger and larger is unfounded, because Regge theory tells us that resonance saturation becomes better and better as $z_s \to \pm\infty$. In our case $|z_s|$ is just large, but not infinite, in the important region. That it is large enough for the saturation by the leading resonances to be excellent is indicated by the following fact : all other Y_0^* resonances in the Rosenfeld table, which lie on lower trajectories $\alpha(s)$ and are less peripheral, each contribute less than 4% of the $5/2^+$ contribution.

At what t values shall we formulate our SCR's ? We have assumed that the cancellation of the resonances is achieved in a different way at small $(-t)$ and at large $(-t)$. Therefore we want to keep the various t regions separate. This means that we are not allowed to enforce a SCR exactly in a continuous interval δt at small $(-t)$. If we did that, then the SCR would be satisfied automatically for all t. We must formulate the SCR's at discrete t values. For the spacing, we choose $\Delta t = 1\ \mathrm{GeV}^2$. This choice corresponds to the typical structure length in experimental differential cross-sections.

For small $(-t)$ we saturate with a small number of direct channel resonances. This is only possible because we use discrete t's. Had we required $S_0(t) = 0$ for continuous t in a small interval then we should have needed an ∞ number of resonances even at small $(-t)$.

The SCR $S_0(t) = \int_0^\infty ds\ \mathrm{Im}\ A(s,t) = 0$ holds when $\alpha(t) < -1$. At $\alpha(t) = -1$ the SC integral is not at all zero, at that point it measures the residue $\beta(t)$ of the (moving) t channel Regge pole. We formulate $S_0(t) = 0$ at infinitely many discrete points separated by $\Delta t = 1\ \mathrm{GeV}^2$ starting at $\alpha(t) = -2$. The next SCR, $S_1(t) = \int_0^\infty ds\ s\ \mathrm{Im}\ A = 0$, holds for $\alpha(t) < -2$. We formulate it at $\alpha(t) = -3, -4, \ldots$ And similarly for $S_2(t)$, $S_3(t)$, etc. At *)

$$\alpha(t_j) = \tfrac{1}{2} + t_j = -j - 1 \qquad j = 1, 2, \ldots \tag{3}$$

we formulate j SCR's : $S_0, S_1, \ldots S_{(j-1)}$. We saturate with those direct-channel poles which have $m^2 = s_i < s(t, 180^\circ)$:

$$s_i < s\,(t_j, \vartheta_s = 180^\circ) = j + 3/2 \tag{4}$$

$$s_i = i + \tfrac{1}{2} \qquad i = 0, 1, \ldots j \tag{5}$$

As we go to larger $(-t)$ we impose more and more SCR's and we saturate them with a larger and larger number of s channel resonances.

For $t = t_j$ we have $(j+1)$ resonances to saturate j SCR's. Therefore, we obtain a unique solution. Had we chosen a grid with $\Delta s < 1/\alpha'(t)$ then there would have been infinitely many solutions. Had we chosen $\Delta s > 1/\alpha'$, there would have been more equations than unknowns, and no solution would have existed.

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*) See footnote p. 2.

The residue function of the pole at $s = s_i$ will be called $R_i(t_j)$, and we treat the values of R_i at the various points t_j as independent unknowns. Our problem is now reduced to the following mathematical form : we have a triangular array (see Fig. 2) of unknowns $R_i(t_j)$ with $j = 1 \dots \infty$ and $i = 0, 1, \dots j$. In the row j there are $(j+1)$ elements labelled by i, and we impose j SCR's :

$$\sum_{i=0}^{j} R_i(t_j)\, i^n = 0 \qquad n = 0, 1, \dots (j-1) \tag{6}$$

The factor i^n corresponds to the moment factor ν^n, where $\nu = (s-u)/2$. The translation of the origin of the weight function :

$$\nu_i = s_i + t_{j/2} \longrightarrow i = s_i - 1/2 \tag{7}$$

is irrelevant as long as we set all moments up to n equal to zero. We solve the system of homogeneous linear equations (6) separately for each row j. A normalization factor for each t_j remains undetermined. We use the residue of the lowest s channel pole as our normalization :

$$R_0(t_j) = N_j \tag{8}$$

The solution to our algebraic problem (6) is the binomial coefficient $\binom{j}{i}$:

$$R_i(t_j) = N_j (-1)^i \binom{j}{i} \tag{9}$$

The proof follows from the definition of the binomial coefficient :

$$(1+z)^j = \sum_{i=0}^{j} z^j \binom{j}{i} \tag{10}$$

We multiply both sides n times by the differential operator $(z \frac{d}{dz})$:

$$\left(z \frac{d}{dz}\right)^n (1+z)^j = \sum_{i=0}^{j} z^i\, i^n \binom{j}{i} \tag{11}$$

and set $z = -1$:

$$0 = \sum_{i=0}^{j} (-1)^i\, i^n \binom{j}{i} \qquad n = 0, 1, \dots (j-1) \tag{12}$$

This proves that the expression (9) solves the system of equations (6).

We choose the normalization N_j by requiring that the pole at $s = \frac{1}{2}$ has a linear residue function, like a superposition of ρ and $\epsilon(750)$. Note that the zero of N_j will appear in all resonances s_i with $i \to \infty$. Because of exchange degeneracy our Regge amplitude must be real for $\alpha(t)$ = integer. This implies that $\mathrm{Im}\, A_{Regge} = 0$ for $\alpha = 0, -1, -2, \ldots$ The zeros at high energy for $\alpha = -1, -2, \ldots$ are automatically produced by the fact that $\binom{j}{i} = 0$ for $i > j$. On the other hand the zero for $\alpha(t_j) = 0$ and $s_i \to \infty$ must be produced by the vanishing of the normalization factor N_j :

$$N_j = -(j+1) = t_j + \tfrac{1}{2} \qquad (13)$$

This completes the solution of our problem (see Fig. 2) :

$$R_i(t_j) = -(j+1)(-1)^i \binom{j}{i} \qquad (14)$$

We now come to a crucial test. The residue functions $R_i(t_j)$ have been determined independently at ∞ many t_j. Are they consistent with being (a) polynomials in t, and (b) polynomials of reasonable order ? Our explicit solution (14) shows that both conditions are fulfilled :

$$R_i(t) = \text{polynomial of } (i+1)^{th} \text{ order in } t \qquad (15)$$

$$j_{max}(s_i) \equiv \alpha(s_i) = i + 1 = \tfrac{1}{2} + s_i \qquad (16)$$

This result is connected to our choice $\Delta t = 1/\alpha'(t)$. It turns out that $j_{max}(s_i)$ is not only reasonable, but we even have obtained the form required by crossing symmetry and Eq. (1), although we never imposed crossing symmetry.

An additional test is the requirement that all coefficients g^2 of the Legendre polynomials be positive. We have not succeeded in giving the general proof that this is the case, but explicit calculations by Dr. F. Wagner for $i = 0, 1, \ldots, 40$ show that $g^2 \geq 0$ for these resonances.

In the spirit of the resonance saturation model we replace t_j by continuous t in (14) :

$$R_i(t) = \frac{\Gamma[1-\alpha(t)]\,(-1)^{\alpha(s_i)}}{\Gamma[\alpha(s_i)]\;\Gamma[1-\alpha(s_i)-\alpha(t)]} \tag{14'}$$

The full amplitude A(s,t) is approximated, for $t < -1.5$, by the sum of these s channel resonances. A(s,t) can be written as a product of the residue function R(s,t) times a function which contains simple poles of unit strength :

$$A(s,t) = R(s,t)\,\frac{\pi\,(-1)^{\alpha(s)+1}}{\sin \pi\alpha(s)} \tag{17}$$

The apparent poles at $\alpha(s) = 0, -1, -2\ldots$ are actually absent because R(s,t) = 0 for those s values. Inserting (14') into (17) we obtain :

$$A(s,t) = -\,\frac{\Gamma[1-\alpha(s)]\;\Gamma[1-\alpha(t)]}{\Gamma[1-\alpha(s)-\alpha(t)]} \tag{18}$$

There is an arbitrary, positive normalization constant multiplying the whole expression.

The expression (18) is nothing else but the Veneziano formula [4] in the form appropriate for $\pi^+\pi^-$ scattering. Veneziano found the formula using the duality concept [5] and making an inspired guess. Here we have arrived at this form by means of a construction, in which the duality concept was never used as an input, since we worked at t = negative. But the duality concept automatically comes out as a consequence at t = positive. We recall that (a) the full amplitude (18) was constructed, for negative t, as a sum of s channel resonances [*], with nothing more added ; (b) this sum of s channel poles automatically and inevitably contains, at positive t, the t channel poles.

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*) For $t<-\frac{1}{2}$ we can write an unsubtracted dispersion relation for expression (18). This dispersion representation of A(s,t) reduces to a sum of s channel poles, with no other terms added (in particular no t channel poles).

Our construction also throws light on the question to what extent the Veneziano ansatz is unique.

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FIGURE CAPTIONS

Figure 1 The exemple of the experimental Y_0^*'s in $\bar{K}N \to \bar{K}N$. It shows (i) that the unphysical region, $z_s < -1$, is all-important for superconvergence, and (ii) that a cut-off at $s = s(t, \vartheta_s = 180^0)$ is an excellent approximation. On the vertical axis we plot the contributions $C[\mathrm{Im}\ B]$ of the Y_0^* resonances to the SCR $\int \mathrm{Im}\ B\ d\nu$ at $t = -3.34\ \mathrm{GeV}^2$. The dashed envelope only serves to guide the eye. The pole parameters are taken from Ref. 6) for the $5/2^+$ and $7/2^-$, from Ref. 7) for the $1/2^+$, and from ref. 8) for all other resonances.

Figure 2 The solution of the superconvergence problem. The triangular array of numbers gives the residues $R_i(t_j)$ of the i^{th} s channel pole at $t = t_j$, $\alpha(t_j) = -j-1$. $R_i(t_j)$ are essentially the binomial coefficients $\binom{j}{i}$, see Eq. (14). At $t = t_j$ the saturation is achieved with those poles which are already in the unphysical region, $\cos \vartheta_s < -1$. (Note that we set $m_\pi = 0$.)

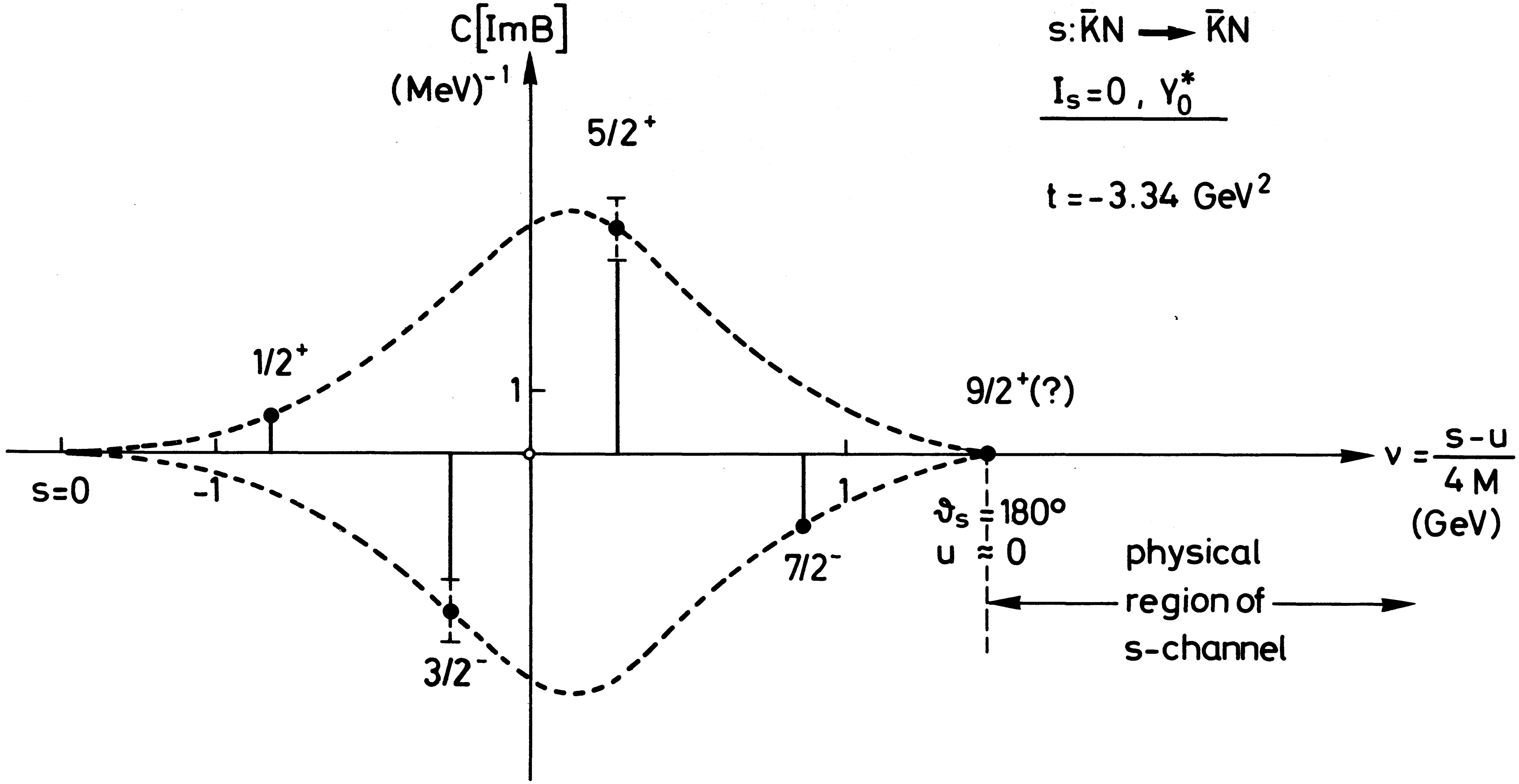
C[ImB]
(MeV)^-1
s:K̄N → K̄N
I_s=0 , Y_0^*
t = -3.34 GeV^2
5/2^+
1/2^+
1
9/2^+(?)
ν = (s-u)/(4M)
(GeV)
s=0
-1
1
ϑ_s = 180°
u ≈ 0
7/2^-
3/2^-
physical
region of
s-channel

FIG. 1

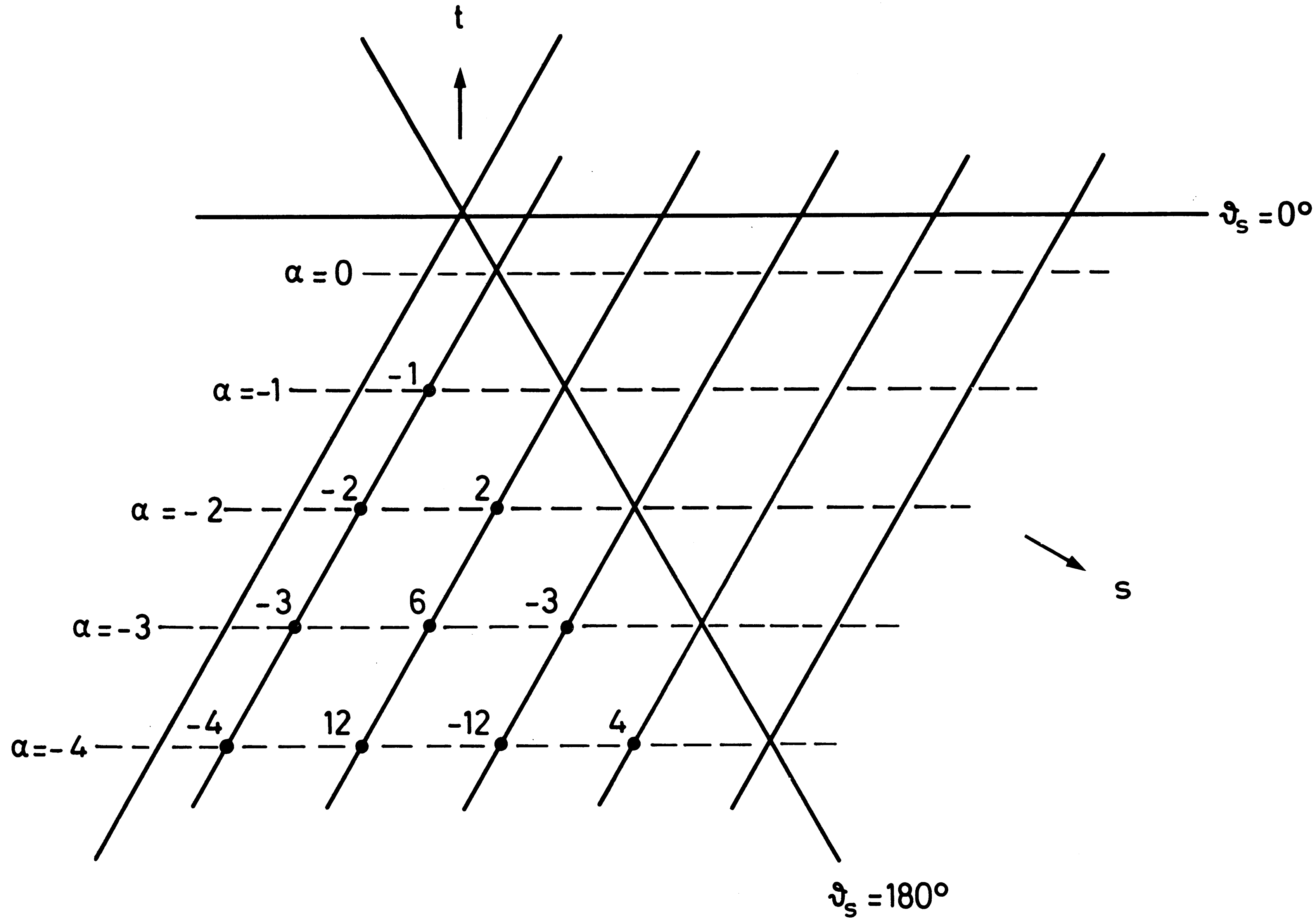

FIG.2