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MISCELLANEOUS TOPICS ON DISPERSION RELATIONS FOR PROBLEMS INVOLVING PHOTONS

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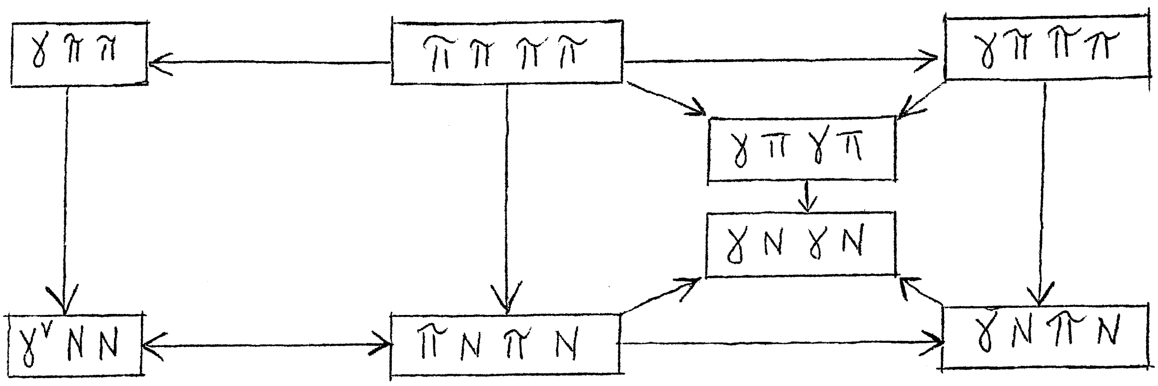
The present notes are the matter of a series of lectures given at the Yugoslavia Theoretical Summer School in August 1961. They are mainly concerned with three applications of the Mandelstam representation: the photoproduction of pions on pions, the nucleon and pion electromagnetic form factors, the photoproduction of pions on nucleons.

Photoproduction of pions on pions is interesting as an intermediate step for the study of more realistic reactions. It is also the more simple way to understand how one can use the Mandelstam representation in practical applications. From a pedagogic point of view, the photoproduction of pions on pions allows to see on an explicit case what kinds of problems are encountered in this aspect of theoretical physics.

The study of nucleon electromagnetic form factors from a theoretical point of view has not been very satisfactory. By using dispersion techniques one can hope to clarify the situation. By taking into account the pions contributions one can reproduce the general features of experiments. Nevertheless, a complete and unambiguous theory does not yet exist and the problem is still open.

The photoproduction of pions on nucleons has been extensively studied by experimentalists. The theory elaborated with the help of dispersion relations techniques is in qualitative agreement with experiments. A completely covariant treatment by using Mandelstam representation is discussed in these notes.

Other problem involving photons, as Compton scattering on pions or on nucleons are not considered here for lack of time but the same techniques can be applied and the general structure is the following



PART I

PHOTOPRODUCTION OF PIONS ON PIONS

I. Introduction

Examination of the processes involving photons, such as pion production on nucleons or Compton scattering on nucleons shows that a necessary intermediate step to apply the Mandelstam technique to these reactions is the preliminary calculation of the process: $\gamma + \pi \rightarrow \pi + \pi$

The interest of this reaction also lies in its simplicity due to the limitation of possible states by conservation laws and symmetry properties.

In Section II, we study the kinematics and the multipole expansion of the reaction amplitude; expression of unitarity condition is given for each multipole amplitude by retaining only a two-pion intermediate state. The Mandelstam representation is used in Section III, following the simplified version proposed by Cini and Fubini [1], to obtain for the magnetic dipole amplitude a Mushkelishvili-Omnès integral equation. Section IV is concerned with the reduction of this singular equation into a Fredholm equation containing an arbitrary multiplicative constant in the inhomogeneous term. In Section V, we find approximate solutions of the equation under the assumption of a sharp pion-pion resonance in the state I=J=1. The same results can be directly obtained by a crude model where the $\pi - \pi$ resonance is simulated by a metastable particle, the bipion, as explained in Section VI. A normalization factor appears in the solution and there is no direct way of determining this arbitrary factor in this approach because the electromagnetic coupling constant of pions never appears and since, in this process the isoscalar part of the photon alone is acting, there is no possibility of using a low energy limit property. We then must relate the $\gamma + \pi \rightarrow \pi + \pi$ process with measurable processes such as photoproduction of pions on nucleons or Compton scattering on nucleons - with as intermediate step the Compton scattering on pions -. These relations are explained in Section VII where also the Chew and Low extrapolation method is applied to the photoproduction of two pions on nucleons.

II. Invariance properties of the matrix elements. Kinematics and Unitarity

1. We define p_1, p_2, p_3 as the incoming four momenta of the pions and $k = -(p_1+p_2+p_3)$ as the incident momentum of the photon.

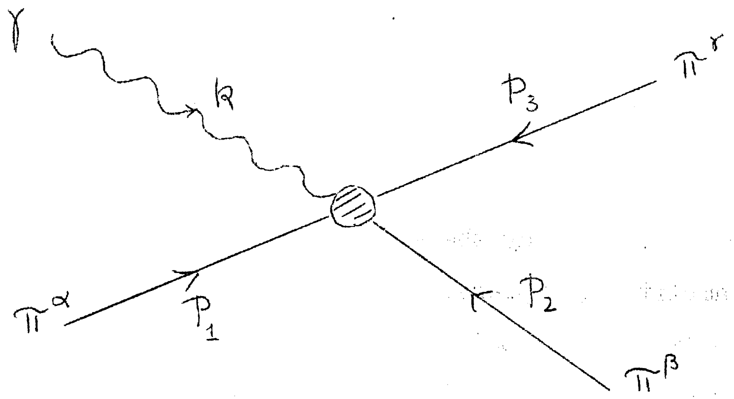


Fig. 1

We define channel I by the reaction $1+\gamma \rightarrow 2+3$ and so on for other channels. As usual we introduce the three scalar quantities :

$$\left\{\begin{aligned} s_1 &= -(k+p_1)^2 \\ s_2 &= -(k+p_2)^2 \\ s_3 &= -(k+p_3)^2 \end{aligned}\right.$$

with, on the energy shell, the relation $s_1+s_2+s_3 = 3\mu^2$. The S matrix element is then related to the T transition amplitude by the definition :

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta_4(k+p_1+p_2+p_3) \frac{1}{4(k_0 p_{10} p_{20} p_{30})^{1/2}} T_{fi}$$

2. From Lorentz invariance and gauge invariance, the T matrix element has the form

$$T_{fi} = \frac{1}{2i} \varepsilon_{\lambda\mu\nu\rho} e_\lambda p_{1\mu} p_{2\nu} p_{3\rho} F(s_1, s_2, s_3) \tag{1}$$

where e_λ is the photon polarization and $\varepsilon_{\lambda\mu\nu\rho}$ the 4th order completely antisymmetric tensor. $F(s_1, s_2, s_3)$ is a scalar function of the three scalar invariants s_1, s_2, s_3.

3. Application of G invariance shows that only the isoscalar part of the photon contributes. Since the $I = 0$ three-pion state is completely antisymmetric in isospace :

$$(\vec{\pi}_1, \vec{\pi}_2, \vec{\pi}_3)$$

The pions two by two are in a $I = 1$ isospin state and their relative orbital angular momentum is odd.

4. It follows from the boson character of the pions that the function $F(s_1, s_2, s_3)$ is completely symmetrical. The problem possesses only one independent channel and the crossed reactions are exactly the same in all channels due to the symmetries of F. The isospin dependence of the T matrix element for the reaction

$$\gamma + \pi^\alpha \to \pi^\beta + \pi^\gamma$$

is then given by the 3th order antisymmetrical tensor $1/\sqrt{2}\, \varepsilon_{\alpha\beta\gamma}$

5. The scalar invariants can be written in the centre-of-mass system

$$\left\{\begin{aligned} s_1 &= (k+\omega_k)^2 = 4\,\omega_q^2 \\ s_2 &= \mu^2 - 2k\omega_q + 2kq\cos\theta \\ s_3 &= \mu^2 - 2k\omega_q - 2kq\cos\theta \end{aligned}\right.$$

where k is the photon momentum, q the relative momentum of the final pions and θ the reaction angle.
The function $F(s_1, s_2, s_3)$ can be considered as a function of q^2 and $\cos\theta$ only

$$F(s_1, s_2, s_3) \Rightarrow A(q^2, \cos\theta)$$

and the T matrix element becomes :

$$T_{fi} = \omega_q(\vec{e}, \vec{k}, \vec{q})\, A(q^2, \cos\theta) \qquad (2)$$

with the particular choice of gauge :

$$e_4 = 0 \qquad\qquad \vec{e}\cdot\vec{k} = 0$$

6. Because of the pseudoscalar character of the π - meson and by parity conservation, only magnetic transitions can occur. By expanding the T matrix in multipole amplitudes $S_J(q^2)$ we obtain :

$$T_{fi} = i \sum_{J}{}' S_J(q^2)\, \frac{\vec{e}\cdot\vec{l}_k}{\sqrt{J(J+1)}}\, \frac{2J+1}{4\pi}\, P_J(\cos\theta)$$

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The operator $i \dfrac{\vec{e}\cdot\vec{\ell}_k}{\sqrt{J(J+1)}}$ is a projector operator for the magnetic multipole and acts on the Legendre polynomials in the following manner :

$$i\ \vec{e}\cdot\vec{\ell}_k\ P_J(\cos\theta) = \frac{\vec{e}\cdot\vec{k}\times\vec{q}}{kq} P'_J(\cos\theta)$$

We then obtain for the transition matrix

$$T_{fi} = (\vec{e},\vec{k},\vec{q})\sum_J{}' \frac{S_J(q^2)}{kq}\ \frac{2J+1}{4\pi}\ \frac{1}{\sqrt{J(J+1)}}\ P'_J(\cos\theta)$$

By comparison with formula (2), the multipole expansion for the scalar amplitude $A(q^2, \cos\theta)$ can be written as :

$$A(q^2,\cos\theta) = \sum_J{}' B_J(q^2)\ P'_J(\cos\theta) \qquad (3)$$

where :

$$B_J(q^2) = \frac{2J+1}{4\pi}\ \frac{1}{\sqrt{J(J+1)}}\ \frac{1}{kq\,\omega_q}\ S_J(q^2)$$

One can invert the relation (3) by using the orthogonality properties of the Legendre polynomials; after some elementary manipulations one obtain :

$$\mathcal{B}_J(q^2) = \frac{2J+1}{2J(J+1)} \int_{-1}^{+1} A(q^2,x)\,(1-x^2)\,P_J^{\prime}(x)\,dx \tag{4}$$

or the equivalent form

$$\mathcal{B}_J(q^2) = (J+\tfrac{1}{2})\,\frac{1}{\sqrt{J(J+1)}} \int_{-1}^{+1} A(q^2,x)\sqrt{1-x^2}\;P_J^1(x)\,dx \tag{5}$$

where $P_J^1(x)$ is the Legendre associated function. The $\sqrt{1-x^2} = \sin\theta$ term is a spherical harmonics of order 1 describing the photon spin.

7. <u>Cross-sections.</u> In the centre-of-mass system, the density of final states is simply

$$\rho = \frac{1}{(2\pi)^6}\;\frac{q\,\omega_q}{2}$$

and the flux of incident particles

$$\phi = 1 + \frac{k}{\omega_k} = 2\,\frac{\omega_q}{\omega_k}$$

The total cross-section averaged over the photon polarizations takes the following form

$$\sigma_T(q^2) = \frac{k q^3}{128\pi} \sum_J \frac{J(J+1)}{2J+1}\,\left|\mathcal{B}_J(q^2)\right|^2 \tag{6}$$

8. Unitarity of the S matrix. Let us define as $| n \rangle$ a complete set of intermediate states. The unitarity of the S matrix corresponds to the following expression

$$\sum_n{}' \langle f | S^* | n \rangle \langle n | S | i \rangle = \delta_{fi}$$

This equality induces for the T matrix a relation

$$i \langle f | T^* - T | i \rangle = \sum_n{}' (2\pi)^4 \delta_4 (\Sigma_i p_f - \Sigma_i p_n) N_n \langle f | T^* | n \rangle \langle n | T | i \rangle \quad (7)$$

where Σp_n is the total energy momentum four vector and N_n the normalization coefficient in the intermediate state $| n \rangle$.

In the energy range between (2μ) and (4μ) the only intermediate state is a two-pion state.

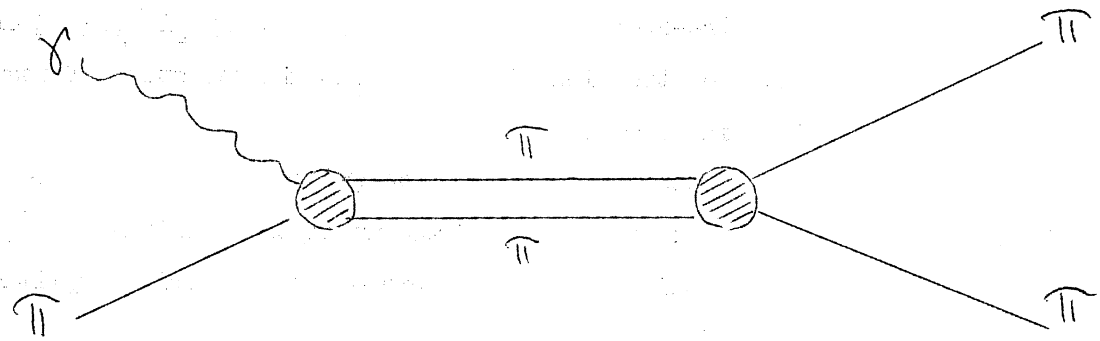

Fig. 2

The $\gamma\pi$ state is disregarded because it corresponds to higher order in the electromagnetic coupling constant.

The unitarity condition takes the simple form

$$i\langle\pi\pi|T^{*}-T|\gamma\pi\rangle=\frac{2}{(8\pi)^{2}}\frac{q}{\omega_{q}}\int d\Omega_{p}\langle\pi\pi|T^{*}|\pi\pi\rangle\langle\pi\pi|T|\gamma\pi\rangle \quad (8)$$

where the angular integration is performed over the pion angles in the intermediate state. When the energy is larger than 4μ, this relation is no longer rigorous but we maintain it because there is no way of taking into account properly the other intermediate states.

A unitarity condition can be derived from (8) for each multipole amplitude $B_J(q^2)$. Let us expand the two-pion scattering amplitude in partial waves

$$\langle\pi\pi|T|\pi\pi\rangle=16\pi\frac{\omega_{q}}{q}\sum_{J}{}'(2J+1)\,h_{J}(q^{2})\,P_{J}(\cos\theta_{pq}) \quad (9)$$

where θ_{pq} is the centre-of-mass angle and $h_J(q^2) = \exp[i\,\delta_J(q^2)]\,\sin\delta_J(q^2)$ the partial amplitude for the pion-pion scattering in the state of angular momentum J and isospin $I = 1$.
By using expansions (3) and (9) into relation (8) we obtain the well-known final state phase theorem: the phase of the multipole amplitude $B_J(q^2)$ is the same as the corresponding one for the pion-pion scattering amplitude in the isospin state $I = 1$

$$\operatorname{Im} B_{J}(q^{2}) = h_{J}^{*}(q^{2})\,B_{J}(q^{2}) \quad (10)$$

III. Mandelstam representation.

1. From G-conjugation, the one-pion intermediate state is missing so that there is no pole in the reaction amplitude.

2. We now assume for the scalar function $F(s_1, s_2, s_3)$ a Mandelstam representation, containing a single weight function, due to the above symmetry properties :

$$F(s_1, s_2, s_3) = f(s_1, s_2) + f(s_2, s_3) + f(s_3, s_1) \tag{11}$$

where

$$f(s_1, s_2) = \frac{1}{\pi^2} \int_{(2\mu)^2}^{\infty} \int_{(2\mu)^2}^{\infty} \frac{\rho(\alpha, \beta)}{(\alpha - s_1)(\beta - s_2)} \, d\alpha \, d\beta$$

The questions of convergence of the integrals and consequently of possible subtractions are disregarded for the moment.

Actually, the two variables α and β cannot reach simultaneously their lower limit, corresponding to the two-pion state, the three-pion state being forbidden. More precisely, if $(2\mu)^2 < \alpha < (4\mu)^2$, the minimum value of β is then $(4\mu)^2$

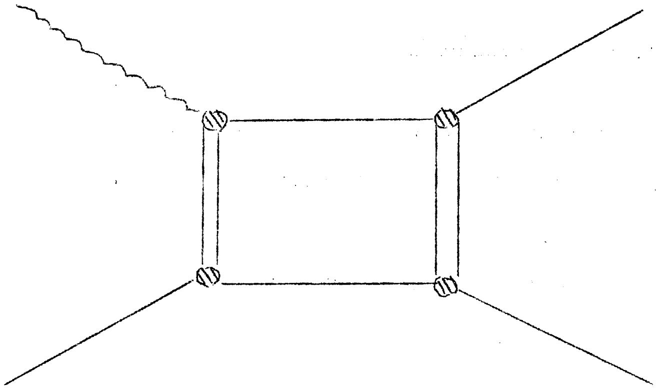

Fig. 3

The Cini-Fubini method of reduction can be applied. Instead of equation (11) we can write:

$$F(s_1, s_2, s_3) = g(s_1, z_1) + g(s_2, z_2) + g(s_3, z_3) \qquad (12)$$

where z_i is the cosine of the centre-of-mass angle in the channel i. The function $g(s,z)$ has a strong dependence on the energy variable s and a weak dependence on the angular variable z; it is an even function of z because F is completely symmetrical with respect to s_1, s_2, s_3. We can expand $g(s,z)$ in Legendre polynomials of z:

$$g(s,z) = \sum_{L \text{ even}}' (2L+1)\, a_L(s)\, P_L(z)$$

with for the one variable functions $a_L(s)$ the spectral representation:

$$a_L(s) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\rho_L(\alpha)}{\alpha - s}\, d\alpha$$

3. We now neglect the z-dependence; the error is of the order z^2. It can be easily seen that this procedure is equivalent to retain only the magnetic dipole contribution in the amplitude $A(q^2, \cos\Theta)$. If it turned out that the octupole transitions are important this would not be permissible.

Then, equation (12) becomes:

$$F(s_1, s_2, s_3) = \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\rho(\alpha)}{\alpha - s_1} d\alpha + \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\rho(\alpha)}{\alpha - s_2} d\alpha + \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\rho(\alpha)}{\alpha - s_3} d\alpha \qquad (13)$$

4. In order to apply the unitarity condition (10), we have to extract the dipole part of $A(q^2, \cos\Theta)$. Equation (4) gives:

$$B(q^2) = \frac{3}{4}\int_{-1}^{+1} A(q^2, x)(1 - x^2)\, dx$$

and we obtain the following representation for $B(q^2)$:

$$B(q^2) = \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\rho(\alpha)\, d\alpha}{\alpha - s_1 - i\varepsilon} + \frac{3}{4\pi}\int_{-1}^{+1}(1 - x^2)\, dx \int_{(2\mu)^2}^{\infty} \rho(\alpha)\, d\alpha \left[\frac{1}{\alpha - s_2} + \frac{1}{\alpha - s_3}\right] \qquad (14)$$

The denominators of the second and third integrals in the r.h.s. can never vanish in the physical region for channel I. The only contribution to the imaginary part of $B(q^2)$ comes from the first term

$$\mathrm{Im}\, B(q^2) = \rho(4q^2 + 4\mu^2)$$

and the unitarity condition gives the spectral function ρ in terms of the amplitude B *)

$$\rho(4q^2 + 4\mu^2) = h^*(q^2)\, B(q^2)$$

With the following change of variable

$$q^2 = \nu \qquad\qquad \alpha = 4(\nu' + \mu^2)$$

we deduce for $B(\nu)$ an integral equation of the Muskhelishvili type depending on the $J=I=1$ pion-pion scattering amplitude

$$B(\nu) = \frac{1}{\pi}\int_0^\infty \frac{h^*(\nu')B(\nu')}{\nu'-\nu-i\varepsilon}\,d\nu' + \frac{3}{4\pi}\int_{-1}^{+1}(1-x^2)\,dx\int_0^\infty h^*(\nu')B(\nu') \times$$

$$\times \frac{2\nu' + \nu + \frac{9}{4}}{\left(\nu' + \frac{\nu}{2} + \frac{9}{8}\right)^2 - \frac{\nu}{\nu+1}\left(\frac{\nu}{2} + \frac{3}{8}\right)^2 x^2}\,d\nu' \qquad (15)$$

where the pion mass μ has been taken to be unity.

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*) The angular momentum index $J = 1$ is suppressed in the following because one amplitude only occurs.

5. We now consider the subtraction problem. Equation (15) is a homogeneous linear equation that possesses the trivial solution $B(\nu) = 0$. Moreover, we have no Born term and no coupling constant can occur in this way. If we consider a perturbation approach of this problem, we can use a Hamiltonian such as

$$\frac{e\Lambda}{\mu^3} \frac{1}{2} \varepsilon_{\lambda\mu\nu\rho} A_\lambda \left(\partial_\mu \vec{\Phi}_\pi , \partial_\nu \vec{\Phi}_\pi , \partial_\rho \vec{\Phi}_\pi \right)$$

where Λ is a dimension less coupling constant.
A manner to introduce this constant in the dispersion relation approach is to write the integral representation with a subtraction and to fix, a priori, the value of the transition amplitude at the subtraction point. For the present case this procedure appears naturally by considering the solutions of the homogeneous truncated integral equation of $B(\nu)$ where channel II and III are not considered. We come back to this point later.

IV. Reduction of the integral equation to a Fredholm type equation.

1. We want to put equation (15) in the form :

$$\mathcal{B}(\nu) = \frac{1}{\pi}\int_0^\infty \frac{h^*(\nu')\,\mathcal{B}(\nu')}{\nu'-\nu-i\varepsilon}d\nu' + \frac{1}{\pi}\int_{-\infty}^{-a} \frac{C(z)\,dz}{\nu-z} \tag{16}$$

Then it is straightforward to transform equation (16) into a Fredholm equation. The problem is to determine the left hand cut and the discontinuity across this cut of the fuction $B(\nu)$. The main difficulty of this transformation is due to kinematical complications coming from the presence of unequal masses.

2. Let us study the kernel $K(\nu, \nu')$ defined as the integral with respect to $\cos\theta$ in the second term of the right hand side of (15)

$$K(\nu,\nu') = \frac{6}{\pi}\int_0^1 (1-x^2)\,\frac{(\nu+1)(\nu+2\nu'+\frac{9}{4})}{(\nu+1)(\nu+2\nu'+\frac{9}{4})^2 - \nu(\nu+\frac{3}{4})^2 x^2}\,dx$$

A direct calculation gives :

$$K(\nu,\nu') = \frac{6}{\pi}\,\frac{(\nu+1)(\nu+2\nu'+\frac{9}{4})}{\nu(\nu+\frac{3}{4})^2} + \frac{3}{\pi}\left(\frac{\nu+1}{\nu}\right)^{\frac{1}{2}}\frac{\nu(\nu+\frac{3}{4})^2-(\nu+1)(\nu+2\nu'+\frac{9}{4})^2}{\nu(\nu+\frac{3}{4})^2}\times \tag{17}$$

$$\times \operatorname{Log}\frac{(\frac{\nu+1}{\nu})^{\frac{1}{2}}(\nu+2\nu'+\frac{9}{4}) + (\nu+\frac{3}{4})}{(\frac{\nu+1}{\nu})^{\frac{1}{2}}(\nu+2\nu'+\frac{9}{4}) - (\nu+\frac{3}{4})}$$

The analytical structure of $K(\nu, \nu')$ with respect to the variable is difficult to analyze on the explicit form (17). Detailed calculations are without any physical interest and we report here only the results by referring to the original work.

The kernel $K(\nu, \nu')$ can be written as :

$$K(\nu,\nu') = \frac{1}{\pi}\left[\int_{-\infty}^{\alpha_M(\nu')} - \int_{\alpha_m(\nu')}^{-1}\right] dz \, \frac{A(\nu',z)}{\nu - z} \tag{18}$$

where the function $A(\nu', z)$ is simply :

$$A(\nu',z) = 12(\nu'+1)\frac{[z-\alpha_M(\nu')][z-\alpha_m(\nu')]}{z(z+\frac{3}{4})^2}\left(\frac{z+1}{z}\right)^{1/2} \cdot \tag{19}$$

$\alpha_M(\nu')$ and $\alpha_m(\nu')$ are the roots in ν of the second order equation

$$(\nu+1)(\nu+2\nu'+\frac{9}{4})^2 - \nu(\nu+\frac{3}{4})^2 = 0$$

From these expressions one immediately gets equation (15) in the form (16) by putting in evidence the discontinuity across the left hand cut

$$C(z) = -\varepsilon(z+\frac{9}{8})\int_0^{\nu'_M(z)} A(\nu',z)\, h^*(\nu')\, B(\nu')\, d\nu' \tag{20}$$

the upper limit of integration is related to ν' by

$$\nu'_M(z) = -\frac{9}{8} - \frac{z}{2} - \frac{1}{2}(z+\frac{3}{4})\left(\frac{z}{z+1}\right)^{\frac{1}{2}}$$

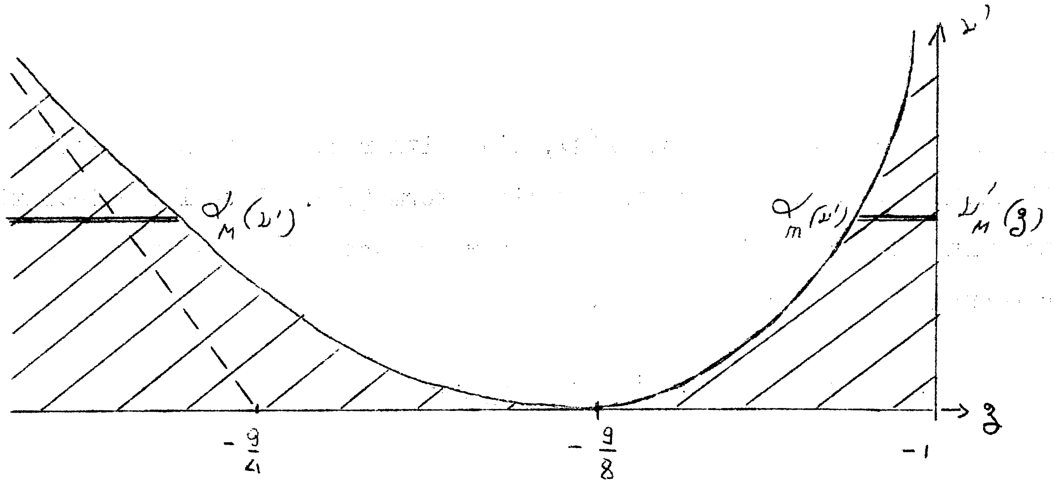

Fig. 4

Domain of integration in the (ν', z) plane.

3. Let us define the function $\rho(\nu)$ related to the pion-pion phase shift by the definition:

$$\rho(\nu) = \frac{P}{\pi}(\nu+1)\int_0^\infty \frac{\delta(\nu')}{(\nu'-\nu)(\nu'+1)}\, d\nu' \tag{21}$$

with the arbitrary normalization $\rho(-1) = 0$.
A particular solution of equation (16) can be written as [2]:

$$B(\nu) = H(\nu) + \exp\left[\rho(\nu)+i\delta(\nu)\right]\frac{1}{\pi}\int_0^\infty \frac{\exp\left[-\rho(\nu')\right]\sin\delta(\nu')\, H(\nu')}{\nu'-\nu-i\varepsilon}\, d\nu' \tag{22}$$

where $H(\nu)$ represents the second integral in the r.h.s. of equation (16). Integration over ν' can be performed by using the contour of integration as indicated in Fig. 5:

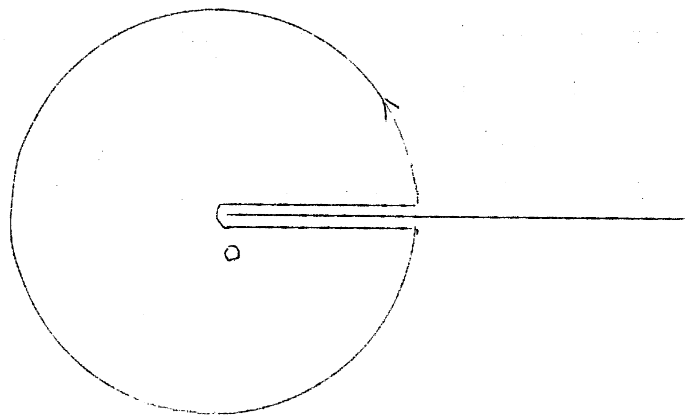

Fig. 5

Contour of integration in the complex ν plane

The solution (22) takes then the following form

$$\mathcal{B}(\nu) = \exp\left[\rho(\nu) + i\delta(\nu)\right] \frac{1}{\pi} \int_{-\infty}^{-1} \frac{\exp\left[-\rho(z)\right] C(z)}{\nu - z} \, dz \tag{23}$$

In order to find the general solution of equation (16) we must add to solution (23) the general solution of the associated homogeneous integral equation

$$\mathcal{B}_H(\nu) = \frac{1}{\pi} \int_0^{\infty} \frac{h^*(\nu') \, \mathcal{B}_H(\nu')}{\nu' - \nu - i\varepsilon} \, d\nu' \tag{24}$$

The possible solutions of (24) are :

$$\mathcal{B}_H(\nu) = \lambda \nu^m \exp\left[\rho(\nu) + i\delta(\nu)\right]$$

where m is an algebraic integer.

If one assumes for the pion-pion J=I=1 phase shift the resonant behaviour $\delta(0)=0$ $\delta(\infty)=\pi$ and that $\delta(\nu)-\pi$ goes to zero fast enough as ν goes to infinity, one sees that $\exp[\rho(\nu)]$ behaves like $1/\nu$ at infinity and the only acceptable solution of equation (24) is

$$B_H(\nu)=\lambda \exp[\rho(\nu)+i\delta(\nu)] \tag{25}$$

Finally, equation (16) is transformed into the following Fredholm equation

$$B(\nu)=\exp[\rho(\nu)+i\delta(\nu)]\left\{\lambda+\int_0^\infty N(\nu,\nu')h^*(\nu')B(\nu')d\nu'\right\} \tag{26}$$

where

$$N(\nu,\nu')=\frac{1}{\pi}\left[\int_{-\infty}^{\alpha_M(\nu')}-\int_{\alpha_m(\nu')}^{-1}\right]dz\,\frac{\exp[-\rho(z)]A(\nu';z)}{\nu-z} \tag{27}$$

The constant λ is an arbitrary parameter which enters as a multiplying factor in the final solution.

4. The particular form of equation (26) suggests to define a new real function $\phi(\nu)$ by :

$$B(\nu)=\exp[\rho(\nu)+i\delta(\nu)]\,\phi(\nu)$$

then

$$\phi(\nu)=\lambda+\int_0^\infty N(\nu,\nu')\exp[\rho(\nu')]\sin\delta(\nu')\phi(\nu')d\nu'$$

This equation is easier to solve, because of more rapid convergence of the integral in a subtracted form

$$\phi(\nu) = \phi(\nu_0) + \int_0^{\infty} \left[N(\nu,\nu') - N(\nu_0,\nu') \right] \exp\left[\rho(\nu')\right] \sin\delta(\nu')\, \phi(\nu')\, d\nu' \tag{29}$$

The unknown parameter λ is eliminated but one introduces another constant, the value of the function at the point of subtraction. It becomes clear that these two constants play an equivalent role.

By using the auxiliary function $F(\nu) = \dfrac{\phi(\nu)}{\nu - \nu_0}$ we put equation (29) into the form :

$$F(\nu) = \frac{\phi(\nu_0)}{\nu - \nu_0} + \int_0^{\infty} M(\nu,\nu';\nu_0) \exp\left[\rho(\nu')\right] \sin\delta(\nu')\, F(\nu')\, d\nu' \tag{30}$$

where the kernel M defined by :

$$M(\nu,\nu';\nu_0) = \frac{\nu' - \nu_0}{\nu - \nu_0} \left[N(\nu,\nu') - N(\nu_0,\nu') \right]$$

has the integral representation :

$$M(\nu,\nu';\nu_0) = (\nu' - \nu_0)\frac{1}{\pi} \left[\int_{-\infty}^{\alpha_M(\nu')} - \int_{\alpha_m(\nu')}^{-1} \right] d\zeta\, \frac{\exp[-\rho(\zeta)]\, A(\nu',\zeta)}{(\nu - \zeta)(\zeta - \nu_0)} \tag{31}$$

5. The position of ν_o is a priori arbitrary. Considerations of symmetry in equation (13) lead us to choose the point $s_1 = s_2 = s_3 = 1$ which corresponds to $\nu_o = -\frac{3}{4}$. It follows that the function $\phi(\nu)$ depends on an arbitrary constant $\phi(-\frac{3}{4})$.

For practical applications, it is more convenient to normalize the function B at the point s=0 corresponding to $\nu = -1$ and we shall define the coupling constant Λ by the relation

$$\frac{e\Lambda}{\mu^3} = B(-1) = \phi(-1)$$

6. Finally, the T amplitude turns out to be

$$T_{fi} = \frac{\varepsilon_{\alpha\beta\gamma}}{\sqrt{2}}\, \omega_q\,(\vec{e}, \vec{k}, \vec{q})\, \exp[\rho(\nu) + i\delta(\nu)]\, \frac{e\Lambda}{\mu^3}\, \frac{\phi(\nu)}{\phi(-1)}$$

and the total can be written as :

$$\sigma_T(q^2) = \frac{\alpha}{48}\, \Lambda^2\, \frac{kq^3}{\mu^6}\, \exp[2\rho(\nu)] \left[\frac{\phi(\nu)}{\phi(-1)}\right]^2$$

V. Approximate solutions

1. We are interested in this section, by the resolution of the Fredholm equation (30). The kernel of this equation depends strongly on the pion-pion interaction and we must calculate the function $\rho(\nu)$ for physical and unphysical values of ν.

There is at present some experimental evidence for a direct pion-pion interaction essentially in the S and P state. By analyzing the experimental results on the isoscalar part of the nucleon electromagnetic form factors, the S wave amplitude for π-meson-nucleon scattering, the multiple π meson production, one can predict the existence of a sharp resonance for the $\pi - \pi$ scattering in the state I=J=1, which is of interest for the actual problem of photoproduction. The position of this resonance seems to be at present [3] 750 MeV and the total width 150 - 200 MeV.

We make the following assumptions for the $\pi - \pi$ phase shift $\delta(\nu)$ in the J=1 state :

a) $\delta(\nu_R) = \frac{\pi}{2} \qquad \delta(0) = 0 \qquad \delta(\infty) = \pi$

b) $\lim\limits_{\nu \to 0} \frac{\delta(\nu)}{\nu^{3/2}} = C^{te}$

c) $\lim\limits_{\nu \to \infty} \nu^{\frac{1}{2}} [\pi - \delta(\nu)] < C^{te}$

Then the function $\rho(\nu)$ is such that :

$$\lim_{\nu \to \infty} [\rho(\nu) + \operatorname{Log} \nu] = C^{te}$$

and the quantity :

$$J(z) = \frac{1}{\pi} \int_0^{\infty} \frac{\exp[\rho(\nu)] \sin\delta(\nu)}{\nu - z} d\nu$$

can be evaluated using the contour of integration C given in Fig. 5. The result, if z is inside C is

$$J(z) = \exp[\rho(z)] \tag{32}$$

This formula will be useful later.

2. As a first crude model, corresponding to a zero width approximation, one can use a step function for the resonant pion-pion scattering phase shift: $\delta(\nu) = \pi\,\theta(\nu - \nu_R)$. With such a form, $\exp[\rho(z)]$ for negative values of z is then given by

$$\exp[\rho(z)] = \frac{\nu_R + 1}{\nu_R + z} \tag{33}$$

In the cases of non zero width, it can be checked, on explicit models that this very simple form can be used on the left hand cut.

3. Approximation "$\cos\theta$" = 0. If we go back to equation (15) and look the second integral in the right hand side, we see that a weighting factor $1-x^2 = \sin^2\theta$ in front of the whole integrand strongly favours $x^2 = \cos^2\theta = 0$. Such an approximation can be directly checked on the explicit form (17) of the kernel $K(\nu, \nu')$ and appears sufficiently good for the present purpose.

Equation (15) becomes :

$$B_a(\nu) = \frac{1}{\pi}\int_0^{\infty} \frac{h^*(\nu')B_a(\nu')}{\nu'-\nu-i\varepsilon}d\nu' + \frac{4}{\pi}\int_0^{\infty}\frac{h^*(\nu')B_a(\nu')}{2\nu'+\nu+\frac{9}{4}}d\nu' \quad (34)$$

and it is easily transformed into the Fredholm equation derived from equation (30) :

$$F_a(\nu) = \frac{\phi(-\frac{3}{4})}{\nu+\frac{3}{4}} - \frac{2}{\pi}\int_0^{\infty} F_a(\nu')\sin\delta(\nu')\frac{\exp[\rho(\nu')-\rho(-2\nu'-\frac{9}{4})]}{2\nu'+\nu+\frac{9}{4}}d\nu' \; . \quad (35)$$

If we first make the approximation that the width of the resonance is extremely small, equation (35) can be put in a form which has an exact solution:

$$F_a(\nu) = \frac{\phi(-\frac{3}{4})}{\nu+\frac{3}{4}} - \frac{2}{\pi}\frac{\exp[-\rho(-2\nu_R-\frac{9}{4})]}{2\nu_R+\nu+\frac{9}{4}}\int_0^{\infty} F_a(\nu')\sin\delta(\nu')\exp[\rho(\nu')]\,d\nu' \; .$$

Making use of the relation (32), we readily get the solution :

$$F_a(\nu) = \phi(-\tfrac{3}{4})\left\{\frac{1}{\nu+\frac{3}{4}} - \frac{2}{3}\frac{\exp[\rho(-\frac{3}{4})-\rho(-2\nu_R-\frac{9}{4})]}{\nu+2\nu_R+\frac{3}{4}}\right\} \quad (36)$$

and if $\exp[\rho(z)]$ is given by the simple form (33), the solution $\phi(\nu)$ becomes :

$$\phi_a(\nu) = \phi(-\tfrac{3}{4})\,\frac{3+8\nu_R-4\nu}{9+8\nu_R+4\nu} \quad (37)$$

4. Possibility of more refined calculations. We go back to equation (30) and using the same arguments as in the precedent section, we can replace equation (30) by an approximate equation :

$$F(\nu) = \frac{\phi(-\frac{3}{4})}{\nu + \frac{3}{4}} + M(\nu, \nu_R; -\tfrac{3}{4}) \int_0^\infty F(\nu') \sin\delta(\nu') \exp[\rho(\nu')] \, d\nu' . \tag{38}$$

This equation has the exact solution :

$$F(\nu) = \frac{\phi(-\frac{3}{4})}{\nu + \frac{3}{4}} + \Gamma M(\nu, \nu_R; -\tfrac{3}{4})$$

where the constant Γ is given by :

$$\Gamma = \int_0^\infty \sin\delta(\nu) \exp[\rho(\nu)] \left\{ \frac{\phi(-\frac{3}{4})}{\nu + \frac{3}{4}} + \Gamma M(\nu, \nu_R; -\tfrac{3}{4}) \right\} d\nu .$$

By using formula (32) and the integral representation (31) for the kernel M, the constant Γ turns out to be independent of ν_R :

$$\Gamma = \frac{\pi}{3} \phi(-\tfrac{3}{4}) \exp[\rho(-\tfrac{3}{4})]$$

and finally our solution can be written as :

$$\phi(\nu) = \phi(-\tfrac{3}{4}) \left\{ 1 + \frac{\pi}{3} \exp[\rho(-\tfrac{3}{4})] (\nu + \tfrac{3}{4}) M(\nu, \nu_R; -\tfrac{3}{4}) \right. \tag{39}$$

We now calculate $M(\nu, \nu_R, -\frac{3}{4})$ by means of the form (33) for $\exp\left[\varrho(z)\right]$. Making use of the identity :

$$\frac{\nu_R - z}{\nu' - z} \equiv 1 + \frac{\nu_R - \nu}{\nu - z}$$

it is easy to relate the kernel $N(\nu, \nu')$ to the kernel $K(\nu, \nu')$

$$N(\nu,\nu') = \frac{\nu_R - \nu}{\nu_R + 1} K(\nu,\nu') + \frac{6}{\pi} \frac{1}{\nu_R + 1}$$

and the function $M(\nu, \nu_R; -\frac{3}{4})$ to $K(\nu, \nu_R)$

$$M(\nu, \nu_R; -\tfrac{3}{4}) = \frac{\nu_R + \frac{3}{4}}{(\nu + \frac{3}{4})(\nu_R + 1)} \left[(\nu_R - \nu) K(\nu,\nu_R) - \frac{2}{\pi}\right]$$

Solution (39) takes then the form

$$\phi(\nu) = \phi(-\tfrac{3}{4}) \; \frac{1}{3}\left[1 + \pi(\nu_R - \nu) K(\nu,\nu_R)\right] \tag{40}$$

It is easy to verify that solution (40) reduces to (37) in the approximation "cos Θ" = 0.

VI. Bipion model approach.

1. In the case of a very sharp resonance, the results given in formula (37) can be derived from a simple model in perturbation theory.

We first assume, for the pion-pion scattering amplitude, in the state I=J=1, a Breit-Wigner form :

$$h(q^2) = e^{i\delta} \sin\delta = \frac{\gamma q^3}{s_R - s - i\gamma q^3}$$

where γ is directly related to the width of the resonance and $s_R = 4(\nu_R + 1)$ the square of the total resonant energy in the centre-of-mass system of the two pions.

We now simulate this resonance by a metastable particle, the bipion of spin 1 and isospin 1. The mass of the bipion is given by $M_B = \sqrt{s_R}$ and the lifetime easily related to γ .

2. In the bipion model, the photoproduction of pions is given as the sum of the three following graphs :

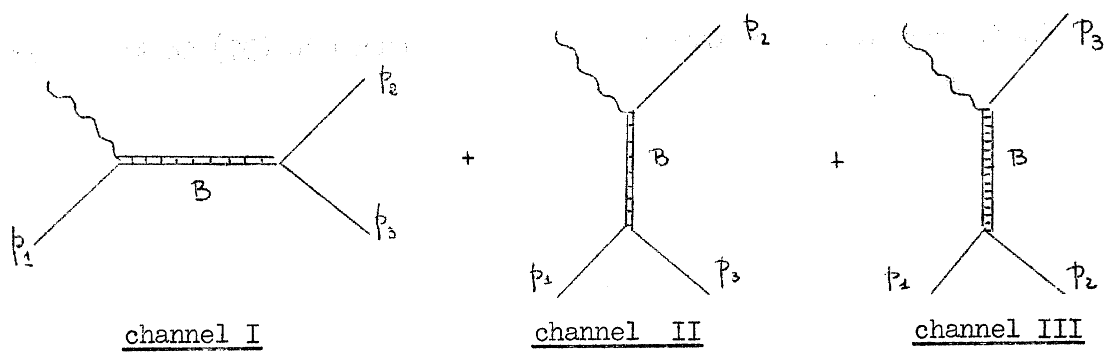

Fig. 6

In order to compute these diagrams we must know :

a) the bipion propagator ;

b) the pion-pion-bipion coupling ;

c) the photon-pion-bipion coupling .

3. The bipion propagator is given by the vector boson field theory. Let us call $B_\mu^\alpha(x)$ the isospin component α of the vector boson field describing the bipion. In order to conserve only three independent components, we add a supplementary condition $\partial_\mu B_\mu(x) = 0$. The bipion propagator is then

$$P_{\mu\nu}^{\alpha\beta} = \delta_{\alpha\beta}\left(\delta_{\mu\nu} + \frac{K_\mu K_\nu}{M_B^2}\right) \frac{1}{K^2 + M_B^2} \qquad (41)$$

where M_B is the bipion mass and K_μ the energy momentum four vector.

4. The pion-pion-bipion interaction must be a scalar with respect to the Lorentz group and to the isospin group :

$$H_1 = \frac{1}{2} \Lambda_1 \frac{\varepsilon_{\alpha\beta\gamma}}{\sqrt{2}} \phi_\pi^\beta(x) \partial_\mu \phi_\pi^\gamma(x) B_\mu^\alpha(x)$$

where $\phi_\pi^\alpha(x)$ is the isospin component α of the pseudoscalar pion field and Λ_1 a dimension less coupling constant.

We are now interested in the matrix element or the bipion current between two pions and the vacuum in the Born approximation .

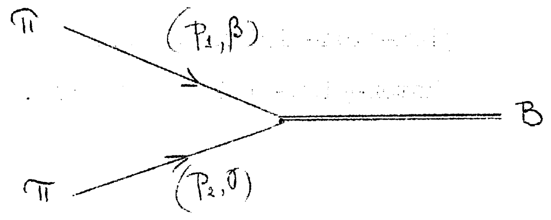

Fig. 7

The π–π–B vertex

With the notations of Fig. 7 we immediately obtain :

$$V^{\alpha}_{\mu} = \frac{1}{2i} \Lambda_1 \frac{\varepsilon_{\alpha\beta\gamma}}{\sqrt{2}} (p_2 - p_1)_{\mu} \tag{42}$$

5. The photon-pion-bipion interaction, by Lorentz invariance and gauge invariance has the following form ;

$$H_2 = \frac{e\Lambda_2}{\mu} \frac{1}{2} \varepsilon_{\lambda\mu\nu\rho} F_{\lambda\mu}(x) \partial_{\nu} \phi^{\alpha}_{\pi}(x) B^{\alpha}_{\rho}(x)$$

where $F_{\lambda\mu}(x)$ is the electromagnetic tensor and Λ_2 a dimensionless coupling constant.

The matrix element for the bipion current is then given by .

$$V_\rho^\beta = \frac{e\Lambda_2}{\mu}\,\frac{1}{2}\,\varepsilon_{\lambda\mu\nu\rho}\, e_\lambda k_\mu p_\nu\, \delta_{\alpha\beta} \tag{43}$$

with the notations given in Fig. 8

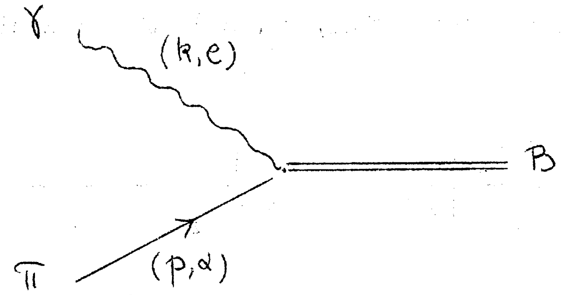

Fig. 8

The γ-π-B vertex .

6. We are now able to compute the Born approximation for the photoproduction. The T matrix element for the diagram represented in Fig. 9 is given by :

$$T^{(1)} = e\,\frac{\Lambda_1\Lambda_2}{\mu^2}\,\frac{1}{4i}\,\varepsilon_{\lambda\mu\nu\rho}\, e_\lambda k_\mu p_{1\nu}(p_2-p_3)_\rho\,\frac{1}{M_B^2-s_1}\,\frac{\varepsilon_{\alpha\beta\gamma}}{\sqrt{2}}$$

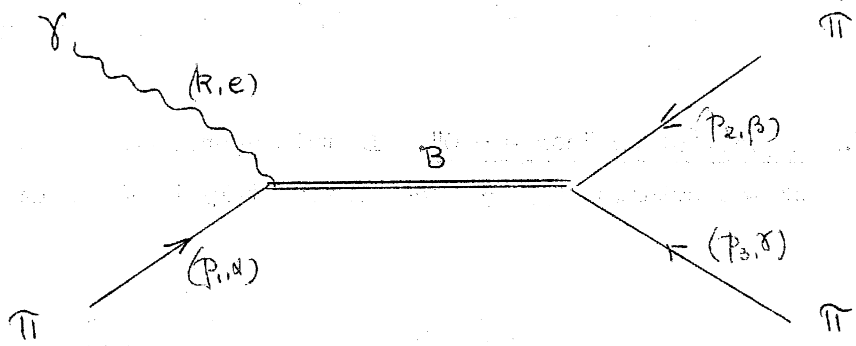

Fig. 9

Conservation of energy momentum leads to the more convenient form:

$$T^{(1)} = \frac{e\Lambda_1\Lambda_2}{\mu} \frac{1}{2i} \varepsilon_{\lambda\mu\nu\rho} e_\lambda P_{1\mu} P_{2\nu} P_{3\rho} \frac{\varepsilon_{\alpha\beta\gamma}}{\sqrt{2}} \frac{1}{M_B^2 - s_1}$$

After comparison with formula (1) and by using the crossing symmetry, we obtain for the scalar function $F(s_1, s_2, s_3)$ the explicit form :

$$F^B(s_1, s_2, s_3) = \frac{e\Lambda_1\Lambda_2}{\mu} \left[\frac{1}{M_B^2 - s_1} + \frac{1}{M_B^2 - s_2} + \frac{1}{M_B^2 - s_3} \right] \tag{44}$$

In the bipion model, the function $\exp\left[\rho(\nu) + i\delta(\nu)\right]$ is simply represented by

$$\frac{\nu_R + 1}{\nu_R - \nu} = \frac{M_B^2}{M_B^2 - s_1}$$

and we can interpret equation (24) into

$$F^B(s_1, s_2, s_3) = \frac{e\Lambda_1\Lambda_2}{\mu M_B^2} \exp\left[\rho(\nu) + i\delta(\nu)\right] \left\{ 1 + \frac{M_B^2 - s_1}{M_B^2 - s_2} + \frac{M_B^2 - s_1}{M_B^2 - s_3} \right\} \tag{45}$$

7. Approximation "cos Θ = 0". In this approximation $s_2 + s_3 = \frac{3 - s_1}{2} = -\frac{1+4\nu}{2}$ and the solution (45) reduces immediately to the form (37) as expected

$$F^B_a(s_1, s_2, s_3) = \exp\left[\rho(\nu) + i\delta(\nu)\right] \phi\left(-\tfrac{3}{4}\right) \frac{3 + 8\nu_R - 4\nu}{9 + 8\nu_R + 4\nu}$$

with

$$\phi\left(-\tfrac{3}{4}\right) = \frac{3\, e\Lambda_1\Lambda_2}{\mu M_B^2}$$

VII. Determination of the coupling constant

1. One has no direct way to measure the coupling constant Λ. We must consider other processes, where the reaction $\gamma + \pi \to \pi + \pi$ can play a role and try to deduce, from experiments, the importance of this reaction and an order of magnitude for the Λ.

One can also use the Chew and Low extrapolation method and then directly measure the total cross-section for the photoproduction of pions on pions.

2. Three measurable processes where the reaction $\gamma + \pi \to \pi + \pi$ enters are the photoproduction of pions on nucleons, the Compton scattering on nucleons and the π^0 lifetime.

The corresponding diagram for the first process is given in Fig. 10

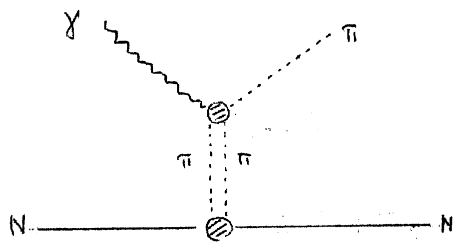

Fig. 10

and will be studied in a next lecture.

In the annihilation channel for Compton scattering we have the π^0 state, considered by Jacob and Matthews [4] and the two-pion state, always neglected. In order to take into account this state, we must know the amplitude for Compton scattering on pions in the annihilation channel. This problem has been solved by Gourdin and Martin [5] in terms of the S $\pi - \pi$ scattering phase shifts and of the $\gamma + \pi \to \pi + \pi$ total cross-section.

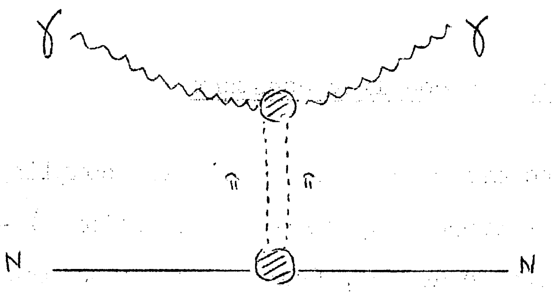

Fig. 11

Intermediate $\pi - \pi$ state in Compton scattering on nucleons

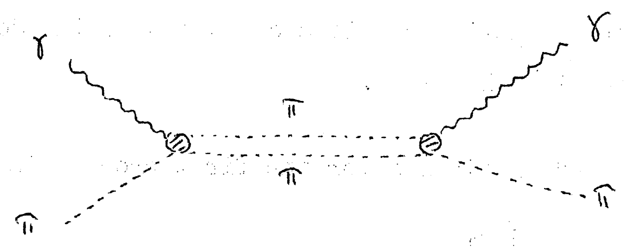

Fig. 12

Amplitude $\gamma + \pi \rightarrow \pi + \pi$ in Compton scattering on pions

Preliminary calculations by Desai [6] show that the two pion term can be important in Compton scattering on nucleons and that in Compton scattering on pions the contributions due to the $\gamma + \pi \rightarrow \pi + \pi$ amplitude are not dominant, in other terms, the Λ coupling constant is not very large.

For the π^{o} lifetime, the $\gamma + \pi \rightarrow \pi + \pi$ amplitude enters in the follwoing manner :

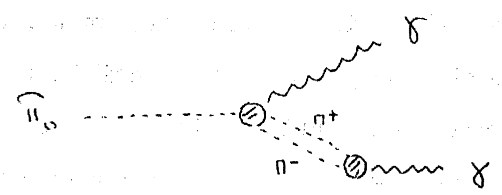

Fig. 13

Contribution from the $\gamma + \pi \rightarrow \pi + \pi$ amplitude to π^{o} lifetime .

H.S. Wong [7] has estimated an upper limit of Λ in this manner, but it appears very difficult to deduce any valuable result, because of the uncertainty of the experimental value of the π^o lifetime.

3. The Chew and Low [8] extrapolation method can be applied to two cases:

a) Production of a pion by a pion in the Coulomb field of a heavy nucleus

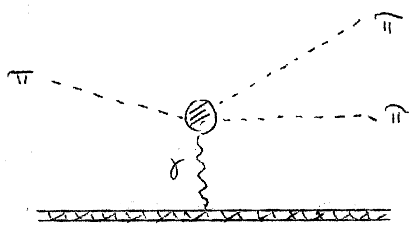

Fig. 14

A possible way of avoiding a difficult extrapolation consists in using various targets and putting in evidence a Z^2 effect in the cross-section.

b) Photoproduction of two pions on a nucleon

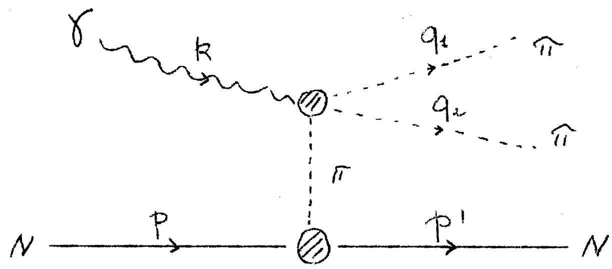

Fig. 15

Photoproduction of two pions on a nucleon

To illustrate the Chew and Low method, we treat completely the photoproduction case. With the notations indicated in Fig. 15 for the energy momentum four vectors, the T matrix element for the diagram indicated in Fig. 15 is given by

$$T_1 = (2\pi)^4 \delta_4 (k+\Delta-q_1-q_2) \left(\frac{1}{4q_{10}q_{20}}\right)^{\frac{1}{2}} \frac{1}{(2k_0)^{\frac{1}{2}}} \frac{M}{(p_0 p'_0)^{\frac{1}{2}}} \frac{\langle q_1 q_2 | j^\alpha | k \rangle \langle p' | j^\alpha | p \rangle}{\Delta^2+\mu^2}$$

where j^α is the pion current and $\Delta = p'-p$ the energy momentum transfert for the nucleon.

The total T matrix for the photoproduction of two pions on nucleons has the form $T = T_1+T_2$ where T_2 corresponds to all other possible diagrams. But, only T_1 has a pole in Δ^2 for $\Delta^2 = -\mu^2$.

The differential cross-section due to T_1 only is then given by

$$d\sigma_1 = \frac{1}{8(2\pi)^5} \frac{|\langle q_1 q_2 | j^\alpha | k \rangle \langle p' | j^\alpha | p \rangle|^2}{(\Delta^2+\mu^2)^2} \frac{M^2}{p\cdot k} \frac{d_3 q_1}{q_{10}} \frac{d_3 q_2}{q_{20}} \frac{d_3 p'}{p'_0} \delta_4(k+\Delta-q_1-q_2)$$

Let us go to the photoproduction of pions on pions. With the same notations, the T matrix element is simply :

$$T_0 = (2\pi)^4 \delta_4 (k+\Delta-q_1-q_2) \left(\frac{1}{4q_{10}q_{20}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2k_0}} \langle q_1 q_2 | j^\alpha | k \rangle$$

and the differential cross-section

$$d\sigma_0 = \frac{1}{8(2\pi)^2} |\langle q_1 q_2 | j^\alpha | k \rangle|^2 \frac{1}{k\cdot\Delta} \frac{d_3 q_1}{q_{10}} \frac{d_3 q_2}{q_{20}} \delta_4(k+\Delta-q_1-q_2)$$

Because the simple isospin dependence of T_0, we can easily relate $d\sigma_1$ to $d\sigma_0$

$$d\sigma_1 = d\sigma_0 \; \frac{M_2^2}{(2\pi)^3} \; \frac{|\langle p'|j^\alpha|p\rangle|^2}{(\Delta^2+\mu^2)^2} \; \frac{|k\cdot\Delta|}{|p\cdot k|} \; \frac{d_3p'}{p'_0}$$

or by using the variable Δ^2 and W^2

$$\Delta^2 = -(p'-p)^2 \qquad W^2 = -(k+\Delta)^2 = -(q_1+q_2)^2$$

$$d\sigma_1 = d\sigma_0 \; \frac{M^2}{(4\pi)^2} \; \frac{|\langle p'|j^\alpha|p\rangle|^2}{(\Delta^2+\mu^2)^2} \; \frac{|k\cdot\Delta|}{(p\cdot k)^2} \; d\Delta^2 \, dW^2$$

The π-N vertex can be calculated in Born approximation using a PS(PS) coupling. After sommation over the nucleon spin we obtain :

$$|\langle p'|j^\alpha|p\rangle|^2 = \frac{g^2}{4M^2} \; \Delta^2 \; \tau_\alpha^2$$

For neutral pion $\tau_\alpha^2 = 1$ and for charged pions $\tau_\alpha^2 = 2$.

We now consider the complete cross-section σ for photoproduction of two pions on nucleons, experimentally measured. Let us define as $F(\Delta^2, W^2)$ the following function :

$$F(\Delta^2, W^2) = \frac{\partial^2\sigma}{\partial\Delta^2\,\partial W^2} \; \frac{(\Delta^2+\mu^2)^2}{\Delta^2} \; \frac{2(8\pi)^2(p\cdot k)^2}{W^2-\mu^2} \; \frac{1}{g^2}$$

At the extrapolation point $\Delta^2 = -\mu^2$, only the amplitude T_1 contributes and we simply obtain :

$$\lim_{\Delta^2+\mu^2\to 0} F(\Delta^2, W^2) = \sigma_0(W^2)$$

R E F E R E N C E S

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PART II

ELECTROMAGNETIC FORM FACTORS

I. General consideration on electromagnetic nucleon form factors .

1. Analysis of experimental data on electro-proton scattering [1)] shows a strong deviation from the Mott cross-section formula calculated in the Born approximation for relativistic electrons

$$\frac{d\sigma_M}{d\Omega} = e^4 \frac{\cos^2\frac{\theta}{2}}{4\, k_0^2 \sin^4\frac{\theta}{2}} \; \frac{1}{1+\frac{2k_0}{M}\sin^2\frac{\theta}{2}}$$

k_0 is the incident electron energy and θ the scattering angle in the laboratory system.

We want now to describe these deviations from Coulomb scattering by assuming for the nucleon an internal electromagnetic structure

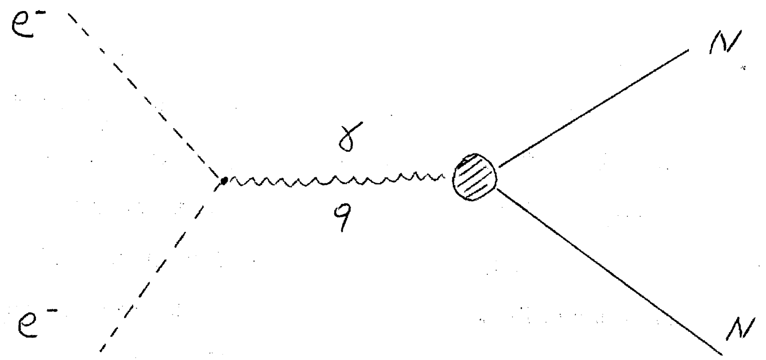

Fig. 1

Electron-nucleon scattering

As it will be seen later, general considerations allow us to define for the proton structure two electromagnetic form factors $F_1^p(q^2)$ and $F_2^p(q^2)$.

Mott's formula can be converted into Rosenbluth's formula:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_M}{d\Omega} \left\{ F_1^2 + \frac{q^2}{4M^2} \left[2(F_1 + \kappa_p F_2)^2 \operatorname{tg}^2 \frac{\theta}{2} + \kappa_p^2 F_2^2 \right] \right\}$$

where κ_p is the anomalous magnetic moment for the proton. For $q^2=0$ the two formulae are identical and we have the normalization

$$F_1^p(0) = F_2^p(0) = 1$$

Experimental data at low momentum transfer can be fitted by the empirical form

$$F_1^p \simeq F_2^p \simeq \left(1 + \frac{a^2 q^2}{12}\right)^{-2} .$$

where $a \simeq 0.8$ fermi.

Such a result is no longer valid for $q^2 > 5\mu^2$ and there is a splitting of F_1^p and F_2^p consistent with the low energy data [2]. By analyzing two measurements of $\frac{d\sigma}{d\Omega}$ at the same q value but at different angles and incident energy one can easily exhibit the different behaviour of F_1^p and F_2^p at high momentum transfer. At each measurement corresponds an ellipse in the F_1^p, F_2^p plane and one finds four possible intersections of ellipses which determine a F_1^p, F_2^p pair; one chooses the more physically reasonable set.

2. The only direct results one can obtain on neutron electromagnetic form factors are deduced from neutron scattering experiments by atomic electrons [3]. One finds for the neutron root mean square radius *) a value consistent with zero :

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*) The r.m.s. radius is defined as

$$F(q^2) = F(0) - \frac{q^2}{6} \langle r_1^2 \rangle + O(q^2)$$

$$\langle r_{1n}^2 \rangle = .003 \pm .007 \quad 10^{-26}\,cm^2$$

One has to use indirect measurements to obtain information about neutron structure at higher momentum transfer.

In an impulse approximation treatment, one can relate the deuteron electromagnetic form factors to the nucleon electromagnetic form factors. Elastic electron-deuteron scattering experiments give information on the iso-scalar combination of the nucleon form factors (deuteron isospin is zero):

$$G_1^S = \frac{e}{2}\left(F_1^p + F_1^n\right) \qquad G_2^S = \frac{e}{4M}\left(\kappa_p F_2^p + \kappa_n F_2^n\right)$$

Unfortunately, the experimental cross-sections are very small ($\simeq 10^{-32}$ cm at $q^2 \simeq 2f^{-2}$) and we have some theoretical uncertainties due to the deuteron structure [4].

By using impulse approximation, one can evaluate the neutron form factors from the value, at the quasi elastic peak, of the cross-sections for electrodisintegration of the deuteron. Actual determinations neglect the two nucleon final state interaction, the interference between neutron and proton contributions, the D part of the deuteron wave function one treats in a non-relativistic approach [5].

Another approach to know the neutron structure is the study of the electroproduction of pions from protons [6]. One can define two inelastic form factors, functions of q^2 and W - the total π-N energy in the c.m. system - [7] related to the nucleon form factors. By looking at the final electron with kinematical conditions corresponding to the 3/2 3/2 resonance, one can see that the dominant contribution is due to the magnetic dipole involving the isovector part of the magnetic form factor G_2^V

$$G_1^V = \frac{e}{2}\left(F_1^p - F_1^n\right) \qquad G_2^V = \frac{e}{4M}\left(\kappa_p F_2^p - \kappa_N F_2^n\right)$$

3. The use of dispersion relations seems to be the more reasonable approach for the theoretical problem of the nucleon electromagnetic structure. In this spirit, we must solve the preliminary problem of pion electromagnetic structure in Section II. The invariance properties for nucleon electromagnetic form factors are given in Section III. Section IV is concerned with the dispersion relation approach of the problem. The isovector part of nucleon form factors are given in terms of pion form factor and π -N scattering amplitudes. A model is given in Section V, corresponding to the bipion metastable particle and tripion bound state which agrees with experimental result.

II. Pion electromagnetic structure .

1. We are first interested by the vertex reproduced in Fig. 2

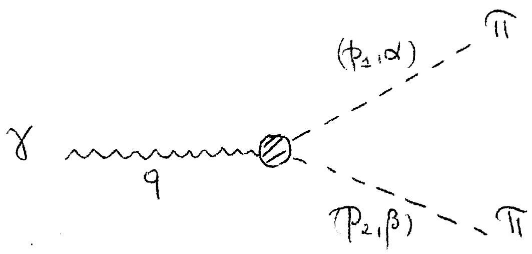

Fig. 2

Pion electromagnetic vertex

Let us put:

$$t = -q^2 = -(p_1 + p_2)^2 .$$

Using parity conservation and charge conjugation invariance, we conclude that the two pion system has the following quantum numbers: I=J=1 and, of course, $I_3=0$.
In other terms, the vertex function must be a vector in Lorentz space and the third component of an isovector.

From gauge invariance, the only vector one can construct is $(p_1-p_2)_\mu$ and the general form of the pion electromagnetic vertex, for two pions on the mass-shell is the following:

$$V_\mu^{\alpha\beta} = \varepsilon_{\alpha\beta 3} \frac{e}{2} (p_1 - p_2)_\mu F_\pi(t) \qquad (1)$$

where the pion form factor $F_\pi(t)$ is normalized as *):

$$F_\pi(0) = 1 \tag{2}$$

2. It has been shown, by a study of Feynman diagrams that a vertex function of one variable satisfy a dispersion relation. The function $F_\pi(t)$ is analytic in the t complex plane except a cut on the real axis from $(2\mu)^2$ to $+\infty$. If we first disregard the problem of possible subtractions. The function $F_\pi(t)$ satisfies the representation:

$$F_\pi'(t) = \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\operatorname{Im} F_\pi(t')}{t'-t-i\varepsilon}\, dt' \tag{3}$$

We use the unitarity condition to calculate the weight function in integral (3). For $4\mu^2 < t < 16\mu^2$, only the two pion intermediate state contributes and we have:

$$\operatorname{Im} F_\pi(t) = F_\pi(t)\, h^*(t) \tag{4}$$

where $h(t)$ is the previously introduced pion-pion scattering amplitude in the state I=J=1, but considered here as a function of $t = 4(\nu + \mu^2)$.

We extend this relation for the values $t > 16\mu^2$ by neglecting inelastic $\pi - \pi$ scattering and other higher contributions as $K\bar{K}$ pairs, N-$\bar{N}$ pairs.

*) This result is immediately obtained by using the Born approximation of the pion electromagnetic vertex.

Under this assumption, equation (3) becomes a Mushkelishvili-Omnès homogeneous equation:

$$F_\pi(t) = \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{h^*(t')\,F_\pi(t')}{t'-t-i\varepsilon}\,dt' \tag{5}$$

one can transform with a subtraction by using condition (2) into :

$$F_\pi(t) = 1 + \frac{t}{\pi}\int_{(2\mu)^2}^{\infty} \frac{h^*(t')\,F_\pi(t')}{t'(t'-t-i\varepsilon)}\,dt' \tag{6}$$

Resolution of equation (6) is straightforward and gives :

$$F_\pi(t) = \exp\left[\rho(t) + i\,\delta(t)\right] \tag{7}$$

where the function $\rho(t)$ is identical to the previous one $\rho(\nu)$ defined in equation (21), Part I.

$$\rho(t) = \frac{t}{\pi}\,P\int_{(2\mu)^2}^{\infty} \frac{\delta(t')}{t'(t'-t)}\,dt'$$

3. Solution 7 is dominated by the $\pi - \pi$ P resonance. In the physical region for electron-π meson (or $e^+ - \pi$) scattering, we have $t < 0$ and we retain only for $F_\pi(t)$ the simple form given in equation(33), Part I:

$$F_\pi(t) = \frac{t_R}{t_R - t} \qquad t < 0$$

The effect of the pion electromagnetic structure is then to reduce the cross-section calculated without pion structure by a factor:

$$\frac{t_R^2}{(t_R - t)^2} = \frac{t_R^2}{\left[t_R + 2k^2(1-\cos\theta)\right]^2}$$

where k and $\cos\Theta$ are the c.m. variables for $e-\pi$ scattering.

4. In the <u>electron-positron</u> annihilation in two pions, however, the cross-section is greatly enhanced by the pion structure. If we use the approximate form:

$$h(q^2) = \frac{\gamma q^3}{t_R - t - i\gamma q^3}$$

for $\pi - \pi$ scattering amplitude, the corresponding one for $F_\pi(t)$ can be written:

$$F_\pi(t) = \frac{t_R - \gamma}{t_R - t - i\gamma q^3}$$

This resonant factor will produce a maximum of the annihilation cross-section for t near t_R; this maximum becomes sharper as the width of the resonance directly related to γ decreases.

III. Invariance properties for nucleon electromagnetic form factors

Let us consider the graph drawn in Fig. 3

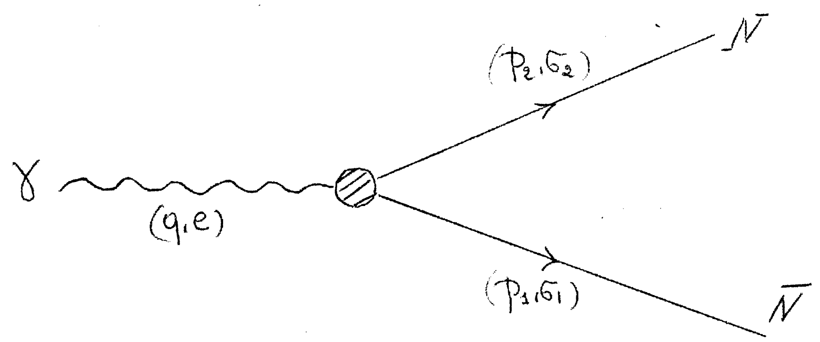

Fig. 3

Nucleon electromagnetic vertex

With nucleons on the mass shell, one can define only one scalar invariant $t = -q^2$.

We are interested in the matrix element of the electromagnetic current between the vacuum and a nucleon-antinucleon state.

$$T_\mu = \langle p_1, \sigma_1 ; p_2 \sigma_2 | J_\mu | 0 \rangle \tag{8}$$

1. Lorentz invariance allows us to write :

$$T_\mu = \sum_\alpha G_{1\alpha}(t)\, \bar{u}_{\sigma_2}(p_2)\, H^\alpha_\mu(p_1, p_2)\, u_{\sigma_1}(-p_1) \tag{9}$$

where $G_\alpha(t)$ are scalar functions at t only and the H^α_μ must be constructed from the sixteen matrices of the Dirac algebra and the energy momenta four vectors $(q = p_1 + p_2)$. Because of parity conservation and time reversal invariance, one can form only two invariant expressions for H^{α}_{μ} if one takes into account the nucleon spin constraints due to the Dirac equation. It is usual to choose [8]:

$$H^{1}_{\mu} = i\gamma_{\mu} \qquad\qquad H^{2}_{\mu} = \frac{1}{2}[\gamma_{\mu}, \gamma\cdot q]$$

Equation (9) becomes:

$$T_{\mu} = \bar{u}_{\sigma_2}(p_2)\left\{ i\gamma_{\mu} G_1(t) + \frac{1}{2}[\gamma_{\mu}, (\gamma\cdot q)]\, G_2(t)\right\} u_{\sigma_1}(-p_1) \qquad (10)$$

The invariant functions $G_1(t)$ and $G_2(t)$ are the nucleon electromagnetic form factors.

This result can be easily transformed into:

$$T_{\mu} = \bar{u}_{\sigma_2}(p_2)\left\{ i\gamma_{\mu}[G_1(t) + 2MG_2(t)] + (p_2-p_1)_{\mu}\, G_2(t)\right\} u_{\sigma_1}(-p_1) \qquad (11)$$

2. The nucleon-antinucleon state can have isospin 0 or 1. The T_{μ} amplitude can be divided into two parts:

$$T_{\mu} = T^{s}_{\mu} + \tau_3\, T^{v}_{\mu}$$

The first one corresponds to the isoscalar part of the electromagnetic current and the second one to the isovector part. The same decomposition is of course very useful for the form factors:

$$G(t) = G^{S}(t) + \tau_3\, G^{V}(t)$$

With those definitions, the proton and neutron form factors are given by

$$
\begin{aligned}
G^{p}(t) &= G^{S}(t) + G^{V}(t) \\
G^{n}(t) &= G^{S}(t) - G^{V}(t)
\end{aligned}
\tag{12}
$$

For a real photon $t = 0$. The Born approximation of the photon-nucleon vertex, calculated with the Dirac and Pauli coupling, allows us to normalize the nucleon form factors as:

$$
\begin{cases}
G_1^{S}(0) = G_1^{V}(0) = \dfrac{e}{2} \\
G_2^{S}(0) = \dfrac{e}{4M}(\kappa_P + \kappa_N) = \dfrac{e}{2M} g_S \\
G_2^{V}(0) = \dfrac{e}{4M}(\kappa_P - \kappa_N) = \dfrac{e}{2M} g_V
\end{cases}
\tag{13}
$$

where e is the electric charge and K_P, K_N the anomalous parts of the proton and neutron magnetic moments in units $e/2M$.

It is also convenient to define form factors $F(t)$ normalized to unity:

$$
\begin{cases}
G_1^{S,V}(t) = \dfrac{e}{2} F_1^{S,V}(t) & G_2^{S,V}(t) = \dfrac{e}{2M} g_{S,V} F_2^{S,V}(t) \\
G_1^{p,n}(t) = e F_1^{p,n}(t) & G_2^{p,n}(t) = \dfrac{e}{2M} \kappa_{P,N} F_2^{p,n}(t)
\end{cases}
$$

Equation (12) becomes:

$$F_1^p(t) = \frac{1}{2}(F_1^S + F_1^V) \qquad F_1^n(t) = \frac{1}{2}(F_1^S - F_1^V)$$

$$F_2^p(t) = \frac{1}{2}(F_2^S + F_2^V) + \frac{\kappa_N}{\kappa_P}\,\frac{1}{2}(F_2^S - F_2^V)$$

$$F_2^n(t) = \frac{1}{2}(F_2^S + F_2^V) + \frac{\kappa_P}{\kappa_N}\,\frac{1}{2}(F_2^S - F_2^V)$$

3. Making use of centre-of-mass variables for the nucleon-anti-nucleon system:

$$P_1 = (-\vec{p}, E) \qquad P_2 = (\vec{p}, E) \qquad t = 4(\vec{p}^2 + M^2) = 4E^2$$

we obtain for $e_\mu T_\mu$ the simple expression:

$$e_\mu T_\mu = \chi_N^* \left\{ \frac{E}{M}\left[G_1 + 2MG_2\right]\vec{\sigma}\vec{e} + \left[2EG_2 - G_1\right]\frac{(\vec{e}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{M(E+M)} \right\} \chi_{\bar{N}} \quad (14)$$

where the gauge is chosen so as $e_4 = 0$.

IV. Dispersion relation approach for nucleon electromagnetic form factors.

1. It can be shown by general considerations that the vertex functions $G_1(t)$ and $G_2(t)$ satisfy dispersion relations. If we first disregard the subtraction problem, we can write the spectral representation:

$$G(t) = \frac{1}{\pi} \int \frac{g(t')}{t'-t}\, dt'$$

where $g(t)$ may be determined from the unitarity condition.

2. The quantum numbers associated to the nucleon-antinucleon state are $J=1$, $\omega = -1$ and the only possible combination corresponds to a $^3S_1 + {}^3D_1$ state.

By using the G invariance, one can immediately see that only even-pion intermediate states contribute to the isovector part and only odd-pion states to the isoscalar part. We neglect, in the present treatment, other possible intermediate states as for example $K\bar{K}$ states, $N\bar{N}$ states, $Y\bar{Y}$ states with an arbitrary number of pions.

3. Let us now consider the isovector part. If there are no bound states with quantum numbers $I=J=1$, $\omega = -1$, the functions $G^V(t)$ are analytic in the t complex plane except a cut from $(2\mu)^2$ to $+\infty$.
For $(2\mu)^2 < t < (4\mu)^2$, the only possible intermediate state is a two-pion $I=J=1$ state:

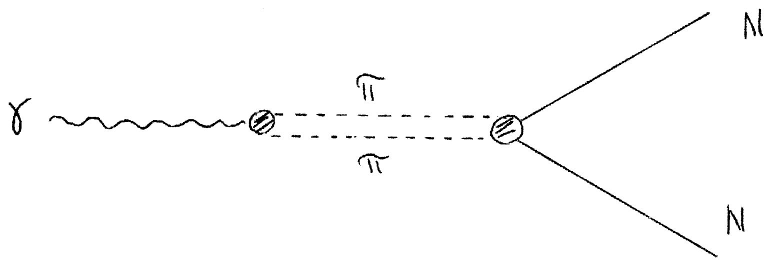

Fig. 4

and the spectral function $g^V(t)$ is given in terms of the pion electromagnetic form factor $F_\pi(t)$ and of the $\pi+\pi \to N+\bar{N}$ transition amplitude in the state I=J=1. We assume this result valid for other values of t.

4. The amplitude $\pi + \pi \to N+\bar{N}$ can be determined in the general framework of π-N scattering [9].

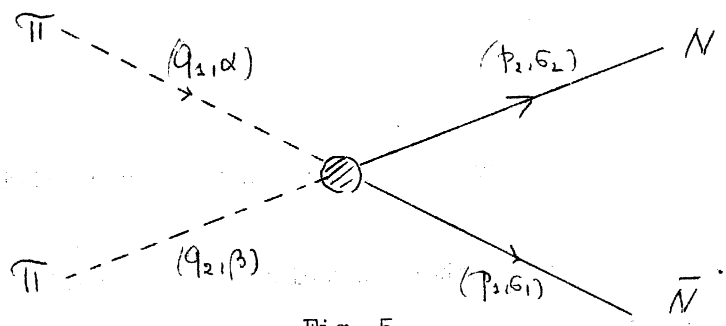

Fig. 5

The $\pi+\pi \to N+\bar{N}$ reaction

By using charge independence, the T matrix element can be written as a 2x2 matrix in the nucleon isospin space:

$$T_{\alpha\beta} = \delta_{\alpha\beta}\, T^{(+)} + \frac{1}{2}\,[\tau_\beta,\tau_\alpha]\, T^{(-)}$$

For a $N\bar{N}$ system of isospin I=1, only the $T^{(-)}$ amplitude occurs. By Lorentz invariance, the $T^{(\pm)}$ amplitudes have the general form

$$T^{\pm} = \bar{u}_{\sigma_2}(p_2)\left[-A^{(\pm)}(s_1,s_2,t) + i\,\frac{\gamma\cdot(q_1-q_2)}{2}\,B^{(\pm)}(s_1,s_2,t)\right]u_{\sigma_1}(-p_1) \quad (15)$$

where

$$s_1 = -(q_1 - p_1)^2$$
$$s_2 = -(q_1 - p_2)^2$$
$$t = -(p_1 + p_2)^2$$

with on the mass shell $s_1+s_2+t = 2M^2+2\mu^2$.
In the centre-of-mass system, by using the following variables:

$$q_1 = (\vec{q}, E) \quad q_2 = (-\vec{q}, E) \quad p_1 = (-\vec{p}, E) \quad p_2 = (\vec{p}, E)$$

equation (15) reduces simply to:

$$T = \chi_N^* \left\{ -\frac{\vec{\sigma}\cdot\vec{p}}{M} A - \frac{(\vec{\sigma}\cdot\vec{p})(\vec{p}\cdot\vec{q})}{M(M+E)} B + \frac{E}{M} \vec{\sigma}\cdot\vec{q}\, B \right\} \chi_{\bar{N}} \tag{16}$$

5. We now apply the unitarity condition to the electromagnetic vertex; the general formula for the case of a two-pion intermediate state has been previously given:

$$i \langle N\bar{N} | T^* - T | \gamma \rangle = \frac{2}{(8\pi)^2} \frac{q}{E} \sum_{\alpha\beta} \int d\Omega_q \langle N\bar{N} | T^* | \pi^\alpha \pi^\beta \rangle \langle \pi^\alpha \pi^\beta | T | \gamma \rangle$$

Summation over pion isospin gives the isovector dependence $2\tau_3$ and we obtain:

$$\frac{E}{M}(\vec{\sigma}\cdot\vec{e})\,\mathrm{Im}\left[G_1^v+2MG_2^v\right]+\frac{(\vec{e}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{M(E+M)}\,\mathrm{Im}\left[2EG_2^v-G_1^v\right]=$$

$$=\frac{2}{(8\pi)^2}\,e\,\frac{q}{E}\,F_\pi^*(t)\int d\Omega_q\,(\vec{e}\cdot\vec{q})\left\{-\frac{\vec{\sigma}\cdot\vec{p}}{M}A^{(-)}-\frac{(\vec{\sigma}\cdot\vec{p})(\vec{p}\cdot\vec{q})}{M(E+M)}B^{(-)}+\frac{E}{M}\vec{\sigma}\cdot\vec{q}\,B^{(-)}\right\}$$

In order to perform the angular integration we expand the scalar functions A and B in Legendre polynomials of the c.m. angle $\vec{p}\cdot\vec{q}=pq\,x$

$$\left\{\begin{aligned}A^{(-)}(t,x)&=\sum_J{}'\left(J+\tfrac{1}{2}\right)A_J^{(-)}(t)\,P_J(x)\\B^{(-)}(t,x)&=\sum_J{}'\left(J+\tfrac{1}{2}\right)B_J^{(-)}(t)\,P_J(x)\end{aligned}\right.$$

and we finally obtain :

$$g_i^v(t)=\mathrm{Im}\,G_i^v(t)=\frac{eq^3}{2E}F_\pi^*(t)\,\Gamma_i(t) \qquad (17)$$

where $\Gamma_i(t)$ are two convenient linear combinations of $A_1^{(-)}$, $B_0^{(-)}$ and $B_2^{(-)}$:

$$\left\{\begin{aligned}\Gamma_1(t)&=\frac{1}{8\pi}\left\{\frac{M}{pq}A_1^{(-)}+\frac{1}{3}B_0^{(-)}-\frac{E^2+2M^2}{3\vec{p}^2}B_2^{(-)}\right\}\\\Gamma_2(t)&=\frac{1}{8\pi}\left\{-\frac{1}{2pq}A_1^{(-)}+\frac{M}{2\vec{p}^2}B_2^{(-)}\right\}\end{aligned}\right. \qquad (18)$$

6. It is easy to verify that the partial amplitudes $\Gamma_i(t)$ do not have kinematical singularities at $p = 0$ $(t = 4M^2)$ or $q = 0$ $(t = 4\mu^2)$. They are analytic in the t complex plane excepted two cuts

a) a right hand cut from $(2\mu)^2$ to $+\infty$ corresponding to the annihilation process ;

b) a left hand cut from $-\infty$ to $\alpha = 4\mu^2 - \frac{\mu^4}{M^2}$ describing the scattering process.

and we can write the spectral representation:

$$\Gamma_i(t) = \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\Gamma_i(t')\, h^*(t')}{t'-t-i\varepsilon}\, dt' + \frac{1}{\pi}\int_{-\infty}^{\alpha} \frac{\operatorname{Im}\Gamma_i(z)}{z-t}\, dz \qquad (19)$$

where the unitarity condition:

$$\operatorname{Im}\Gamma_i(t) = h^*(t)\, \Gamma_i(t)$$

corresponding to a two-pion $I=J=1$ intermediate state

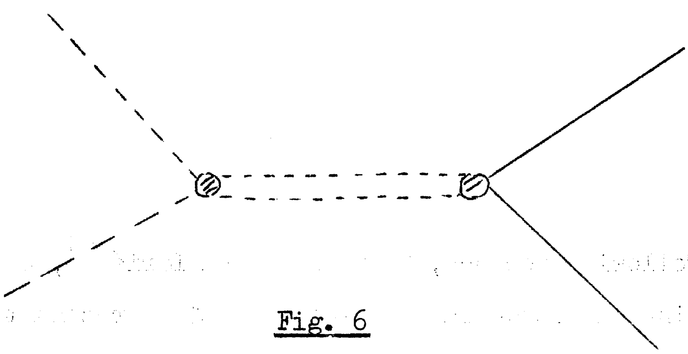

Fig. 6

has been applied in the entire physical range $t > 4\mu^2$. The spectral functions on the left hand cut, $\operatorname{Im}\Gamma_i(z)$, are given in terms of the π-N scattering amplitudes. They contain, in particular, contributions due to the Born terms and the 3/2 3/2 resonance.

Equation (19) can be solved by the Omnès method and we obtain the formal solution:

$$\Gamma_i(t) = \exp\left[\rho(t) + i\delta(t)\right] \frac{1}{\pi} \int_{-\infty}^{\alpha} \frac{\mathrm{Im}\, \Gamma_i(z) \exp\left[-\rho(z)\right]}{z - t}\, dz$$

By putting

$$\Gamma_i(t) = F_\pi(t)\, J_i(t)$$

we transform equation (17) into:

$$\mathrm{Im}\, G_i^V(t) = \frac{e q^3}{2t} \left| F_\pi(t) \right|^2 J_i(t) \tag{20}$$

with

$$J_i(t) = \frac{1}{\pi} \int_{-\infty}^{\alpha} \frac{\mathrm{Im}\, \Gamma_i(z) \exp\left[-\rho(z)\right]}{z - t}\, dz \tag{21}$$

7. Following Bowcock, Cottingham and Lurié [9], we assume for the π-π scattering amplitude in the state I=J=1 a resonant form:

$$h(t) = \frac{\gamma q^3}{t_R - t - i\gamma q^3}$$

The pion electromagnetic form factor can then be written as:

$$F_\pi(t) = \frac{t_R - \gamma}{t_R - t - i\gamma q^3}$$

The integrals $J_i(t)$ are smooth functions of t with respect to $F_\pi(t)$. We can take $J_i(t)$ as a constant given by the π-N scattering:

$$J_i = \frac{C_i}{t_R - \gamma}$$

Equation (20) becomes:

$$\operatorname{Im} G_i^V(t) = \frac{eq^3}{2E} \frac{t_R - \gamma}{(t_R - \gamma)^2 + \gamma^2 q^6} C_i \tag{22}$$

In the case of a narrow resonance, one can perform the following substitution:

$$\frac{\gamma q^3}{(t_R - t)^2 + \gamma^2 q^6} \Rightarrow \pi\, \delta(t_R - t)$$

and equation (22) takes the more practical form:

$$\operatorname{Im} G_i^V(t) = \frac{e}{2} \frac{C_i (t_R - \gamma)}{E_R\, \gamma} \pi\, \delta(t_R - t) \tag{23}$$

We now consider the spectral representation for the form factors $G_1^V(t)$ and $G_2^V(t)$ in the subtracted form. We consider the constants of subtraction as describing higher energies contributions and we adjust these at t=0

$$G_i^V(t) = G_i^V(0) + \frac{t}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\operatorname{Im} G_i^V(t')}{t'(t'-t-i\epsilon)} dt' \tag{24}$$

By using expression (24) for the spectral functions $G_i^V(t)$ we obtain the isovector nucleon form factors in the simple form:

$$\begin{aligned} G_1^V(t) &= \frac{e}{2}\left(1 + \frac{at}{t_R - t}\right) \\ G_2^V(t) &= \frac{e g_V}{2M}\left(1 + \frac{bt}{t_R - t}\right) \end{aligned} \tag{25}$$

where the constants a and b are given by :

$$a = \frac{C_1(t_R - \gamma)}{E_R \gamma t_R} \qquad b = \frac{M}{g_V} \frac{C_2(t_R - \gamma)}{E_R \gamma t_R} \tag{26}$$

8. The same approach is very difficult to use for the isoscalai part. If they are no bound states with quantum numbers I=0, J=1, $\omega = -1$, the functions $G^S(t)$ are analytic in the t complex plane, except a cut from $(3\mu)^2$ to + oo.

For $(3\mu)^2 < t < (5\mu)^2$, the only possible intermediate state is a three-pion I=0 J=1 state

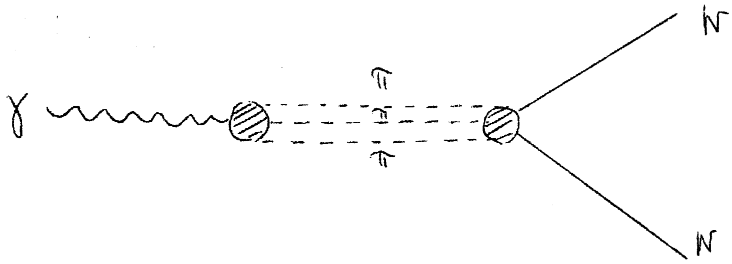

Fig. 7

and the spectral function $g^s(t)$ is given in terms of the $\gamma \rightarrow 3\pi$ vertex and of the $\pi + \pi + \pi \rightarrow N+\bar{N}$ transition amplitude. Unfortunately, nothing is known about this last amplitude and it appears as very difficult to treat the isosclar form factors in a similar way as the isovector form factors and the problem is still open.

V. A model for the nucleon electromagnetic form factors

1. Experimental data exhibit a rapid variation of the electromagnetic nucleon form factor $G(t)$ with the momentum transfer t [10]. If we assume a spectral representation:

$$G(t) = \frac{1}{\pi} \int \frac{g(t')}{t'-t} dt'$$

we easily deduce that the weight function $g(t')$ must be dominated by the little values of t'.

If there is no strong correlation between the intermediate pions, $g(t)$ is related to the statistical weight of the many particle state and will be a smooth function of t'. Only strong correlations between the pions, as for example a resonance, can lead to a more satisfactory understanding of experimental data [11].

2. The low momentum transfer data can be filled by a model proposed three years ago by Clementel and Villi [12]

$$F_1^p(t) \simeq F_2^p(t) \simeq F_2^n(t) \simeq -.2 + \frac{1.2}{1 - t/22\mu^2} \tag{27}$$

$$F_1^n(t) = 0$$

Bergia, Stanghellini, Fubini and Villi [13] suggest to extend this form for the isovector form factors:

$$\begin{aligned} F_1^v(t) &= (1-a_v) + \frac{a_v}{1 - t/t_v} \\ F_2^v(t) &= (1-b_v) + \frac{b_v}{1 - t/t_v} \end{aligned} \tag{28}$$

and to interpret the pole occuring in equation (28) on the basis of a resonant two-pion state: $t_V = t_R$.
We can remark that the form (28) is identical to the result of equation (25) obtained by Bowcock, Cottingham and Lurié on the basis of a dispersion relation treatment under the assumption of a narrow two-pion resonance. The bipion, of course, gives immediately the same result.

3. The isoscalar electric form factor $F_1^S(t)$ must be of the same order of magnitude of $F_1^V(t)$ at low momentum transfer in order to give, for the neutron, a vanishing r.m.s. radius. We have then to expect also a weight function $g^S(t')$ dominated by little values of t'.

At high momentum transfers, it is impossible to fit the experimental data on electron-proton scattering with a form of type (27) where one pole only occurs. The deviations of $F_1^p(t)$ from a Clementel and Villi form are due to the isoscalar form factor $F_1^S(t)$.

As a result of this interpretation of experimental data Bergia, Stanghellini, Fubini and Villi deduce that $F_1^S(t)$ must be formed of a large constant part and a part which vanishes much faster than $F_1^V(t)$:

$$F_1^S(t) = (1 - a_s) + \frac{a_s}{1 - t/t_s} \qquad (29)$$

with

$$\frac{a_s}{t_s} \simeq \frac{a_v}{t_v} \qquad \text{for} \quad \langle r_n^2 \rangle = 0$$

$$a_s < a_v \qquad\qquad t_s < t_v$$

It follows that the neutron charge form factor $F_1^N(t)$ will be positive at larger values of t.

4. Little is known about the isoscalar magnetic form factor $F_2^S(t)$ and we assume for it a form analogous to (29) :

$$F_2^s(t) = (1-b_s) + \frac{b_s}{1-t/t_s} \tag{30}$$

The pole $t=t_s$ can be interpreted as a resonance (or bound state if $t_s < 9\mu^2$) in the I=0, J=1 three-pion state simulated by a "tripion" particle. Formulae (29) and (30) are given by Gourdin, Lurié and Martin [14] as an illustration of the "tripion model" introduced in the photoproduction of pions on nucleons.

Such a bound state can be an explanation of the peak experimentally observed in the He^3 momentum distribution for the reaction $p+d \rightleftharpoons He^3 + \omega_0$ and corresponding to a t_s value of $5\mu^2$ [15].

5. Numerical computations, on the basis of formulae (28), (29), and (30) were performed by Bergia and Stanghellini [16]. It appears that the model is not contradictory with experiments, but the existing data are not sufficient to allow a unique determination of the parameters $a_s b_s t_s$, $a_v b_v t_v$. It is possible, in particular, to find a set of parameters compatible with the experimental information on the two and three- pion resonances ($t_R \simeq 28\mu^2$; $t_s \sim 5\mu^2$). Experimental information, particularly at low momentum transfers, are needed with greater accuracy in order to select the most convenient solutions. On the other hand, deduction of neutron data from deuteron cross-sections requires some progress in the theoretical analysis of the phenomena. The problem of understanding electromagnetic nucleon structure is not yet resolved but we may hope with such a model to clarify the present situation.

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PART III

PHOTOPRODUCTION OF PIONS ON NUCLEONS

I. Introduction

The present lectures are concerned with the effect of the pion-pion I=J=1 resonance on pion photoproduction on nucleons. It is hoped, furthermore, that the existing discrepancies between the experimental data for this process and the theoretical prediction of Chew, Low, Goldberger and Nambu [1] could be adequately explained by taking the pion-pion interaction, and particularly the I=J=1 resonance, into account.

In Section II we study the kinematics and write down the fundamental invariant forms as given by CGLN. Section III is concerned with the multiple expansion in the photoproduction channel of the T reaction matrix: the complete connection between the invariant CGLN amplitudes and the multipole amplitudes is given and a reflection property in the total energy variable established. The unitarity condition for the isosclar amplitude is examined in Section IV; it is shown that in channel III for the reaction $\gamma + \pi \rightarrow N+\bar{N}$, the imagninary part of these amplitudes can be expressed in terms of the $\gamma + \pi \rightarrow N + \bar{N}$ amplitude and the imaginary parts of the isovector nucleon form factors. In Section V, the expressions for the isoscalar amplitudes resulting from the Cini-Fubini [2] form of the Mandelstam representation combined with unitarity in channel III are written down under the assumption that the $I=\frac{1}{2}$ pion-nucleon phase shifts may be neglected; possible corrections to this last assumption are indicated. In Section VI it is shown that a simple model involving a I=J=1 intermediate particle or bipion is able to reproduce the main features of the dispersion treatment of the preceding section. An analogous model, involving a J=1, I=0 intermediate particle or "tripion", is applied in Section VII to the isovector amplitudes and ordinary dispersion relations are written down in this case.

II. Invariance properties of the matrix elements

1. It is convenient to define the three scalar invariants :

$$s_1 = -(k+p_1)^2$$
$$s_2 = -(k+p_2)^2$$
$$t = -(p_1+p_2)^2$$

where the notations are indicated on Fig. 1

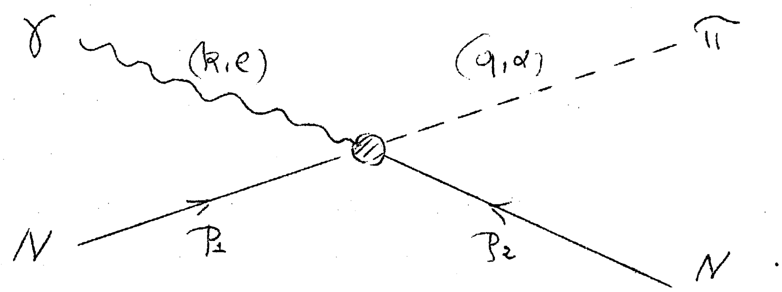

Fig. 1

Conservation of energy momentum: $k+q+p_1+p_2 = 0$ leads, on the mass shell, to: $s_1+s_2+t = 2M^2+\mu^2$, where M is the nucleon mass and μ the pion mass.

The S matrix elements for the processes described by diagram 1 are given by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta_4(p_1+p_2+k+q) \frac{M}{2(p_{10}\, p_{20}\, k_0\, q_0)^{1/2}} T_{fi} \qquad (1)$$

2. Because of the non-conservation of isospin in electromagnetic reactions, we must introduce three invariant quantities for the photoproduction of one pion with isospin :

$$I^{(+)}_\alpha = \delta_{3\alpha} \qquad I^{(-)}_\alpha = \frac{1}{2}[\tau_\alpha, \tau_3] \qquad I^{(0)}_\alpha = \tau_\alpha$$

The first two correspond to the isovector part of the electromagnetic current and the latter to the isoscalar part.

We then can write

$$T_{fi} = T^{(+)} I_{\alpha}^{(+)} + T^{(-)} I_{\alpha}^{(-)} + T^{(0)} I_{\alpha}^{(0)} \tag{2}$$

3. The causal amplitudes T will be functions of

a) the scalar invariants s_1, s_2, t with $s_1+s_2+t = 2M^2+\mu^2$;

b) the energy momenta four vectors p_1, p_2, k, q with $p_1+p_2+k+q = 0$;

c) the spin parameters of the nucleons σ_1, σ_2 ;

d) the photon polaization e_μ .

Lorentz invariance, allows to write amplitude T in the form :

$$T = \sum_{\delta} A_{\delta}(s_1, s_2, t)\, M_{\delta}(p_1, p_2, k, q; \sigma_1, \sigma_2; e_\mu) \tag{3}$$

and our aim is to construct an independent set of scalar functions M_δ , Lorentz invariant, gauge invariant and, T, C, P invariant. The scalar functions $A_\delta(s_1, s_2, t)$, independent of spin and polarization, will be given by complicated integral equations as it would be seen later.

The calculation is performed at the lowest order in the electromagnetic coupling constant and the M_δ must be linear and homogeneous in e_μ : $M_\delta = e_\mu M_\delta^\mu$. By gauge invariance, we have $k_\mu M^\mu = 0$ and we are interested only in the case $e_\mu k_\mu = 0$.

The nucleon spin dependence of M_δ can be written as:

$$M_\delta = \bar{u}_{\sigma_2}(-p_2)\; H_\delta \; u_{\sigma_1}(p_1)$$

where $u_\sigma(p)$ is a free Dirac spinor normalized in a Lorentz invariant way, solution of the Dirac equation.
Summarizing these results, one obtains:

$$T = \sum_\delta A_\delta(s_1, s_2, t)\; e_\mu\, \bar{u}_{\sigma_2}(-p_2)\; H^\mu_\delta \; u_{\sigma_1}(p_1) \tag{4}$$

and we now construct the H_δ from the sixteen irreducible matrices of the Dirac algebra and the energy momenta four vectors. Because of the pseudoscalar character of the pion, the only Dirac matrices which can enter are: $i\gamma_5$, $\gamma_5\gamma_\mu$, $i\gamma_5\gamma_\mu\gamma_\nu$.

With the nucleon spin constraint:

$$(i\gamma\cdot p_1 + M)\, u_{\sigma_1}(p_1) = 0 = \bar{u}_{\sigma_2}(-p_2)(i\gamma\cdot p_2 - M)$$

one can form six invariant forms for H^μ_δ

$$H^\mu_1 = \gamma_5\gamma_\mu \qquad H^\mu_2 = i\gamma_5 P_{1\mu} \qquad H^\mu_3 = i\gamma_5 P_{2\mu}$$

$$H^\mu_4 = \gamma_5(\gamma\cdot k)P_{1\mu} \qquad H^\mu_5 = \gamma_5(\gamma\cdot k)P_{2\mu} \qquad H^\mu_6 = i\gamma_5\gamma_\mu(\gamma\cdot k)$$

The gauge condition $k_\mu M^\mu = 0$ leads to relations between the scalar functions A_δ:

$$A_1 + (p_1 \cdot k) A_4 + (p_2 \cdot k) A_5 = 0$$
$$(p_1 \cdot k) A_2 + (p_2 \cdot k) A_3 = 0 \qquad (5)$$

If we now introduce the CGLN gauge invariant independent forms :

$$\begin{cases} M_A = i \gamma_5 (\gamma \cdot e)(\gamma \cdot k) = e_\mu H_6^\mu & (6) \\ M_B = i\gamma_5 [(e \cdot p_2)(k \cdot p_1) - (e \cdot p_1)(k \cdot p_2)] = (k \cdot p_1) e_\mu H_3^\mu - (k \cdot p_2) e_\mu H_2^\mu \\ M_C = \gamma_5 [(\gamma \cdot e)(p_1 + p_2) \cdot k - (\gamma \cdot k)(p_1 + p_2) \cdot e] = (p_1 + p_2) \cdot k \, e_\mu H_1^\mu - e_\mu (H_4^\mu + H_5^\mu) \\ M_D = \gamma_5 [(\gamma \cdot e)(p_1 - p_2) \cdot k - (\gamma \cdot k)(p_1 - p_2) \cdot e] - 2MM_A = (p_1 - p_2) \cdot k \, e_\mu H_1^\mu - e_\mu (H_4^\mu - H_5^\mu) - 2M e_\mu H_6^\mu \end{cases}$$

we can expand the T amplitude in the following way ;

$$T = \bar{u}_{\sigma_2}(-p_2) \{ A M_A + B M_B + C M_C + D M_D \} u_{\sigma_1}(-p_1) \qquad (7)$$

with the correspondence between the A and the new functions deduced from the gauge condition (5):

$$\begin{cases} A = M(A_4 - A_5) + A_6 \\ B = -\dfrac{A_2}{k \cdot p_1} = \dfrac{A_3}{k \cdot p_2} & (8) \\ C = -\dfrac{A_4 + A_5}{2} \\ D = -\dfrac{A_4 - A_5}{2} \end{cases}$$

4. It can be shown by using general arguments due to Hearn [3] that the set of amplitudes $A_\delta(s_1, s_2, t)$ is free from kinematical singularities. Equations (8) exhibit a singularity for the B amplitude as it was pointed out first by Ball [4] and it is a general fact, due to the gauge condition in the processes involving an odd number of photons. However, it is possible to write, for this B amplitude a subtracted dispersion relation with the correct properties of analyticity.

5. The complete photoproduction amplitude is then described by 12 scalar functions. The T_{fi} matrix is invariant under the exchange of the two nucleons and we deduce, for the scalar functions the following crossing properties: $A^{(+,o)}$, $B^{(+,o)}$, $C^{(-)}$, $D^{(+,o)}$ are even under the transformation $s_1 \longleftrightarrow s_2$, $t \longleftrightarrow t$, whereas $A^{(-)}$, $B^{(-)}$, $C^{(+,o)}$, $D^{(-)}$ are odd.

III. Multipole expansion in the photoproduction channel

1. We use the following notations in c.m. system of channel I

$$k = (\vec{k}, k) \qquad p_1 = (-\vec{k}, E_1) \qquad q = (-\vec{q}, \omega_q) \qquad p_2 = (\vec{q}, -E_2)$$

$$W = k + E_1 = \omega_q + E_2 \qquad \vec{k}.\vec{q} = kq \cos\theta$$

where

$$E_1 = \sqrt{\vec{k}^2 + M^2} = \frac{W^2 + M^2}{2W} \qquad E_2 = \sqrt{\vec{q}^2 + M^2} = \frac{W^2 + M^2 - \mu^2}{2W}$$

$$k = \frac{W^2 - M^2}{2W} \qquad \omega_q = \sqrt{\vec{q}^2 + \mu^2} = \frac{W^2 - M^2 + \mu^2}{2W}$$

The scalar quantities s_1, s_2, t become:

$$\begin{cases} s_1 = W^2 \\ s_2 = M^2 - 2k E_2 - 2kq \cos\theta \\ t = \mu^2 - 2k\omega_q + 2kq \cos\theta \end{cases}$$

The gauge is chosen as: $e_4 = 0 \quad \vec{e}.\vec{k} = 0.$

2. We can reduce the quantities $u(-p_2) M_\mathcal{J} u(p_1)$ - with $\mathcal{J}$ = A, B, C, D - to the form $\chi_f^* N_\mathcal{J} \chi_i$ where χ_i and χ_f are two component Pauli spinors describing the initial and final nucleon spin in the c.m. system and $N_\mathcal{J}$ a 2x2 matrix in the nucleon spin space.

It is convenient to introduce the following independent set of matrices

$$
\left[\begin{aligned}
I_1 &= i\,\vec{\sigma}\cdot\vec{e} \\
I_2 &= \frac{(\vec{\sigma}\cdot\vec{q})(\vec{\sigma},\vec{k},\vec{e})}{kq} \\
I_3 &= i\,\frac{(\vec{\sigma}\cdot\vec{k})(\vec{e}\cdot\vec{q})}{kq} \\
I_4 &= i\,\frac{(\vec{\sigma}\cdot\vec{q})(\vec{e}\cdot\vec{q})}{q^2}
\end{aligned}\right.
\qquad (9)
$$

and to expand the N_δ on the I_α :

$$
\left[\begin{aligned}
N_A &= \frac{\sqrt{(E_1+M)(E_2+M)}}{2M}\left[(W-M)I_1 - \frac{q(W-M)}{E_2+M}I_2\right] \\
N_B &= \frac{\sqrt{(E_1+M)(E_2+M)}}{2M}\left[q(W-M)^2 I_3 - \frac{q^2(W^2-M^2)}{E_2+M}I_4\right] \\
N_C &= \frac{\sqrt{(E_1+M)(E_2+M)}}{2M}\left[(q\cdot k)I_1 + \frac{q(W-M)}{(W+M)(E_2+M)}(q\cdot k)I_2 + q(W-M)I_3 + \frac{q^2(W-M)}{E_2+M}I_4\right] \\
N_D &= \frac{\sqrt{(E_1+M)(E_2+M)}}{2M}\left[\left[(W-M)^2-q\cdot k\right]I_1 + \frac{\left[(W+M)^2-q\cdot k\right]q(W-M)}{(W+M)(E_2+M)}I_2 - q(W-M)I_3 - \frac{q^2(W-M)}{E_2+M}I_4\right]
\end{aligned}\right.
$$

The differential cross-section has then the following form:

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left| \sum_j \chi_f^* \mathcal{F}_j \, I_j \, \chi_i \right|^2 \tag{10}$$

where the scalar functions $\mathcal{F}_j$ are linear combinations of A, B, C, D:

$$\begin{cases} \mathcal{F}_1 = \dfrac{W^2-M^2}{8\pi W}\left(\dfrac{E_2+M}{2W}\right)^{1/2}\left[A+(W-M)D+\dfrac{q\cdot k}{W-M}(C-D)\right] \\ \mathcal{F}_2 = \dfrac{W^2-M^2}{8\pi W}\left(\dfrac{E_2-M}{2W}\right)^{1/2}\left[-A+(W+M)D+\dfrac{q\cdot k}{W+M}(C-D)\right] \\ \mathcal{F}_3 = \dfrac{W^2-M^2}{8\pi W}\, q\left(\dfrac{E_2+M}{2W}\right)^{1/2}\left[(W-M)B+C-D\right] \\ \mathcal{F}_4 = \dfrac{W^2-M^2}{8\pi W}\, q\left(\dfrac{E_2-M}{2W}\right)^{1/2}\left[-(W+M)B+C-D\right] \end{cases} \tag{11}$$

3. The selection rules for the transitions from a multipole ℓ to a π-nucleon states (j, ℓ_π) are given by conservation laws of total angular momentum j and parity ω.

For a given value of ℓ, we have four transitions

$\ell_\pi = \ell \qquad j = \ell + \frac{1}{2} \qquad \omega = (-1)^{\ell+1}$ magnetic $\quad M_\ell^+$

$\ell_\pi = \ell \qquad j = \ell - \frac{1}{2} \qquad \omega = (-1)^{\ell+1}$ magnetic $\quad M_\ell^-$

$$\ell_\pi = \ell+1 \qquad j = \ell + \tfrac{1}{2} \qquad \omega = (-1)^{\ell} \qquad \text{electric} \qquad E^{-}_{\ell+1}$$

$$\ell_\pi = \ell-1 \qquad j = \ell - \tfrac{1}{2} \qquad \omega = (-1)^{\ell} \qquad \text{electric} \qquad E^{+}_{\ell-1}$$

In order to expand the T matrix in multipole amplitudes, we use projection operators corresponding to the four possible states *):

a) magnetic transition $\quad \dfrac{\vec{e}\,\vec{\ell}_k}{\sqrt{\ell(\ell+1)}}$

b) electric transition $\quad \dfrac{(\vec{e},\vec{k},\vec{\ell}_k)}{k\sqrt{\ell(\ell+1)}}$

c) π-N state $\quad j = \ell_\pi + \frac{1}{2} \qquad P_+ = \dfrac{\ell_\pi + 1 + \vec{\sigma}\cdot\vec{\ell}_q}{2\ell_\pi + 1}$

d) π-N state $\quad j = \ell_\pi - \frac{1}{2} \qquad P_- = \dfrac{\ell_\pi - \vec{\sigma}\cdot\vec{\ell}_q}{2\ell_\pi + 1}$

We then obtain:

$$T = \sum_{\ell} \Big\{ \frac{1}{i}(\ell+1+\vec{\sigma}\cdot\vec{\ell}_q)(\vec{e}\cdot\vec{\ell}_k) M_\ell^{+} + \frac{1}{i}(\ell-\vec{\sigma}\cdot\vec{\ell}_q)(\vec{e}\cdot\vec{\ell}_k) M_\ell^{-} - \frac{1}{kq}(\vec{\sigma}\cdot\vec{q})(\ell+1+\vec{\sigma}\cdot\vec{\ell}_q)(\vec{e},\vec{k},\vec{\ell}_k) E^{-}_{\ell+1} + \frac{1}{kq}(\vec{\sigma}\cdot\vec{q})(\ell-\vec{\sigma}\cdot\vec{\ell}_q)(\vec{e},\vec{k},\vec{\ell}_k) E^{+}_{\ell+1} \Big\} P_\ell(\cos\theta) \tag{12}$$

*) See for example R. Stora, Seminar given in the Istituto di Fisica dell'Università, Bologna (1960)

The pseudoscalar operator $i\frac{\vec{\sigma}\cdot\vec{q}}{q}$ introduced for electric transition describes the change of orbital parity corresponding to $\ell_\gamma = \ell \pm 1$.

We now write explicitly the action of the operators $\vec{\ell}_q$ and $\vec{\ell}_k$ on the Legendre polynomials $P_\ell(\cos\theta_{kq})$ and obtain immediately a multipolar expansion for the general amplitudes $\mathcal{F}_\sigma$:

$$\left\{\begin{aligned}
\mathcal{F}_1 &= \sum_\ell \left\{(\ell-1)M^+_{\ell-1} + (\ell+2)M^-_{\ell+1} + E^+_{\ell-1} + E^-_{\ell+1}\right\} P'_\ell(\cos\theta) \\
\mathcal{F}_2 &= \sum_\ell \left\{(\ell+1)M^+_\ell + \ell M^-_\ell\right\} P'_\ell(\cos\theta) \\
\mathcal{F}_3 &= \sum_\ell \left\{-M^+_{\ell-1} + M^-_{\ell+1} + E^+_{\ell-1} + E^-_{\ell+1}\right\} P''_\ell(\cos\theta) \\
\mathcal{F}_4 &= \sum_\ell \left\{M^+_\ell - M^-_\ell - E^+_\ell - E^-_\ell\right\} P''_\ell(\cos\theta)
\end{aligned}\right. \tag{13}$$

4. Relations (13) can be inverted by using orthogonality properties of the projection operators. From (12) one deduces:

$$\left\{\begin{aligned}
M^+_\ell &= \frac{1}{4\pi\,\ell(\ell+1)^2} \sum_{\varepsilon,\sigma} i\int P_\ell(\cos\theta)(\vec{\varepsilon}\cdot\vec{\ell}_k)(\ell+1+\vec{\sigma}\cdot\vec{\ell}_q)\, T\, d\Omega \\
M^-_\ell &= \frac{1}{4\pi\,\ell^2(\ell+1)} \sum_{\varepsilon,\sigma} i\int P_\ell(\cos\theta)(\vec{\varepsilon}\cdot\vec{\ell}_k)(\ell-\vec{\sigma}\cdot\vec{\ell}_q)\, T\, d\Omega \\
E^+_{\ell-1} &= \frac{1}{4\pi\,\ell^2(\ell+1)} \sum_{\varepsilon,\sigma} \int P_\ell(\cos\theta)(\vec{\varepsilon}\cdot\vec{k},\vec{\ell}_k)(\ell-\vec{\sigma}\cdot\vec{\ell}_q)(\vec{\sigma}\cdot\vec{q})\frac{T}{kq}\, d\Omega \\
E^-_{\ell+1} &= \frac{-1}{4\pi\,\ell(\ell+1)^2} \sum_{\varepsilon,\sigma} \int P_\ell(\cos\theta)(\vec{\varepsilon},\vec{k},\vec{\ell}_k)(\ell+1+\vec{\sigma}\cdot\vec{\ell}_q)(\vec{\sigma}\cdot\vec{q})\frac{T}{kq}\, d\Omega
\end{aligned}\right.$$

By writing T in the form: $T = \sum_{\alpha} \mathcal{F}_{\alpha} I_{\alpha}$ one can reduce the inversion problem to an integral over $\cos\theta$:

$$
\left\{
\begin{aligned}
M_{\ell}^{+} &= \frac{1}{2(\ell+1)} \int_{-1}^{+1} \left\{ \mathcal{F}_1 P_{\ell} - \mathcal{F}_2 P_{\ell+1} - \mathcal{F}_3 \frac{P_{\ell-1} - P_{\ell+1}}{2\ell+1} \right\} dx \\
M_{\ell}^{-} &= \frac{1}{2\ell} \int_{-1}^{+1} \left\{ -\mathcal{F}_1 P_{\ell} + \mathcal{F}_2 P_{\ell-1} + \mathcal{F}_3 \frac{P_{\ell-1} - P_{\ell+1}}{2\ell+1} \right\} dx \\
E_{\ell-1}^{+} &= \frac{1}{2\ell} \int_{-1}^{+1} \left\{ \mathcal{F}_1 P_{\ell-1} - \mathcal{F}_2 P_{\ell} + (\ell-1) \mathcal{F}_3 \frac{P_{\ell-2} - P_{\ell}}{2\ell-1} + \mathcal{F}_4 \, \ell \, \frac{P_{\ell-1} - P_{\ell+1}}{2\ell+1} \right\} dx \\
E_{\ell+1}^{-} &= \frac{1}{2(\ell+1)} \int_{-1}^{+1} \left\{ \mathcal{F}_1 P_{\ell+1} - \mathcal{F}_2 P_{\ell} - \mathcal{F}_3 (\ell+2) \frac{P_{\ell} - P_{\ell+2}}{2\ell+3} - \mathcal{F}_4 (\ell+1) \frac{P_{\ell-1} - P_{\ell+1}}{2\ell+1} \right\} dx
\end{aligned}
\right. \tag{14}
$$

5. Reflection properties with respect to W can be established for the $\mathcal{F}_{\alpha}(W)$ amplitudes from equations (11)

$$
\left\{
\begin{aligned}
\mathcal{F}_1(-W) &= \mathcal{F}_2(W) \\
\mathcal{F}_3(-W) &= \mathcal{F}_4(W)
\end{aligned}
\right.
$$

This induces on the multipole amplitudes the following relations:

$$\left\{\begin{aligned} M_\ell^+(-W) &= \frac{1}{\ell+1}\left[(\ell+2)\,M_{\ell+1}^-(W) + E_{\ell+1}^-(W)\right] \\ M_\ell^-(-W) &= \frac{1}{\ell}\left[(\ell-1)\,M_{\ell-1}^+(W) + E_{\ell-1}^+(W)\right] \\ E_{\ell-1}^+(-W) &= \frac{1}{\ell}\left[M_\ell^-(W) - (\ell-1)\,E_\ell^-(W)\right] \\ E_{\ell+1}^-(-W) &= \frac{1}{\ell+1}\left[M_\ell^+(W) - (\ell+2)\,E_\ell^+(W)\right] \end{aligned}\right. \qquad (15)$$

IV. Unitarity condition for the isoscalar amplitude

1. In the photoproduction channel, it is well-known that the phases of the multipole amplitudes are given by the pion-nucleon scattering phase shifts in the corresponding quantum states. For a given eigenstate of isospin I, we simply have :

$$
\begin{aligned}
\mathrm{Im}\, M_\ell^{\pm} &= M_\ell^{\pm}\, e^{-i\delta_\ell^{\pm}} \sin\delta_\ell^{\pm} \\
\mathrm{Im}\, E_\ell^{\pm} &= E_\ell^{\pm}\, e^{-i\delta_\ell^{\pm}} \sin\delta_\ell^{\pm}
\end{aligned}
\tag{16}
$$

where $\delta_\ell^{\pm}$ is the pion-nucleon phase shift ($J = \ell \pm \frac{1}{2}$).

2. We now consider the reaction $\gamma + \pi \longrightarrow N + \bar{N}$. Introducing the centre-of-mass variables

$$k = (\vec{K}, K) \qquad q = (-\vec{K}, \omega) \qquad p_1 = (-\vec{p}, -E) \qquad p_2 = (\vec{p}, -E)$$

where

$$
\begin{aligned}
E &= \sqrt{\vec{p}^{\,2} + M^2} \\
\omega &= \sqrt{K^2 + \mu^2}
\end{aligned}
$$

and setting $e_4 = 0$ $\vec{e} \cdot \vec{K} = 0$ by a suitable choice of gauge we can first reduce the quantities $\bar{u}(-p_2) M_i\, u(p_1)$ to a 2x2 matrix in the nucleon spin space:

$$
\left\{
\begin{aligned}
\bar{u}\, M_A\, u &\Rightarrow i\, \vec{e}\cdot\vec{p}\, \frac{K}{M} + (\vec{\sigma}, \vec{e}, \vec{K}) + \frac{1}{M(E+M)} (\vec{\sigma}\cdot\vec{p})(\vec{p}, \vec{e}, \vec{K}) \\
\bar{u}\, M_B\, u &\Rightarrow i\, \vec{e}\cdot\vec{p}\, \frac{4KE^2}{M} \\
\bar{u}\, M_C\, u &\Rightarrow \frac{2KE}{M} (\vec{\sigma}, \vec{e}, \vec{p}) \\
\bar{u}\, M_D\, u &\Rightarrow -\frac{2E^2}{M} (\vec{\sigma}, \vec{e}, \vec{K}) + \frac{2E}{M(E+M)} (\vec{\sigma}\cdot\vec{p})(\vec{p}, \vec{e}, \vec{K})
\end{aligned}
\right.
$$

The T matrix element is also a matrix in the nucleon spin space:

$$T = i\,\vec{e}\cdot\vec{p}\,\frac{K}{M}(A + tB) + \frac{2KE}{M}(\vec{\sigma},\vec{e},\vec{p})C + (\vec{\sigma},\vec{e},\vec{K})\left(A - \frac{t}{2M}D\right) + \tag{17}$$
$$+ \frac{1}{M(E+M)}(\vec{\sigma}\cdot\vec{p})(\vec{e},\vec{K},\vec{p})(A + 2ED)$$

where the index (0) has been dropped. A, B, C and D are functions of $t = 4E^2$ and $\cos\varphi = \frac{\vec{K}\cdot\vec{p}}{Kp}$.

3. The unitarity condition may be written as:

$$\langle f|T^{*} - T|i\rangle = \frac{1}{i}\sum_{n}(2\pi)^4\,\delta_4\left(\Sigma p_f - \Sigma p_n\right)N_n\,\langle f|T^{*}|n\rangle\langle n|T|i\rangle$$

In the energy region $(2\mu)^2 < t < (2M)^2$, this equation is to be understood as an analytic continuation of the unitarity condition from the region of physical energies. Retaining only the two-pion contributions one can express the imaginary parts of the scalar functions A, B, C, D in terms of the $\gamma + \pi \rightarrow \pi + \pi$ and $\pi + \pi \rightarrow N+\bar{N}$ reaction amplitudes:

$$\langle N\bar{N}|T^{*} - T|\gamma\pi\rangle = \frac{1}{i}\,\frac{2}{(8\pi)^2}\,\frac{p}{E}\int d\Omega_p\,\langle N\bar{N}|T^{*}|\pi\pi\rangle\langle\pi\pi|T|\gamma\pi\rangle$$

We note, however, that the same I=J=1 two-pion intermediate state occurs in the expression of the imaginary part of the isovector nucleon form factors. As the same $\pi\pi \rightarrow N+\bar{N}$ amplitude appears therefore in both processes, we can eliminate this amplitude and re-express the imaginary parts of A, B, C and D in terms of the imaginary parts of the isovector nucleon form factors.

More precisely, the $\gamma \longrightarrow N+\bar{N}$ vertex function may be written in the centre of mass of the $N\bar{N}$ pair:

$$e_\mu \langle N\bar{N} | j_\mu | 0 \rangle = \chi_N^* \Big\{ \frac{E}{M} (\vec{\sigma}\cdot\vec{e}) [G_1^v + 2M G_2^v] + \\ + \frac{1}{M(E+M)} (\vec{\sigma}\cdot\vec{p})(\vec{e}\cdot\vec{p}) [2E\, G_2^v - G_1^v] \Big\} \chi_{\bar{N}} \tag{18}$$

where we have used the same notations as for the $\gamma + \pi \rightarrow N+\bar{N}$ reaction. The invariant functions $G_1^V(t)$ and $G_2^V(t)$ are the ordinary isovector form factors for the nucleon. On the other hand, the $\gamma \rightarrow \pi^\alpha + \pi^\beta$ vertex and the $\gamma + \pi^\alpha \rightarrow \pi^\beta + \pi^\gamma$ reaction amplitudes are given respectively by

$$e_\mu \langle \pi^\alpha \pi^\beta | j_\mu | 0 \rangle = e \quad \vec{e}\cdot\vec{q} \; F_\pi(t) \; \varepsilon_{3\alpha\beta} \tag{19}$$

where the pion form factor $F_\pi(t)$ is normalized as $F_\pi(0) = 1$, and :

$$e_\mu \langle \pi^\beta \pi^\gamma | j_\mu | \pi^\alpha \rangle = \frac{e\Lambda}{\mu^3} \; \frac{\varepsilon_{\alpha\beta\gamma}}{\sqrt{2}} \; \omega_q \; (\vec{e}, \vec{k}, \vec{q}) \; F_\pi(t) \, \phi(t) \tag{20}$$

where $\phi(t)$ is a real smooth function previously considered and Λ a dimensionless coupling constant corresponding to the normalization $\phi(0) = 1$. The formal substitution :

$$\vec{e} \Rightarrow \vec{e} \times \vec{k}$$

allows one to establish a correspondence between equations (17) and (18) and equations (19) and (20) and to write :

$$
\begin{aligned}
\mathrm{Im}\, A &= \frac{1}{4\sqrt{2}} \frac{\Lambda}{\mu^3}\, t\, \phi(t)\, \mathrm{Im}\, G_2^v(t) \\
\mathrm{Im}\, B &= -\frac{1}{4\sqrt{2}} \frac{\Lambda}{\mu^3}\, \phi(t)\, \mathrm{Im}\, G_2^v(t) \\
\mathrm{Im}\, C &= 0 \\
\mathrm{Im}\, D &= -\frac{1}{4\sqrt{2}} \frac{\Lambda}{\mu^3}\, \phi(t)\, \mathrm{Im}\, G_1^v(t)
\end{aligned}
\qquad (21)
$$

V. Mandelstam representation for the isoscalar amplitudes

1. We assume for the scalar functions A, B, C, D a Mandelstam representation of the form:

$$\text{Born terms} + \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} dx \int_{(2\mu)^2}^{\infty} dy \; \frac{\rho(x,y)}{y-t}\left[\frac{1}{x-s_1} + \varepsilon\,\frac{1}{x-s_2}\right] + \tag{22}$$

$$\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{(M+\mu)^2}^{\infty} dx\,dy \,\frac{\sigma(x,y)}{(x-s_1)(y-s_2)}$$

where the Born terms are those calculated by CGLN and where, as a result of crossing, $\varepsilon\,\sigma(x,y) = \sigma(y,x)$ and $\varepsilon = +1$ for A, B, D and $\varepsilon = -1$ for C.

It is easily seen that, owing to the isoscalar nature of the photon current, one may apply the Cini-Fubini arguments and write the representation (22) in one-dimensional form where the corresponding weight functions are directly related to the absorptive parts of the amplitudes in the various channels.

2. As shown by equation (21), furthermore, the absorptive parts of the amplitudes in the $\gamma + \pi \rightarrow N+\bar{N}$ channel are simply functions of t in the approximation where only the I=J=1 two-pion intermediate state is retained. These considerations lead us to write the two-pion contribution to the $T^{(0)}$ amplitude (7) using (21) as follows

$$T_{2\pi} = \frac{1}{4\sqrt{2}}\,\frac{\Lambda}{\mu^3}\left[M_A\,\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}\frac{t'\phi(t')\,\mathrm{Im}\,G_2^V(t')}{t'-t}\,dt' - M_B\,\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\phi(t')\,\mathrm{Im}\,G_2^V(t')}{t'-t}\,dt'\right. \tag{23}$$

$$\left. - M_D\,\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\phi(t')\,\mathrm{Im}\,G_1^V(t')}{t'-t}\,dt'\right]$$

where possible subtractions have been omitted.

The form of the first integral in equation (23) would seem to suggest that a subtraction might be effected in the A amplitude so as to cast this integral into the same form as the second one. If one does this, equation (23) becomes

$$T_{2H} = \frac{1}{4\sqrt{2}} \frac{\Lambda}{\mu^3} \Big[A_0^? M_A + (tM_A - M_B)\frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\phi(t')\, \mathrm{Im}\, G_2^V(t')}{t'-t} dt' - M_D \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\phi(t')\, \mathrm{Im}\, G_1^V(t')}{t'-t} dt' \Big] \tag{24}$$

3. At this point, we make the assumption of a sharp pion-pion resonance. Writing the I=J=1 pion-pion scattering amplitude as previously in the resonant form :

$$\exp[i\delta] \sin\delta = \frac{\gamma q^3}{t_R - t - i\gamma q^3}$$

we deduce :

$$\mathrm{Im}\ e^{i\delta}\sin\delta = \sin^2\delta = \frac{(\gamma q^3)^2}{(t_R - t)^2 + \gamma^2 q^6}$$

If the resonance width, proportional to γ is not too large, we shall make the approximation of replacing $\sin^2\delta$ by $\pi\, \gamma q_R^3\, \delta(t-t_R)$. With this approximation, Bowcock, Cottingham and Lurié [5)] have shown that the imaginary parts of the isovector nucleon form factors are of the form :

$$\begin{cases} \mathrm{Im}\, G_1^V(t) = \frac{e}{2}\pi a\, t_R\, \delta(t-t_R) \\ \mathrm{Im}\, G_2^V(t) = \frac{e g_V}{2M}\pi b\, t_R\, \delta(t-t_R) \end{cases} \tag{25}$$

where g_V is the isovector part of the anomalous gyromagnetic ratio. The constant a and b are related to γ, t_R and to new constants, C_1 and C_2, describing in the same approximation the I=J=1 $\pi+\pi \rightarrow N+\bar{N}$ reaction amplitude

$$a = 2\frac{C_1(t_R-\gamma)}{\gamma\, t_R^{3/2}} \qquad\qquad b = 2\,\frac{M}{g_V}\,\frac{C_2(t_R-\gamma)}{\gamma\, t_R^{3/2}}$$

Insertion of (25) into (24) yields the simple form :

$$T_{2\pi} = \frac{1}{8\sqrt{2}}\,\frac{e\Lambda}{\mu^3}\,\phi(t_R)\,t_R\left[A_0 M_A + \frac{g_V}{M}\, b\,\frac{tM_A - M_B}{t_R - t} - a\,\frac{M_D}{t_R - t}\right] \qquad (26)$$

4. For isoscalar amplitude, the isospin in the photoproduction channel is $I = \frac{1}{2}$. The corresponding π-nucleon phase shift is very small at low energy (except perhaps the S phase shift δ_1). Following CGLN, one may, at low energy, neglect the imaginary part of $A^{(o)}$ $B^{(o)}$ $C^{(o)}$ and $D^{(o)}$ in the photoproduction channels. The $T^{(o)}$ amplitude is then simply given by:

$$T^{(o)} = T^{(o)}_{BORN} + T^{(o)}_{2\pi} \qquad (27)$$

At higher energies, of course (above the I=J=3/2 P resonance for example) such an approximation is no longer valid.

In order to check the validity of formula (27), we can calculate the modifications due to the $I=J=\frac{1}{2}$ S π-nucleon phase shift δ_1. The only partial amplitude affected is the electric dipole E_0^+ and we calculate this amplitude by resolving a Mushkelishvili-Omnès equation where the inhomogeneous term is the electric dipole part of the Born and bipion terms:

$$E(W) = E_B(W) + E_{2\pi}(W) + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{E(W')\, e^{-i\delta_1} \sin\delta_1}{W'^2 - W^2 - i\varepsilon}\, dW'^2 + \frac{1}{2\pi}\int_{-1}^{+1} d\cos\theta \int_{(M+\mu)^2}^{\infty} \frac{E(W')\, e^{-i\delta_1} \sin\delta}{W'^2 - s_2}\, dW'^2$$

where the indices $(+)$ and (0) are dropped. The Born term $E_B(W)$ and the two-pion term $E_{2\pi}(W)$ can be easily calculated by using formula (14) from the total expression given respectively in ref. [1] and equation (26).

One can neglect the crossed term because the π-N scattering amplitude is small and the crossing corresponds only to a correction of the present one. With this assumption, we obtain an explicit expression for $E_0^+(W)$ in terms of the singularities contained in the bipion and Born terms:

$$E(W) = E_B(W)\left[1 + \Delta_B(W)\right] + E_{2\pi}(W)\left[1 + \Delta_{2\pi}(W)\right]$$

where the corrections Δ_B and $\Delta_{2\pi}$ are given by:

$$\Delta_B(W) = \frac{1}{E_B(W)} \exp\left[\rho_1(W) + i\delta_1(W)\right] \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{\exp\left[-\rho_1(W')\right] \sin\delta_1(W')\, E_B(W')}{W'^2 - W^2 - i\varepsilon}\, dW'^2$$

$$\Delta_{2\pi}(W) = \frac{1}{E_{2\pi}(W)} \exp\left[\rho_1(W) + i\delta_1(W)\right] \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{\exp\left[-\rho_1(W')\right] \sin\delta_1(W')\, E_{2\pi}(W')}{W'^2 - W^2 - i\varepsilon}\, dW'^2$$

Explicit calculations performed by Warburton [6] show that the corrections are small up to 300 MeV.

VI. Bipion model approach for isoscalar amplitudes

1. The bipion model was introduced in the first part of these lectures where in particular the boson propagator and the photon-pion-bipion interactions are studied.

In the photoproduction problem, we are interested in the diagram indicated in Fig. 2

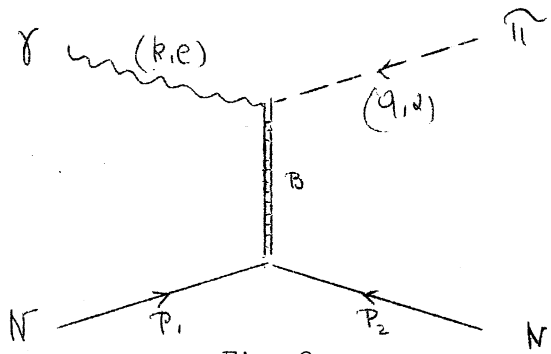

Fig. 2

and we must know the nucleon-nucleon-bipion coupling to compute this diagram.

2. By using arguments very similar to those of electrodynamics one can introduce two forms of coupling:

$$H_{\underline{\mathrm{III}}} = i\,\mathcal{C}_1 \bar{\Psi}_N(x)\,\tau_\alpha \gamma_\mu \psi_N(x)\, B^\alpha_\mu(x) + i\,\mathcal{C}_2 \bar{\Psi}_N(x)\,\tau_\alpha \sigma_{\mu\nu} \psi_N(x)\,\partial_\nu B^\alpha_\mu(x) \tag{28}$$

where $\psi_N(x)$ is the spinor nucleon field and $\mathcal{C}_1$, $\mathcal{C}_2$ two coupling constants analogous to the electric charge and the magnetic moment. The matrix element of the bipion current between two nucleons can easily be computed in first Born approximation:

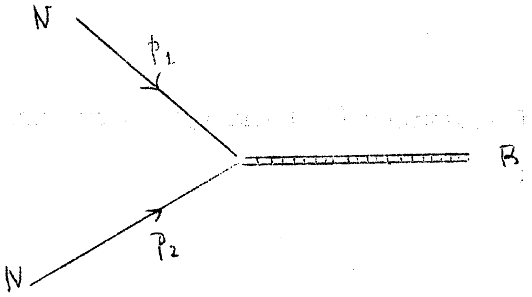

Fig. 3

With the notations of Fig. 3 we obtain immediately:

$$V_\mu^\alpha = i\tau_\alpha\, \bar{u}(-p_2)\left[\mathcal{C}_1\gamma_\mu - i\mathcal{C}_2\sigma_{\mu\nu}(p_1+p_2)_\nu\right]u(p_1) \tag{29}$$

Making use of the Dirac equation formula (29) becomes:

$$V_\mu^\alpha = i\tau_\alpha\, \bar{u}(-p_2)\left[(\mathcal{C}_1+2M\mathcal{C}_2)\gamma_\mu + i\mathcal{C}_2(p_1-p_2)_\mu\right]u(p_1) \tag{30}$$

3. The matrix element for the bipion contribution to photoproduction is then given by

$$T_b^{(0)} = \frac{ie\Lambda_2}{2\mu}\,\frac{1}{t_R-t}\,\varepsilon_{\lambda\mu\nu\rho}\,e_\lambda k_\mu q_\nu\,\bar{u}(-p_2)\left[(\mathcal{C}_1+2M\mathcal{C}_2)\gamma_\mu + i\mathcal{C}_2(p_1-p_2)_\mu\right]u(p_1)$$

After some algebraic manupulations to introduce the CGLN invariant form, the final result can be written as

$$T^{(0)}_{bipion} = \frac{\lambda_b}{t_R-t}\left[\mathcal{C}_1 M_D - \mathcal{C}_2(tM_A - M_B)\right] \tag{31}$$

Inspection of equations (26) and (31) shows that they are of the same form if one takes *):

$$\frac{\mathcal{C}_1}{\mathcal{C}_2} = \frac{C_1}{C_2}$$

The constant A_0, which enters in formula (26) can be interpreted as high energy contributions corresponding to other intermediate states: this parameter cannot appear in formula (31) where only one diagram is retained. Formula (31) has independently been given by de Tollis, Ferrari, Munczek [7]. They use the simple relation

$$\frac{\mathcal{C}_2}{\mathcal{C}_1} = \frac{g_V}{M}$$

which corresponds for the nucleon form factors to $F_1^V = F_2^V$ (normalized to 1 for $t=0$) and introduce an arbitrary constant A_0 in numerical computations.

VII. Isovector amplitudes

1. If one now considers the Mandelstam representation for the isovector amplitudes $A^{(\pm)}$, $B^{(\pm)}$, $C^{(\pm)}$ and $D^{(\pm)}$, one finds that owing to G invariance one cannot directly carry through the Cini-Fubini reduction. Indeed, one sees that for the graph drawn in Fig. 4a the variables s_1 and t may both reach their lower limits of integration in the Mandelstam integrals. The same situation holds for the variables s_1 and s_2 if one considers the graph drawn in Fig. 4b.

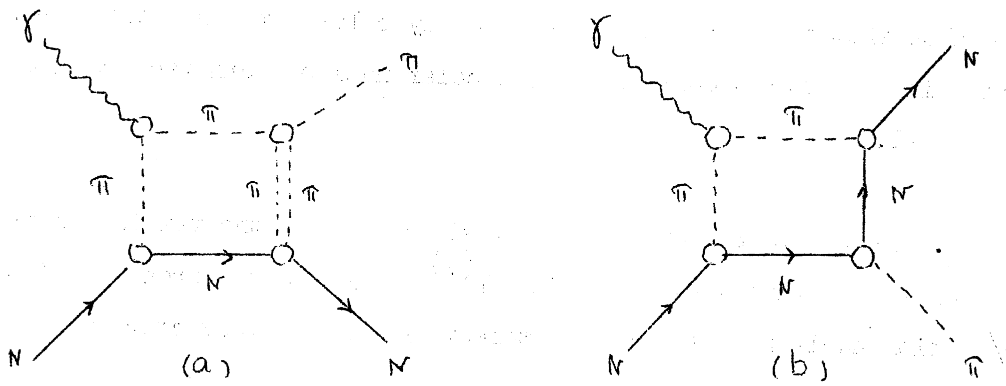

Fig. 4

2. Under those circumstances we shall consider a very naïve model in which the three pion intermediate state is replaced by a "tripion" particle. It has been shown by Chew [8] that in a I=0, J=1 three-pion state all three pairs of pions may be in an I=J=1 state. Conversely, if one requires that all three pairs of pions be in an I=J=1 state, it can be shown that then total isospin and total angular momentum will be I=0 and J=1.

The existence of a strong pion-pion resonance leads us to retain only the three-pion state characterized by these quantum numbers. As a result, the three-pion state will affect only the amplitudes $A^{(+)}$, $B^{(+)}$, $C^{(+)}$ and $D^{(+)}$.

3. As we are again dealing with a vector boson, we may write in complete analogy with Section VI

$$T^{(+)}_{\text{tripion}} = \frac{\lambda_t}{t_s - t}\left[\mathcal{C}_1' M_D - \mathcal{C}_2'(t M_A - M_B)\right] \tag{32}$$

where λ_t is related to the coupling constant corresponding to the photon-pion-tripion vertex and where $\mathcal{C}_1'$ and $\mathcal{C}_2'$ are the coupling constants of the tripion with the nucleon; t_s is the square of the tripion mass. It is easy to show that these constants are closely related to the parameters appearing in the calculation of the isoscalar nucleon form factors with a similar model.

4. It is clear that one cannot simply add on the tripion contribution to the solution obtained by CGLN for $T^{(+)}$ owing to the presence of the $I=J=3/2$ pion nucleon final state interactions in the isovector amplitudes.

If we now make the tripion approxiamtion, the cut in the complex t plane is replaced by a pole $t=t_s$ so that only the cuts in s_1 and s_2 remain. The amplitude $T^{(+)}$ may then be written:

$$T^{(+)} = T^{(+)}_{\text{BORN}} + T^{(+)}_{\text{TRIPION}} + \sum_j M_j \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}\int_{(M+\mu)^2}^{\infty} \frac{\rho_j(x,y)}{(x-s_1)(y-s_2)}\,dx\,dy$$

Taking a fixed value of t we can reduce the double integral to a sum of two one-dimensional integrals :

$$A^{(+)}(s_1,t) = A^{(+)}_{\text{BORN}}(s_1,t) + A^{(+)}_{\text{TRIPION}}(t) + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dx\, \operatorname{Im} A^{(+)}(x,t)\left\{\frac{1}{x-s_1-i\varepsilon} + \frac{1}{x+s_1+t-2M^2-\mu^2}\right\}$$

Similar representations satisfying crossing symmetry hold for $B^{(+)}$, $C^{(+)}$, $C^{(+)}$, $D^{(+)}$.

It may easily be seen that the amplitudes $A^{(-)}$, $B^{(-)}$, $C^{(-)}$ and $D^{(-)}$ satisfy representations of the same form with the exception that the tripion terms will be absent; these are simply the equations written down by CGLN.

5. The unitarity condition in the photoproduction channel has the form given in equation (16) for an eigenstate of isospin I. Representation (33) leads to a system of coupled Mushkelishvili-Omnès equations in the multipole amplitudes, involving as a result of crossing both the (+) and (-) amplitudes. By well-known techniques as explained in the first part this system may be transformed into a system of Fredholm type integral equations.

VIII. Comparison with experiments

1. The amplitudes for photoproduction of mesons with definite charge are linear combinations of $T^{(+)}$, $T^{(-)}$, $T^{(o)}$ given by equation (2)

$$\gamma + p \Rightarrow n + \pi^{+} \qquad \sqrt{2}\left(T^{(-)} - T^{(o)}\right)$$

$$\gamma + p \Rightarrow p + \pi^{o} \qquad T^{(+)} + T^{(o)}$$

$$\gamma + n \Rightarrow p + \pi^{-} \qquad -\sqrt{2}\left(T^{(-)} - T^{(o)}\right)$$

$$\gamma + n \Rightarrow n + \pi^{o} \qquad T^{(+)} - T^{(o)}$$

The amplitudes for photoproduction of charged pions are mainly affected by the bipion term and the CGLN terms; the contribution from the tripion, occuring by crossing will be less important. For π^{o} photoproduction, however, we have bipion, tripion contributions as well as CGLN terms.

The energy dependence of the experimentally measured ratio

$$\rho = \frac{d\sigma^{-}}{d\sigma^{+}} = \frac{|T^{(-)} - T^{(o)}|^2}{|T^{(-)} + T^{(o)}|^2}$$

at various angles is in disagreement with the theoretical predictions based on CGLN calculations. It should be pointed out that, since the $T^{(-)}$ amplitude is strongly affected by the 3/2 3/2 resonance in the range of validity of the present theory, whereas $T^{(o)}$ is not, the ratio ρ will be relatively insensitive to small corrections on $T^{(-)}$ amplitudes as for example tripion contributions. On the other hand, ρ depends strongly on the two constants A_o and Λ introduced in formula (26) and these circumstances are rather fortunate insofar as the comparison of this aspect of our theory with experiment is concerned. Preliminary calculations were performed by Ferrari [9]. It appears as very difficult to find a unique set of values for A_o and Λ that fits the experimental data at various angles.

3. Another way of comparison with experiments is the study of the π^+ photoproduction. No definite answer can be given for the actual problem.

The same situation holds for the π^o photoproduction where it is very difficult to analyze experimental results because of the presence of various unknown contributions.

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