REVISED VERSION

ANALYSIS OF REACTION CROSS-SECTIONS IN PARTIAL WAVES OF A CROSSED CHANNEL *), **)

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SOMMAIRE

Nous donnons la forme générale de la section efficace différentielle pour des réactions entre particules de spin arbitraire où un nombre limité J_{max} de moments angulaires intervient dans une des voies croisées, ce qui constitue une généralisation du cas où une seule particule (ou résonance) est échangée entre la particule incidente et la cible. Nous montrons alors que la section efficace, mis à part des facteurs cinématiques connus, est un polynôme de degré maximum $2J_{max}$ en s, carré de l'énergie totale dans le système du centre de masse, dont les coefficients sont des fonctions du transfert d'impulsion lorsque l'état final ne renferme que deux corps. Une généralisation est faite au cas où l'état final comporte plus de deux corps. Nous montrons dans quelles situations le nombre de fonctions de structure indépendantes peut se réduire. Ces considérations sont illustrées par divers cas particuliers: diffusion élastique et inélastique électron-noyau, compte tenu de la possibilité d'échange de plusieurs photons; réactions induites par des neutrinos et des antineutrinos dans l'hypothèse d'un courant leptonique vectoriel et pseudovectoriel; désintégration d'une particule en trois corps; applications à la physique nucléaire et aux interactions fortes des particules élémentaires. Dans ce dernier cas des analyses quantitatives des résultats expérimentaux seront données ultérieurement.

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I. INTRODUCTION

It has been noticed that in various reactions the differential cross-section has a simple structure when the reaction is due to a single particle exchange. This is the case, for instance, in electron-nucleus elastic [1] and inelastic [2),3] scattering; the same happens in neutrino reactions via intermediate bosons or in local Fermi interactions; finally, this simplicity character of the cross-section appears also in peripheral collisions via one-pion exchange [5]. We wish, in this paper, to unify and generalize these results and to exhibit the deep character of their simplicity. We want to throw away all unnecessary assumptions, for instance neglect of the form factors and of the off-shell behaviour of the propagator of the exchanged particle, narrow width (if the particle exchanged is a resonance). It is clear that if the reaction $A+B \longrightarrow C+D$ is dominated by the exchange of a particle of spin J coupled to $A\bar{C}$ and $B\bar{D}$, the possible angular momenta, in the reaction $A+\bar{C} \longrightarrow \bar{B}+D$ are :

$$J, \quad J-1, \quad J-2, \quad \ldots\ldots\ldots \; 0 \; (\tfrac{1}{2})$$

As an example, a vector boson induces $J=1$ and 0 angular momenta.

In the following we shall only assume that one or several states of given total angular momentum J and, possibly, of given parity ω, otherwise unspecified, are exchanged between the incident particle and the target. It will turn out that the simplicity of the cross-section is essentially independent of the spins of the ingoing and outgoing particles. However, some further restrictions occur in special cases.

Though one could construct the expression of the cross-section directly, we prefer to use the crossed channel where Lorentz invariance can be reduced to three-dimensional rotation invariance in the centre-of-mass system. In this crossed channel, the above limitations restrict the number of partial waves. Then one can use well-known results on the expansion of the cross-section in Legendre polynomials of the centre-of-mass angle. The most convenient formalism seems to be the helicity formalism introduced by Jacob and Wick [6]. The maximum value of J, determines the number of structure functions of the differential cross-section, which is at most $2J+1$. An analogous result is well known in an ordinary partial wave expansion, where the maximum power of $\cos \phi$ is the minimum value of the three quantities :

$$2J, \quad 2L_i, \quad 2L_f$$

where L_i and L_f are the initial and final angular momenta [7),8)]. We shall not use the restrictions due to L_i and L_f because no centrifugal barrier argument can be used in the present situation.

The unpolarized cross-section is a function of the square of the momentum transfer t and of the square of c.m. energy s. In ordinary partial wave expansions, apart from trivial kinematical factors, it is expanded in powers of t with coefficients depending only on s. In the present approach, we expand it in powers of s with coefficients depending on t. We can disregard the question of convergence of such an expansion because we shall restrict ourselves to cases where a finite number of exchanged angular momenta dominate the process. It is clear that very high-energy phenomena cannot be described by such a limited expansion without violating general asymptotic behaviour of cross-sections deduced from analyticity and unitarity arguments [9].

Sections II and III contain the mathematical formalism necessary to establish the shape of the cross-section; they can be omitted by the reader only interested in practical applications; further, Section III deals with important particular cases where the number of independent structure functions can be restricted. In Section IV we study the electron-nucleus elastic and inelastic scattering; we construct the general formula which replaces the Rosenbluth formula when more than one photon is exchanged and we give a possible interpretation of the deviations observed in electron-proton scattering. In Section V neutrino induced reactions are analysed; in particular we show that three structure functions only are necessary to describe unpolarized cross-sections for production of leptons of non-zero mass on arbitrary nuclei by neutrinos; the same three functions describe also the antilepton production by antineutrinos. Section VI is an extension to three-body decay processes in which one has information on the relative angular momentum of two of the decay products. Finally, in Section VII we consider applications to strongly interacting particles leading to some kind of generalized peripherism; in particular we suggest applications both to nuclear physics and to elementary particle physics; quantitative analyses are postponed to a later publication.

II. GENERAL FORM OF THE DIFFERENTIAL CROSS-SECTION EXPANDED IN PARTIAL WAVES IN THE CROSSED CHANNEL

We consider the reaction

$$A + B \longrightarrow C + D \tag{I}$$

and define the following scalar invariants

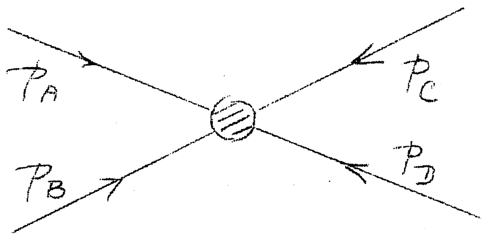

Fig. 1

$$\begin{aligned} s &= -(p_A + p_B)^2 \\ t &= -(p_A + p_C)^2 \\ u &= -(p_A + p_D)^2 \end{aligned} \tag{1}$$

The crossed channel we shall consider, corresponds to the reaction

$$A + \bar{C} \longrightarrow \bar{B} + D \tag{II}$$

so s is the square of the c.m. total energy for the crossed reaction. We can write the differential cross-section for reaction (I) in the following invariant form :

$$\frac{d\sigma}{dt} = \frac{1}{\left[s-(M_A+M_B)^2\right]\left[s-(M_A-M_B)^2\right]} \, \frac{1}{(2s_A+1)(2s_B+1)} \sum_{\text{polarizations}}' |T|^2 \tag{2}$$

where the invariant T matrix corresponds to fixed values of the four polarizations. At this point we notice that :

$$\sum_{\text{polarizations}}' |T|^2 \tag{3}$$

is a Lorentz invariant quantity depending only on s and t which can be prolongated in some way from channel (I) to channel (II). We take advantage of the invariance properties to evaluate (3) in the c.m. system of reaction (II), and we use the Jacob-Wick helicity formalism [6] to expand the corresponding T matrix elements in partial wave amplitudes.

$$\langle \lambda_{\bar{B}} \lambda_D | T | \lambda_A \lambda_{\bar{C}} \rangle = \sum_J{}' (J+\tfrac{1}{2}) \langle \lambda_{\bar{B}} \lambda_D | T^{(J)}(t) | \lambda_A \lambda_{\bar{C}} \rangle \, D^{*J}_{\lambda\mu}(\alpha, \phi, -\alpha) \tag{4}$$

where λ_i is the helicity of the i^{th}-particle, λ and μ the relative initial and final helicities defined by :

$$\lambda = \lambda_A - \lambda_{\bar{C}}$$
$$\mu = \lambda_{\bar{B}} - \lambda_D ,$$

J the total angular momentum and ϕ, α the reaction angles in channel (II). As a result of the properties of the Wigner D functions, expression (3) takes the form

$$\sum{}' |T|^2 = \sum_{\lambda,\mu}{}' \sum_{J_1 J_2}{}' (J_1+\tfrac{1}{2})(J_2+\tfrac{1}{2})(-1)^{\lambda-\mu} \langle \lambda_{\bar{B}} \lambda_D | T^{J_1}(t) | \lambda_A \lambda_{\bar{C}} \rangle \times$$
$$\times \langle \lambda_{\bar{B}} \lambda_D | T^{J_2}(t) | \lambda_A \lambda_{\bar{C}} \rangle^* \sum_{\ell=|J_1-J_2|}^{\ell=J_1+J_2}{}' C_{J_1 J_2}(\ell, 0; \lambda, -\lambda) C_{J_1 J_2}(\ell, 0; \mu, -\mu) P_\ell(\cos\phi) \tag{5}$$

or, equivalently,

$$\sum{}' |T|^2 = \sum_{\ell}{}' A_\ell(t)\, P_\ell(\cos\phi) \tag{6}$$

with

$$A_\ell(t) = \sum_{\mu}{}' \sum_{J_1 J_2} (J_1 + \tfrac{1}{2})(J_2 + \tfrac{1}{2})(-1)^{-\mu} C_{J_1 J_2}(\ell, 0, \mu - \mu)\, A_\ell^{J_1 J_2}(\lambda_{\bar{B}} \lambda_D) \tag{7}$$

$$A_\ell^{J_1 J_2}(\lambda_{\bar{B}} \lambda_D) = \sum_{\lambda}{}' (-1)^{\lambda} C_{J_1 J_2}(\ell, 0, \lambda - \lambda) \langle \lambda_{\bar{B}} \lambda_D | T^{J_2}(t) | \lambda_A \lambda_{\bar{C}} \rangle^* \langle \lambda_{\bar{B}} \lambda_D | T^{J_1}(t) | \lambda_A \lambda_{\bar{C}} \rangle \tag{8}$$

Formula (6) exhibits the expansion of the differential cross-section in Legendre polynomials of the c.m. angle for the crossed reaction. In terms of the scalar invariants s,t,u $\cos\phi$ is given by :

$$\cos\phi = \frac{1}{\alpha(t)} \left[t(s-u) - (M_A^2 - M_C^2)(M_B^2 - M_D^2) \right] \tag{9}$$

with

$$\alpha^2(t) = \left[t - (M_A + M_C)^2 \right] \left[t - (M_A - M_C)^2 \right] \left[t - (M_B + M_D)^2 \right] \left[t - (M_B - M_D)^2 \right]$$

Let us now assume that all angular momenta in the crossed channel are smaller or equal to a given value J_{max}. Formulae (5) and (7) show that in this case the maximum value of ℓ is $2J_{max}$. Using the linear dependence of $\cos \phi$ with respect to s for fixed t, we get :

Theorem 'A'

> if the maximum angular momentum in the crossed channel is J_{max}, $\sum |T|^2$ is a polynomial function of a of degree $2J_{max}$ whose coefficients are functions of t.

Therefore the cross-section (2) may be written

$$\frac{d\sigma}{dt} = \frac{1}{\left[s-(m_A+m_B)^2\right]\left[s-(m_A-m_B)^2\right]} \sum_{0}^{2J_{max}}{}' B_\ell(t)\, s^\ell \qquad (10)$$

III. FURTHER RESTRICTIONS ON THE CROSS-SECTION

So far we have only considered one limitation of the coefficients $A_\ell(t)$ in expansion (6). However, it may happen that some of the coefficients vanish under certain general circumstances. To study this problem we first look at the symmetry properties of the Clebsch-Gordan coefficients. We shall just make use of the relation

$$C_{J_1 J_2}(\ell, 0; -\lambda, \lambda) = (-1)^{J_1+J_2+\ell} C_{J_1 J_2}(\ell, 0; \lambda, -\lambda) \tag{11}$$

This relation is the only one which can be useful when one does not want to look at the detailed mechanism of the reaction.

An immediate consequence of (11) is $C_{J_1 J_2}(\ell, 0; 0, 0) = 0$ if $J_1 + J_2 + \ell$ is odd and we get the following theorem :

Theorem 'B'

if particles A and C have both spin 0

$$A_\ell^{J_1 J_2}(\lambda_{\bar{B}} \lambda_D) = 0$$

if $J_1 + J_2 + \ell$ is odd.

We get immediately the following corollary :

Corollary 'a'

if particles A and C (or B and D) have both spin zero and if the values of the angular momenta in the crossed channel are all even (or all odd), the coefficients $A_\ell(t)$ in expansion (6) vanish for odd values of ℓ.

This corollary takes an especially simple form, according to Eq. (9), when $M_A = M_C$ (or $M_B = M_D$) :

$$\frac{d\sigma}{dt} = \frac{1}{\left[s-(M_A+M_B)^2\right]\left[s-(M_A-M_B)^2\right]} \sum_{0}^{J_{max}} C_\ell(t)\,(s-u)^{2\ell} \tag{12}$$

Of course, if a single angular momentum is exchanged, corollary 'a' can be immediately applied.

To get further information we have to use the symmetry properties of the reduced T matrix elements with respect to the exchange $\lambda \rightarrow -\lambda$.

We now want to relate T matrix elements corresponding to helicities $\lambda_A, \lambda_{\bar{C}}$ and $-\lambda_A, -\lambda_{\bar{C}}$. Then it is convenient to work with eigenstates of parity which are linear combinations of helicity states corresponding to opposite values of λ. Following Jacob and Wick [6], we write

$$|\lambda_A, \lambda_{\bar{c}}, \omega\rangle = \frac{1}{\sqrt{2}}\left[|\lambda_A, \lambda_{\bar{c}}\rangle + \omega\,\eta_A\,\eta_{\bar{c}}\,(-1)^{J+s_A+s_c}\,|-\lambda_A, -\lambda_{\bar{c}}\rangle\right] \tag{13}$$

with

$$P\,|\lambda_A, \lambda_{\bar{c}}; \omega\rangle = \omega\,|\lambda_A, \lambda_{\bar{c}}; \omega\rangle$$

where P is the parity operator, and $\eta_A\ \eta_{\bar{C}}$ the intrinsic parities of A and $\bar{C}$.

From relation (13), one gets :

$$|-\lambda_A, -\lambda_{\bar{c}}; \omega\rangle = \omega\,\eta_A\,\eta_{\bar{c}}\,(-1)^{J+s_A+s_c}\,|\lambda_A, \lambda_c; \omega\rangle \tag{14}$$

After some algebraic manipulations, the coefficient $A_{\ell}^{J_1J_2}(\lambda_{\bar{B}},\lambda_D)$ given in equation (8) can be written as

$$A_{\ell}^{J_1J_2}(\lambda_{\bar{B}},\lambda_D)=\frac{1}{2}\sum_{\omega_1=\pm1}{}' \sum_{\omega_2=\pm1} \sum_{\lambda} (-1)^{\lambda} C_{J_1J_2}(\ell,0;\lambda,-\lambda) \times$$

$$\times \langle \lambda_{\bar{B}}\lambda_D | T^{J_2}(+) | \lambda_A \lambda_{\bar{C}} \omega_2 \rangle^{*} \langle \lambda_{\bar{B}} \lambda_D | T^{J_1}(t) | \lambda_A, \lambda_{\bar{C}}, \omega_1 \rangle$$

$$=\frac{1}{2}\sum_{\omega_1,\omega_2}{}' A_{\ell}^{J_1J_2}(\lambda_{\bar{B}},\lambda_D,\omega_1\omega_2) \qquad (15)$$

In the summation over helicity states occurring in Eq. (15), it is convenient to associate the couple of values $\lambda_A,\lambda_{\bar{C}}$, $-\lambda_A,-\lambda_{\bar{C}}$ corresponding to opposite values of λ. Then, taking into account the symmetry relations (11) and (14), one arrives at the following theorem :

Theorem 'C'

> the coefficient $A_{\ell}^{J_1J_2}(\lambda_{\bar{B}},\lambda_D;\omega_1,\omega_2)$ vanishes if $(-1)^{\ell} = -\omega_1\,\omega_2$.

We notice that Theorem 'B' is a particular case of Theorem 'C'; indeed if particles A and C have both spin zero, the product of parities $\omega_1\,\omega_2$ is equal to $(-1)^{J_1+J_2}$.

Theorem 'D' in its general form does not imply parity conservation and does not bring any restriction on $A^{J_1J_2}(\lambda_{\bar{B}},\lambda_D)$. The only interesting physical situation is the one where in the summation over ω_1 and ω_2 only one term is present; in other words, a single value of ω_i corresponds to a given value of J_i. We can then derive under this hypothesis the following corollaries :

Corollary 'b'

> if only one angular momentum J is exchanged and if the parity of $A\bar{C}$ is defined $A_\ell(t) = 0$ if ℓ is odd.

Such a result must be used with some care because in the case of half integral spin exchange, we know that the parity of a virtual particle is not defined without any supplementary assumption.

Corollary 'c'

> if several angular momenta J are exchanged and if the parity of $A\bar{C}$ is uniquely defined $A_\ell(t) = 0$ if ℓ is odd.

Before closing this section, it is worthwhile to notice that the above results are still valid when one of the particles, say D, is replaced by a group of particles of total energy W in their own centre-of-mass system and the differential cross-section corresponding to the detection of particle C only is given now by

$$\frac{\partial^2\sigma}{\partial t\,\partial W^2} = \frac{1}{\left[s-(m_A+m_B)^2\right]\left[s-(m_A-m_B)^2\right]}\sum_\ell{}' A_\ell(t,W^2)\,P_\ell(\cos\phi) \tag{16}$$

IV. APPLICATION TO HIGH-ENERGY ELECTRON SCATTERING

a) General considerations

We wish to analyse, as an illustration of the general method, elastic scattering of electrons by nuclei, and, in particular electron-proton scattering.

We first express the invariants s, t, and u in terms of laboratory variables of the electron.

We define as E, E', and θ, respectively, the initial energy, final energy and scattering angle of the electron in the laboratory system.

The cosine of the c.m. angle ϕ in channel (II) is given by

$$\cos\phi = \frac{s-u}{\sqrt{t(t-4M^2)}} = \frac{2M(E+E')}{\sqrt{q^2(q^2+4M^2)}} \tag{17}$$

where M is the mass of the target nucleus, with $q^2 = -t = 4EE' \sin^2\frac{\theta}{2}$, neglecting the electron mass.

The scattering angle in the lab. system is related to electron energies by

$$E + E' = \sqrt{\frac{q^2(q^2+4M^2)}{4M^2} + q^2 \operatorname{ctn}^2\frac{\theta}{2}} \tag{18}$$

We can now write the differential cross-section in the laboratory system, when the maximum angular momentum exchanged between the electron and the nucleus is J_{max}:

$$\frac{d\sigma}{d\Omega_{lab}} = \left(\frac{E'}{E}\right)^2 \sum_{\ell=0}^{2J_{max}} A_\ell(q^2)(E+E')^\ell \tag{19}$$

Using formula (18) we can re-express the sum in (19) in terms of q^2 and $tg^2\frac{\theta}{2}$ only. When only the even values of ℓ contribute (for instance, if the exchanged states have all the same parity), the cross-section can be expanded in powers of $ctn^2\frac{\theta}{2}$:

$$\frac{d\sigma}{d\Omega_{lab}} = \frac{d\sigma_0}{d\Omega_{lab}}\left[B_0(q^2)tn^2\frac{\theta}{2} + B_1(q^2) + B_2(q^2)ctn^2\frac{\theta}{2} + \dots + B_{J_{max}}(q^2)ctn^{2(J_{max}-1)}\frac{\theta}{2}\right] \quad (20)$$

where the Mott cross-section is given by :

$$\frac{d\sigma_0}{d\Omega_{lab}} = \frac{E'}{E}\,\frac{\alpha^2\cos^2\frac{\theta}{2}}{4E^2\sin^4\frac{\theta}{2}}$$

We see that formula (20) contains as a special case the Rosenbluth formula. In the one-photon exchange assumption only $J = 1$ $\omega = -1$ is present, because conservation of current at the electron electromagnetic vertex suppresses a possible $J = 0$ state.

However, it is not obvious in Eq. (20) that the Rosenbluth formula reduces to a single term when the target has zero spin. The origin of the simplification is helicity conservation for high-energy electrons by the interaction $\bar{\psi}_{e'}\gamma_\mu\psi_e A_\mu$. Let us consider the one-photon case. It follows that the value $\lambda = 0$ for the relative e^+e^- helicity is excluded. In the case of a zero spin target, there is only one reduced T amplitude, and making use of the properties of the Clebsch-Gordan coefficients, one shows that $B_0(q^2)$ in Eq. (20) vanishes as expected.

If we now consider positron-nucleus elastic scattering instead of electron-nucleus elastic scattering, we see that the reaction in channel (II) is in both cases $e^+e^- \to \mathcal{N} + \bar{\mathcal{N}}$, and the only change in the differential cross-section consists in replacing ϕ by $\pi - \phi$, so that the general form of cross-section (19) becomes, in the laboratory system,

$$\frac{d\sigma}{d\Omega_{lab}} = \left(\frac{E'}{E}\right)^2 \sum_{\ell=0}^{2J_{max}} (-1)^\ell A_\ell(q^2)(E+E')^\ell \tag{21}$$

irrespective of the number of photons exchanged.

b) Electron-proton scattering

There is at present some interest in the shape of the electron-proton scattering cross-sections, since one might observe, in future experiments for large momentum transfers, deviations from the Rosenbluth formula. The discussion which follows does not depend on the spin of the target and could apply as well to other cases. In this paragraph we shall try to find the simplest possible origins of deviations from the Rosenbluth formula. The main source of deviation from the Rosenbluth formula certainly comes from two-photon exchange, and more precisely from the interference of the two-photon exchange amplitude with the one-photon exchange amplitude. A first estimate of these effects has been made by Drell and Fubini [10], taking into account the rescattering due to the 3/2 3/2 pion-nucleon resonance; the two-photon contribution they get is almost purely imaginary and therefore the interference term in the differential cross-section is very small (whereas the polarization effects might be large). Any excited state of the nucleon will give the same type of results. So the last remaining source of deviations may be due to an interaction of the two photons with the nucleon through a strongly interacting particle or resonance.

Inspection of formula (20) shows clearly that an angular momentum larger than the one for the two-photon state gives a fixed q^2 cross-section which is more singular in the high-energy limit than the Mott cross-section, which for $E \to \infty$, behaves like E^2. Though, in this case, due to the zero mass of the photon, it seems difficult to make any rigorous statement on the asymptotic behaviour of the cross-section, analogous to the one given by Froissart [9] for the non-zero mass case, we believe that a too singular behaviour is not physically acceptable. So two cases may occur, either only $J = 0$ and $J = 1$ angular momenta are present, or all the angular momenta contribute in such a way that the singularity is cancelled after summation. In the latter case our analysis breaks down; however, one could try then to take into account complex angular momenta. Here we shall restrict ourselves to $J \leqslant 1$. According to the values of parity and angular momentum, we have four possible states :

$J = 0^-$ two-photon state, which may be coupled to the nucleon through a π^0.

$J = 0^+$ two-photon state, which may be coupled to the nucleon through a two-pion s state, which might be the one observed by Abashian et al. [11].

$J = 1^-$ one-photon state. The two-photon state is forbidden by charge conjugation invariance.

$J = 1^+$ two-photon state. Notice that the selection rule $J \neq 1$ for two real photons does not hold for two virtual photons *).

Let us first add to the one-photon exchange the $J = 0^{\pm}$ two-photon states. From Theorem 'C' we have no interference between the states 0^- and 1^- and between the states 0^+ and 0^-. We shall now show that the conservation of helicity of the electron in the 1^- amplitude suppresses also the interference between 0^+ and 1^-. We have already seen that in the one-photon amplitude the relative helicity λ of e^+e^- cannot be zero and therefore the Clebsch-Gordan coefficient $C_{10}(\ell, 0; \lambda, -\lambda)$ in the interference term vanishes. Hence, the cross-section has the following form :

$$\frac{d\sigma}{d\Omega_{lab}} = \frac{d\sigma_0}{d\Omega_{lab}} \left[B_0(q^2) + tn^2\frac{\theta}{2} + B_1(q^2) \right] \qquad (22)$$

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*) Notice that if this 1^+, $C = +1$, state, let us say $\mathcal{E}$, was interpreted as a resonance, it would not be too difficult to explain that it was not yet observed because the decays $\mathcal{E}_0 \to 2\gamma$, $\mathcal{E}_0 \to \pi_0 + \gamma$, $\mathcal{E} \to 2\pi$ are forbidden by gauge invariance, charge conjugation and parity conservation. The most likely decay mode will be $\mathcal{E} \to 3\pi$ but this may be forbidden if the mass is too low. Such a particle has been postulated by P. Dennery and H. Primakoff in the study of form factors in weak interactions (preprint 1962).

where :

B_1 is the same coefficient as in the one-photon case;

B_2 is the sum of three non-interfering positive contributions from the three states.

Eq. (22) simulates the Rosenbluth formula but in the present case we do not have the restriction on B_1 and B_0 due to the reality of form factors :

$$\frac{B_0}{B_1} \leqslant 2 + \frac{q^2}{2M^2} \qquad (23)$$

Here the slope of $d\sigma/d\sigma_0$ versus $\tan^2\frac{\theta}{2}$ is arbitrary, but the two parameters entering in the linear function must be positive. It must be kept in mind, however, that the 0^+ and 0^- contributions are expected to be small because they are of order α^2 with respect to the one-photon contribution to the cross-section.

We now take into account the 1^+ amplitude. Because of the interferences 1^+1^- and 1^+0^-, formula (22) is no longer valid and we must take expansion (19), which, using (18), may be written as

$$\frac{d\sigma}{d\Omega_{lab}} = \frac{d\sigma_0}{d\Omega_{lab}}\left[B_0(q^2)\tan^2\frac{\theta}{2} + B_1(q^2) + C(q^2)\tan^2\frac{\theta}{2}\sqrt{1 + \frac{4M^2}{4M^2+q^2}\operatorname{ctn}^2\frac{\theta}{2}}\right. \qquad (24)$$

Though formula (24) looks different from (22), one can approximate the bracket by a linear function of $\tan^2\frac{\theta}{2}$ in the range $\tan^2\frac{\theta}{2} > 2$. A deviation from linearity of $d\sigma/d\sigma_0$ will only appear at small angles. Since deviations to the Rosenbluth formula are expected for large momentum transfer only, this necessitates a very high initial energy. However, the existence of the C term can be detected by taking into account the requirement that the cross-section should be positive at all angles.

V. APPLICATION TO NEUTRINO INDUCED REACTIONS

The structure of the cross-sections for lepton production by neutrinos has been extensively studied by various authors [4),12),13),14)] under the hypothesis of a point structure of the lepton current. Nevertheless, the present approach can give a more elementary derivation of the general formula, independent of the spin of the target, and, as will appear later, exhibits a simpler structure of the cross-sections for production of unpolarized massive leptons. We replace the hypothesis of point structure for lepton current with V-A interaction by the slightly more general assumption that the only angular momentum exchanged between the leptons and the target is $J = 1$.

We first consider the reactions

$$\nu + A \to B + \ell^- \tag{25a}$$

$$\bar{\nu} + B \to A + \ell^+ \tag{25b}$$

where ℓ 's stand for the electron or muon. A and B are two particles of arbitrary spin, such that charge and baryonic number are conserved in the reaction. We can think of ν and $\bar{\nu}$ as unpolarized four-component neutrinos, the interaction taking care of the fact that only two components play a role. In doing so we just neglect a trivial statistical factor. According to Theorem 'A' of

Section II., the differential cross-section for the production of unpolarized ℓ^- is given by :

$$\frac{d\sigma(\nu+A\to B+\ell^-)}{dq^2} = \frac{1}{k_\nu^2}\left[f_1(q^2) + f_2(q^2)\cos\phi + f_3(q^2)\cos^2\phi \right] \tag{26}$$

with the following definitions (Fig. 2) :

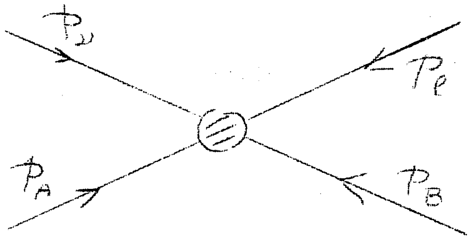

Fig. 2

$$\begin{aligned} s &= -(p_\nu + p_A)^2 \\ u &= -(p_\nu + p_B)^2 \\ t &= -q^2 = -(p_A + p_B)^2 \end{aligned} \tag{27}$$

k_ν is the energy of the incident neutrino in the rest system of particle A. $\cos\phi$ is given by formula (9)

$$\cos\phi = \frac{t(s-u) + m_\ell^2 (M_A^2 - M_B^2)}{(t - m_\ell^2)\sqrt{[t-(M_A+M_B)^2][t-(M_A-M_B)^2]}} \tag{28}$$

We now compare reactions (25a) and (25b). Assuming CP invariance, one immediately sees that the same structure functions f_1, f_2, f_3 describe the reaction induced by antineutrinos; more precisely, the bracket in Eq. (26), expressed with the <u>same</u> invariants (27) and the same angle (28), is unchanged. The only alteration is the replacement of k_ν by $k_{\bar\nu}$, energy of the antineutrino in the rest system of particle B.

In the special case $M_A = M_B = M$ (for instance single nucleon target), the expressions of s,u,t in terms of the laboratory energies of the neutrino and the lepton for the first reaction (25a), are respectively formally identical to those of u,s,t in terms of the laboratory energies of the antineutrino and the antilepton for the second reaction (25b). Hence, the two cross-sections may be written simultaneously as :

$$\frac{d\sigma\begin{bmatrix}\nu + A \to B + \ell^- \\ \bar\nu + B \to A + \ell^+\end{bmatrix}}{dq^2} = \frac{1}{k_\nu^2}\Big[g_1(q^2) \pm g_2(q^2)\left[2M(E_\ell + k_\nu) - m_\ell^2\right] + g_3(q^2)\left[2M(E_\ell + k_\nu) - m_\ell^2\right]^2\Big] \tag{29}$$

with :

k_ν = laboratory energy of the (anti)-neutrino.

E_ℓ = laboratory energy of the (anti)-lepton of mass m_ℓ.

$q^2 = 2M(k_\nu - E_\ell)$.

The structure functions g_i are easily connected with the f_i's.

Such a simple structure is not obvious on the formulae given by Lee and Yang in Ref. [14] in the case $m_\ell \neq 0$, because the cross-sections for production of polarized leptons contain five independent structure functions, assuming only time reversal invariance. Our result can also be derived after tedious algebraic manipulations from Lee and Yang's formulae, but does not use the $|\Delta \vec{I}| = 1$ rule for the baryon current.

When the lepton is an electron, one can neglect its mass m_ℓ and then the function x used by Lee and Yang is given by the relation :

$$x = \frac{\cos\phi - 1}{\cos\phi + 1}$$

These considerations apply as well to inelastic reactions. Then, if one observes only the outgoing lepton

$$\frac{\partial^2\sigma}{\partial q^2\,\partial W^2} = \frac{1}{k_\nu^2}\left[h_1(q^2,W^2) + h_2(q^2,W^2)\,k_\nu + h_3(q^2,W^2)\,k_\nu^2 \right] \qquad (30)$$

where q^2 and w^2 can be expressed in terms of the lepton laboratory variables θ and E_ℓ :

$$\begin{cases} q^2 = -m_\ell^2 + 2k_\nu \left(E_\ell - k_\ell \cos\theta \right) \\ 2M_A \left(k_\nu - E_\ell \right) = W^2 - M_A^2 + q^2 \end{cases}$$

with $E_\ell^2 = k_\ell^2 + m_\ell^2$. In this latter case, there is no relation between neutrino reactions and antineutrino reactions.

VI. APPLICATION TO THREE-BODY DECAYS

We consider decay processes of the type

$$A \longrightarrow 1 + 2 + 3 \quad .$$

If the relative angular momentum of particles 2 and 3 is known, our formalism furnishes restrictions on the differential spectrum. This happens, for instance, if particles 2 and 3 are emitted via an intermediate particle. Then, defining the three scalar invariants

$$s_i = -(p_A - p_i)^2$$

with

$$\sum s_i = M_A^2 + \sum M_i^2 \quad ,$$

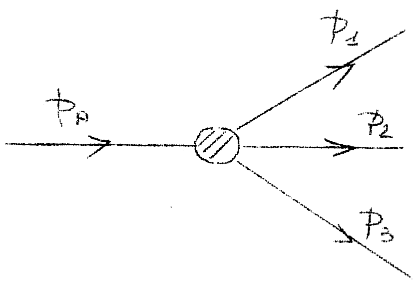

Fig. 3

we get the spectrum in invariant form :

$$\frac{\partial^2 N}{\partial s_1 \partial s_2} = \sum_{0}^{2J_{max}} A_\ell(s_1)\, P_\ell(\cos\varphi) \qquad (31)$$

where J_{max} is the maximum relative angular momentum of 2 and 3, and $\cos\phi$ is the reaction angle in the c.m. system of particles 2 and 3 [see Eq. (9)]. In the rest frame of the decaying particle, the general structure of the spectrum is given by

$$\frac{\partial^2 N}{\partial E_1 \partial E_2} = \sum_{\ell=0}^{2J_{max}} B_\ell(E_1)\, E_2^{\ell} \tag{32}$$

This formula applies to $\beta^{\pm}$ decay, (neglecting Coulomb interaction), where, due to the V-A interaction, $J_{max} = 1$, and the β^+ and β^- decays are described by the same 3 structure functions; in this case E_1 is the nucleon recoil energy and E_2 the electron energy. Similarly, we find the structure of the spectrum of

$$K \to \pi + \ell^{\pm} + \nu^{\mp} \qquad \ell = e, \mu$$

In this particular case, assuming the V-A character of the lepton current, we deduce $J^{max} = 1$, and there are only three structure functions B_0, B_1, B_2.*) In the special case where the leptons are electrons, it is legitimate to neglect their mass; then since V-A interaction conserves helicity, the only non-vanishing reduced matrix element in the crossed channel is

$$\langle 0,0 \mid T^{(1)} \mid \lambda_{\mu^+} = -\tfrac{1}{2}, \lambda_{e^-} = +\tfrac{1}{2} \rangle$$

so that only one structure function determines the decay spectrum.

We realize that the examples considered in this section have been already treated by other methods [4), 15)], but it seemed to us interesting to present them in a unified manner.

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*) It is clear, in this approach, that a final state interaction between the lepton and the neutrino will not change the result. In the special case where the lepton mass is neglected this follows, as was noticed by Professor Pais, from γ_5 invariance but, in fact, it still holds if the lepton mass is not zero.

VII. SOME APPLICATIONS TO STRONG INTERACTIONS

We shall not consider the case where a strong interaction between two particles can be described by the exchange of a limited number of angular momenta. This category of reactions contains, as a particular case, the single pion exchange model which has been extensively studied recently [5),16),17),18)]. In the more general model, we are considering here, we shall meet the same type of difficulties as in the one-pion exchange model.

Consider a reaction

$$A + B \longrightarrow C + D.$$

Our model is valid if only the intermediate states in one of the crossed channels, for instance $A+\bar{C} \longrightarrow \bar{B}+D$, are important. So the intermediate states, as well as the initial and final interactions, in the direct channel, must be negligible and one of the crossed channels must predominate with respect to the other one. This was the case for the examples given in the previous sections, where the magnitude of the various interactions permitted to retain only one channel. If the above conditions are fulfilled, the experimental test of the exchange of a maximum angular momentum J_{max} consists in studying the energy dependence of the differential cross-section for fixed value of the momentum transfer t according to formula (10). The same information can be obtained for processes in which D, for instance, consists of a group of particles, if one observes only particle C; in the latter case one has to find, for each experimental situation, arguments to classify the outgoing particles.

One may think of applying these considerations to nucleon-nucleon scattering. In some region of energy, one may hope that the interaction is essentially due to exchange of single pions and composite resonant system of pions. However, there exists a complete symmetry between the two crossed channels and the only possibility to extract valuable information from this type of analysis is to restrict oneself to a limited range of momentum transfer to eliminate one of the two crossed channels. This will be studied later in detail.

If instead of taking hydrogen as a target, we use a nucleus, the symmetry between the two crossed channels is destroyed and, if intermediate states in channel (I) are not too important, we can try to use our formalism, to analyse (p,p') experiments :

$$p + N \longrightarrow p' + N'$$

N' may be the initial nucleus or an excited state of this nucleus.

In the $|\Delta \vec{I}| = 1$ *) case, one may have the exchange of a pion $(J = 0^-)$ [19] or a ρ particle $(J = 1^-)$ between the proton and the nucleus. If only one pion is exchanged, the cross-section is described by one structure function, whilst if also the ρ particle plays a role, we have only two structure functions according to corollary 'c', and because of time reversal invariance of the $\rho pp'$ vertex.

If N and N' have both isospin 0, for instance in $p+He^4 \longrightarrow p+He^4$, only $|\Delta \vec{I}| = 0$ is present and the single pion and the ρ particle are forbidden. The possible candidates

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*) This refers to the isospin of the nucleus.

According to experimental results [21], the $\bar{Y}$ are generally produced in the forward direction, which seems to indicate a peripheral mechanism via a K or a K^*. Analysis of the energy dependence of this reaction, when sufficient statistics are available, should give an indication on the spin of the exchanged particle [22]. If two structure functions are necessary, this will be an indication that the K^* has spin 1 and plays a role in the reaction. If only one structure function is needed, either K^* has zero spin or it plays no role in the reaction.

We would like to add the following remark. If the events under consideration are rare, instead of considering a fixed value for momentum transfer t, one can add up all the events in a strip $t_1 < t < t_2$ and apply the above considerations to an averaged expansion of type (10) in powers of s.

VIII. CONCLUDING REMARKS

The above analysis shows the importance of performing experiments at various energies for fixed value of the momentum transfer, in order to know if the crossed channel partial wave expansion is meaningful for the process under consideration, and, if it is the case, to determine the angular momenta exchanged in order to give a basis for further theoretical analysis. An example of this method has been previously given in Ref. [3].

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