On the Anticyclotomic Iwasawa's Main Conjecture for Hilbert Modular forms

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Abstract

This paper generalizes to the totally real case the previous work of Bertolini and Darmon [BD3] on Anticyclotomic Iwasawa's Main Conjecture for modular forms over $\mathbb{Q}$ with coefficients in $\mathbb{Z}_p$. It contains the definition of anticyclotomic p-adic L-functions attached to Hilbert modular forms and the generalization of the main result of [BD3] to this context. The main feature of the totally real case is the possibility of defining several p-adic L functions (each in several variables) corresponding to different divisors $\mathfrak{p}$ of p: the paper also explores the relations between these different p-adic L-functions.

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1 Introduction

Let $F/\mathbb{Q}$ be totally real of degree $d := [F : \mathbb{Q}]$ and let K/F be quadratic imaginary. Fix a rational prime p and assume that p does not ramify in $K/\mathbb{Q}$. Let $f_0 \in S_2(\mathfrak{n}_0)$ be a Hilbert modular form for the $\Gamma_0(\mathfrak{n}_0)$ level structure (where $\mathfrak{n}_0 \subseteq \mathcal{O}_F$ is an integral ideal of the ring of integers of F), with trivial central character and parallel weight 2. Suppose that f_0 verifies the following condition:

Assumption 1. *1. The ideal $\mathfrak{n}_0$ is prime to the discriminant $\mathfrak{d}_{K/\mathbb{Q}}$ of $K/\mathbb{Q}$.*

2. f_0 is an eigenform for the Hecke algebra $\mathbb{T}_{\mathfrak{n}_0}$ acting faithfully on $S_2(\mathfrak{n}_0)$. Denote by $\mathcal{O}_{f_0}$ the ring of integers of the finite extension of $\mathbb{Q}$ containing the eigenvalues of the action of the Hecke operators on f_0 and, for any prime $\mathfrak{q} \subseteq \mathcal{O}_F$, denote by $a_{\mathfrak{q}}$ the eigenvalue of the Hecke operator $\mathrm{T}_{\mathfrak{q}}$ for $\mathfrak{q} \nmid \mathfrak{n}_0$ or $\mathrm{U}_{\mathfrak{q}}$ for $\mathfrak{q} \mid \mathfrak{n}_0$;

3. There exists a square-free divisor $\mathfrak{n}^-$ of $\mathfrak{n}_0$ such that:

(a) $\mathfrak{n}^-$ is prime to p;

(b) f_0 arises from a newform of level divisible by $\mathfrak{n}^-$;

(c) The number prime ideals $\mathfrak{q} \subseteq \mathcal{O}_F$ dividing $\mathfrak{n}^-$ and $d = [F : \mathbb{Q}]$ have the same parity;

(d) Any prime ideal $\mathfrak{q} \subseteq \mathcal{O}_F$ dividing $\mathfrak{n}^-$ is inert in K/F.

4. There is a prime ideal $\pi \subseteq \mathcal{O}_{f_0}$ dividing p such that f_0 is π-ordinary at p, that is, for any prime $\wp \subseteq \mathcal{O}_F$ dividing p, there exists a unit root $\alpha_\wp$ of $X^2 - a_\wp X + |\wp|$, where $|\wp|$ is the norm of $\wp$.

Fix an ideal (prime or not) $\mathfrak{p} \subseteq \mathcal{O}_F$ which divides p. To such a modular form f_0 is possible to associate a $\mathfrak{p}$-stabilized modular form, that is, a modular form $f \in S_2(\mathfrak{n})$ where $\mathfrak{n} := \mathfrak{n}_0 \prod_{\wp|\mathfrak{p},\wp\nmid\mathfrak{n}_0} \wp$, such that $\mathrm{U}_\wp(f) = \alpha_\wp f$ for $\wp \mid \mathfrak{p}$ and, of course, $\mathrm{T}_\mathfrak{q} f = a_\mathfrak{q} f$ for $\mathfrak{q} \nmid \mathfrak{n}$, $\mathrm{U}_\mathfrak{q} f = a_\mathfrak{q} f$ for $\mathfrak{q} \mid (\mathfrak{n}/\mathfrak{p})$. Using the notion of Gross points it is then possible to associate to f a p-adic L-function $L_\mathfrak{p}(f, K)$ relative to the prime $\mathfrak{p}$. This is an element of $\Lambda_{\mathfrak{p}^\infty} := \mathcal{O}_{f,\pi}[\![G_{\mathfrak{p}^\infty}]\!]$, where $\mathcal{O}_{f,\pi}$ is the completion of $\mathcal{O}_{f_0}$ at π (note that the eigenvalues of the action of $\mathbb{T}_\mathfrak{n}$ on f are contained in this ring because $\alpha_\wp$ are unit roots) and $G_{\mathfrak{p}^\infty} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \simeq \mathbb{Z}_p^{\deg(\mathfrak{p})}$ is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension associated to $\mathfrak{p}$ (as defined in Section 2.1).

On the other hand, there is a notion of Selmer group attached to f. Denote by $\rho_f = \rho_{f,\pi} : G_F \to \mathrm{GL}_2(\mathcal{O}_{f,\pi})$ the π-adic Galois representation attached to f and by T_f the associated G_F-module. Let $V_f := T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $A_{f,1} := (V_f/T_f)[\pi]$. To define the Selmer group it is convenient to assume that the modular form f satisfies the following condition:

Assumption 2. *Let $G_{F_\mathfrak{q}} \subseteq G_F$ be a decomposition group at $\mathfrak{q}$. For any prime $\mathfrak{q} \subseteq \mathcal{O}_F$ dividing $\mathfrak{n}$ exactly and not dividing p, there is an exact sequence of $G_{F_\mathfrak{q}}$-modules: $0 \to A_{f,1}^{(\mathfrak{q})} \to A_{f,1} \to A_{f,1}^{(1)} \to 0$ such that $G_{F_\mathfrak{q}}$ acts on the one-dimensional $\mathcal{O}_{f,\pi}/(\pi)$-vector space $A_{f,1}^{(\mathfrak{q})}$ as multiplication by ϵ or $-\epsilon$, where $\epsilon : G_F \to \mathbb{Z}_p^\times$ is the cyclotomic character.*

The Selmer group is then defined in the usual way by requiring suitable local conditions to global cohomology classes. This lead to a notion of a compact Selmer group $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty}) \subseteq H^1(K_{\mathfrak{p}^\infty}, T_{f,\infty})$, which is naturally a module over the Iwasawa algebra $\Lambda_{\mathfrak{p}^\infty}$. The theory of $\Lambda_{\mathfrak{p}^\infty}$-modules implies that the Pontryagin dual $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})^\vee$ of this Selmer group admits a characteristic power series $\mathrm{Char}_{\mathfrak{p}^\infty}(f, K) \in \Lambda_{\mathfrak{p}^\infty}$.

Anticyclotomic Iwasawa's Main Conjecture. *The equality $L_p(f, k) = u\mathrm{Char}_\mathfrak{p}(f, K)$ holds in $\Lambda_{\mathfrak{p}^\infty}$, where $u \in \Lambda_{\mathfrak{p}^\infty}^\times$ is a unit.*

To state the main result of this paper, suppose that the following conditions on f are verified:

Assumption 3. *1. ρ_f is is residually irreducible.*

2. Let $k := \mathcal{O}_{f,\pi}/(\pi)$ and define $\mathfrak{m}_{f,\pi}$ to be kernel of the natural morphism $\mathbb{T}_\mathfrak{n} \to k$ associated to f. The completion $\mathbb{T}_f$ of $\mathbb{T}_\mathfrak{n}$ at $\mathfrak{m}_{f,\pi}$ is isomorphic to $\mathcal{O}_{f,\pi}$. If this condition holds, say that f is π-isolated.

Theorem. *Under the above Assumptions 1, 2 and 3, the characteristic power series $\mathrm{Char}_\mathfrak{p}(f, K)$ divides the p-adic L-function $L_\mathfrak{p}(f, K)$.*

Remark. While Assumptions 1 and 3 are fundamental for the arguments, Assumption 2 could, in principle, be replaced by any Selmer-type condition at the primes dividing $\mathfrak{n}$ and not dividing p.

The proof of this result is a generalization [BD3], where the case of $F = \mathbb{Q}$ and $\mathcal{O}_{f,\pi} = \mathbb{Z}_p$ is considered. In Section 4 the main steps of the proof are recalled and the necessary adaptations are performed.

The main new feature of the totally real case is that it is possible to define a p-adic L-function for all divisors $\mathfrak{p} \mid p$. In particular, suppose that $\mathfrak{p}_1 \mid \mathfrak{p}_2 \mid p$, with $\mathfrak{p}_1 \neq \mathfrak{p}_2$ ideals of $\mathcal{O}_F$. Then there are two p-adic L-functions $L_{\mathfrak{p}_1}(f, K)$ and $L_{\mathfrak{p}_2}(f, K)$ and a natural restriction map $\lambda_{\mathfrak{p}_2/\mathfrak{p}_1} : \Lambda_{\mathfrak{p}_2^\infty} \to \Lambda_{\mathfrak{p}_1^\infty}$. It is then possible to investigate the relationship between $L_{\mathfrak{p}_1}(f, K)$ and $\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(L_{\mathfrak{p}_2}(f, K))$. The result is that they are equal up to an explicit factor which depends on the eigenvalues $\alpha_{\mathfrak{p}_i} = \prod_{\wp \mid \mathfrak{p}_i} \alpha_\wp$ for $i = 1, 2$ and on the behavior of the primes $\wp$ dividing $\mathfrak{p}_2/\mathfrak{p}_1$ in the extension K/F (see Corollary 2.7). In particular, if some of these primes are split in K/F then it may be possible that this factor relating $L_{\mathfrak{p}_1}(f, K)$ and $\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(L_{\mathfrak{p}_2}(f, K))$ is zero (see Corollary 2.7 and the Example after it). It is then conjectured that the same relation holds between the characteristic power series $\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(\mathrm{Char}_{\mathfrak{p}_2}(f, K))$ and $\mathrm{Char}_{\mathfrak{p}_1}(f, K)$.

Notations. The following notations will be used throughout the paper:

- F is a fixed totally real number field, with ring of integers $\mathcal{O}_F$, and p a fixed prime ideal of $\mathbb{Z}$. The letters $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}'$ and $\mathfrak{p}_*$ will denote integral ideals, *not necessarily prime*, of $\mathcal{O}_F$ which divide p, while the letter $\wp$ will denote a *prime* ideal of $\mathcal{O}_F$ which divides p.

- For E a number field, denote by $\mathcal{O}_E$ its ring of integers; for any place v of E denote by E_v the completion of E at v and, if v is finite, denote by $\mathcal{O}_{E,v}$ the completion of $\mathcal{O}$ at v. The letter $\mathfrak{q}$ will denote prime ideals in $\mathcal{O}_E$. If $\mathfrak{q}$ s such an ideal, denote by $|\mathfrak{q}|$ its norm. If D is a $\mathcal{O}_E$-algebra and $\mathfrak{q} \subseteq \mathcal{O}_E$ is a prime ideal, define $D_\mathfrak{q} := D \otimes_{\mathcal{O}_E} \mathcal{O}_{E,\mathfrak{q}}$, while, if $v \mid \infty$, define $D_v := D \otimes_E E_v$.

- For any field E, let $G_E = \mathrm{Gal}(\bar{E}/E)$ be the absolute Galois group, where $\bar{E}$ is an algebraic closure of E. For any G_E-module M, denote by $H^r(E, M)$ the continuous cohomology groups $H^r(G_E, M)$. For any extension E'/E, denote by $H^r(E'/E, M)$ the continuous cohomology groups $H^r(\mathrm{Gal}(E'/E), M)$. If E is a number field, $\mathfrak{q} \subseteq \mathcal{O}_F$ a prime ideal and E'/E an extension, define $H^r(E'_\mathfrak{q}, M) := \oplus_{\mathfrak{q}' \mid \mathfrak{q}} H^r(E'_{\mathfrak{q}'}, M)$, where the direct sum is taken over all prime ideals $\mathfrak{q}' \subseteq \mathcal{O}_{E'}$ dividing $\mathfrak{q}$.

- For any $\mathbb{Z}$-algebra E, denote by $\widehat{E} := E \otimes_{\mathbb{Z}} \prod_q \mathbb{Z}_q$ the profinite completion of E, where q ranges over the set of prime ideals of $\mathbb{Z}$.

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2 p-adic L-functions

2.1 Anticyclotomic $\mathbb{Z}_p$-extensions

Let $F/\mathbb{Q}$ be totally real of degree d over $\mathbb{Q}$ and K/F quadratic imaginary. Fix a rational prime p, prime to the discriminant $\mathfrak{d}_{K/\mathbb{Q}}$ of K over $\mathbb{Q}$. For any integral ideal $\mathfrak{f} \subseteq \mathcal{O}_F$, let $\mathcal{O}_\mathfrak{f} := \mathcal{O}_F + \mathfrak{f}\mathcal{O}_K$ be the order of conductor $\mathfrak{f}$ in K and let $\mathrm{RCF}(\mathfrak{f})/K$ be the ring class field of K of conductor $\mathfrak{f}$, that is, the Galois extension so that $\mathrm{Gal}(\mathrm{RCF}(\mathfrak{f})/K) \simeq \mathrm{Pic}(\mathcal{O}_\mathfrak{f}) \simeq \widehat{K}^\times/(\widehat{\mathcal{O}}_\mathfrak{f}^\times K^\times)$, where $\mathrm{Pic}(\mathcal{O}_\mathfrak{f})$ is the Picard group of $\mathcal{O}_\mathfrak{f}$. By the formula of Dedekind:

$$h_\mathfrak{f} := \#\mathrm{Pic}(\mathcal{O}_\mathfrak{f}) = \frac{h_{(1)}|\mathfrak{f}| \prod_{\mathfrak{q}|\mathfrak{f}} \left(1 - \left(\frac{K}{\mathfrak{q}}\right) |\mathfrak{q}|^{-1}\right)}{[\mathcal{O}_K^\times : \mathcal{O}_\mathfrak{f}^\times]}, \tag{1}$$

where $\left(\frac{K}{\mathfrak{q}}\right) = 1$ (respectively, $-1, 0$) if $\mathfrak{q}$ is split (respectively, inert, ramified) in K/F. For any m, any ideal (prime or not) $\mathfrak{p} \subseteq \mathcal{O}_F$ dividing p and any ideal $\mathfrak{c} \subseteq \mathcal{O}_F$ prime to $\mathfrak{p}$, define via class field theory the extension $\tilde{K}_{\mathfrak{c}\mathfrak{p}^m}/K$ contained in $\mathrm{RCF}(\mathfrak{c}\mathfrak{p}^m)$ by requiring that

$$\mathrm{Gal}(\tilde{K}_{\mathfrak{c}\mathfrak{p}^m}/K) \simeq \widehat{K}^\times/(\widehat{\mathcal{O}}_{\mathfrak{c}\mathfrak{p}^m}^\times \widehat{\mathcal{O}}_F^\times K^\times) \simeq \mathrm{Pic}(\mathcal{O}_{\mathfrak{c}\mathfrak{p}^m})/\mathrm{Pic}(\mathcal{O}_F),$$

where $\mathrm{Pic}(\mathcal{O}_F) \simeq \widehat{F}^\times/(\widehat{\mathcal{O}}_F^\times F^\times)$ is the Picard group of $\mathcal{O}_F$. Note that $\tilde{K}_{\mathfrak{c}\mathfrak{p}^m} \subseteq \tilde{K}_{\mathfrak{c}p^m}$ for any $\mathfrak{p} \mid p$ and that $\tilde{K}_{\mathfrak{c}\mathfrak{p}^m}$ is unramified outside the places dividing $\mathfrak{c}\mathfrak{p}$. Define $\tilde{K}_{\mathfrak{c}\mathfrak{p}^\infty} := \lim_{\to m} \tilde{K}_{\mathfrak{c}\mathfrak{p}^m}$. Then:

$$\tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty} := \mathrm{Gal}(K_{\mathfrak{c}\mathfrak{p}^\infty}/K) \simeq \widehat{K}^\times/(\prod_{\mathfrak{q}\nmid\mathfrak{p}} \mathcal{O}_{K,\mathfrak{q}}^\times \widehat{F}^\times K^\times). \tag{2}$$

There is a non-canonical isomorphism

$$\mathrm{Gal}(\tilde{K}_{\mathfrak{c}\mathfrak{p}^\infty}/K) \simeq \mathbb{Z}_p^{\deg(\mathfrak{p})} \times \Delta_{\mathfrak{c}\mathfrak{p}^\infty},$$

where $\Delta_{\mathfrak{c}\mathfrak{p}^\infty}$ is the finite torsion subgroups of $\mathrm{Gal}(\tilde{K}_{\mathfrak{c}\mathfrak{p}^\infty}/K)$ and, if $\wp$ is prime, $\deg(\wp) := [F_\wp : \mathbb{Q}_p]$, while if $\mathfrak{p} = \prod \wp_1 \ldots \wp_s$ with $\wp_j$, $j = 1, \ldots s \leq d$ (different) primes, $\deg(\mathfrak{p}) := \sum_{j=1}^s \deg(\wp_j)$ (so that, since p is unramified in K/F, $p^{\deg(\mathfrak{p})} = |\mathfrak{p}|$). In particular, note that $\mathrm{Gal}(\tilde{K}_{\mathfrak{c}p^\infty}/K) \simeq \mathbb{Z}_p^d \times \Delta_{\mathfrak{c}p^\infty}$.

Definition 2.1. *The* $\mathfrak{p}$-anticyclotomic $\mathbb{Z}_p$-extension $K_{\mathfrak{p}^\infty}/K$ *is defined to be the subfield of* $\tilde{K}_{\mathfrak{c}\mathfrak{p}^\infty}$ *so that*

$$G_{\mathfrak{p}^\infty} := \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \simeq \mathbb{Z}_p^{\deg(\mathfrak{p})}.$$

The extension $K_{\mathfrak{p}^\infty}/K$ does not depend on the choice of $\mathfrak{c}$, is Galois over F and the quotient $\mathrm{Gal}(K/F)$ acts by conjugation on the normal subgroup $\mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)$ by the formula $\sigma \mapsto \tau\sigma\tau = \sigma^{-1}$, where τ is the choice of a complex conjugation raising the non trivial automorphism of $\mathrm{Gal}(K/F)$. For any integer $m \geq 1$, define the extension $K_{\mathfrak{p}^m}/K$ by requiring that $G_{\mathfrak{p}^m} := \mathrm{Gal}(K_{\mathfrak{p}^m}/K) \simeq (\mathbb{Z}/p^m\mathbb{Z})^{\deg(\mathfrak{p})}$.

2.2 Modular forms

For any integral ideal $\mathfrak{r}$, denote by $S_2(\mathfrak{r})$ the $\mathbb{C}$-vector space of Hilbert modular forms of parallel weight 2 and trivial central character with respect to the $\Gamma_0(\mathfrak{r})$-level structure and by $\mathbb{T}_{\mathfrak{r}}$ the Hecke algebra acting faithfully on $S_2(\mathfrak{r})$ (see [Zh1, Section 3.1] for precise definitions). For any quaternion algebra D/F of discriminant $\mathfrak{d}$ which is ramified at all archimedean places, any ideal $\mathfrak{s} \subseteq \mathcal{O}_F$ prime to $\mathfrak{d}$ and any ring C, denote by $S_2^D(\mathfrak{s}, C)$ the C-module of functions:

$$\widehat{O}^\times \widehat{F}^\times \backslash \widehat{D}^\times / D^\times \to C,$$

where $O \subseteq D$ is an Eichler order of level $\mathfrak{s}$. There is an action of Hecke algebra $\mathbb{T}_{\mathfrak{s}\mathfrak{d}}$ on $S_2^D(\mathfrak{s}, C)$ defined as in [Sh] via double cosets. If $\mathfrak{d}$ is square-free, the Jacquet-Langlands correspondence [JL] yields a $\mathbb{T}_{\mathfrak{s}\mathfrak{d}}$-equivariant isomorphism between $S_2^D(\mathfrak{s}, C)$ and the C-module of forms in $S_2(\mathfrak{s}\mathfrak{d})$ which are new at $\mathfrak{d}$ and whose Fourier coefficients belong (after a suitable normalization) to C.

Let $f_0 \in S_2(\mathfrak{n}_0)$ be a Hilbert modular form. As in the Introduction fix an ideal (prime or not) $\mathfrak{p} \subseteq \mathcal{O}_F$ dividing p and assume that f_0 satisfies the conditions in Assumption 1 (where K is the field in Section 2.1). From now on use the same notations as in Assumption 1. Define as in the Introduction $\mathfrak{n} := \mathfrak{n}_0 \prod_{\wp|\mathfrak{p}, \mathfrak{p}\nmid\mathfrak{n}_0} \wp$ and $\mathfrak{n}^+ := \mathfrak{n}/\mathfrak{n}^-$.

Let B/F be the quaternion algebra of discriminant $\mathfrak{n}^-$ which is ramified at all archimedean places. Fix an Eichler order $R \subseteq B$ of level $\mathfrak{n}^+$. Since f_0 is π-ordinary at $\mathfrak{p}$ by Assumption 1, the Jacquet-Langlands correspondence can be used as in [BD3, Propositions 1.3, 1.4] to show that there exists an unique (up to multiplication by a number in $\mathcal{O}_{f,\pi}^\times$) modular form $f = f_{\mathfrak{p}} \in S_2^B(\mathfrak{n}^+, \mathcal{O}_{f,\pi})$, where $\mathcal{O}_{f,\pi} := \mathcal{O}_{f_0,\pi}$, such that:

- $f \notin \pi S_2^B(\mathfrak{n}^+, \mathcal{O}_{f,\pi})$;
- $\mathrm{T}_{\mathfrak{q}} f = a_{\mathfrak{q}} f$ for all prime ideals $\mathfrak{q} \subseteq \mathcal{O}_F$ with $\mathfrak{q} \nmid \mathfrak{n}$;

- $\mathrm{U}_{\mathfrak{q}} f = a_{\mathfrak{q}} f$ for all prime ideals $\mathfrak{q} \subseteq \mathcal{O}_F$ with $\mathfrak{q} \mid (\mathfrak{n}/\mathfrak{p})$;
- $\mathrm{U}_{\wp} f = \alpha_{\wp} f$ for all prime ideals $\wp \subseteq \mathcal{O}_F$ with $\wp \mid \mathfrak{p}$.

In fact, f is viewed via strong approximation ([Vi, Chapitre III]) as a function:

$$f : (\prod_{\wp|\mathfrak{p}} \overrightarrow{\mathcal{E}}_{\wp})/\Gamma_{\mathfrak{p}} \times \{1, \dots, t\} \longrightarrow \mathcal{O}_{f,\pi}$$

where $\overrightarrow{\mathcal{E}}_{\wp}$ is the set of oriented edges of the Bruhat-Tits tree $\mathcal{T}_{\wp}$ associated to $\mathrm{PGL}_2(F_{\wp})$,

$$\Gamma_{\mathfrak{p}} := (B^\times \cap (\prod_{\mathfrak{q}\nmid\mathfrak{p}} R_{\mathfrak{q}}^\times \prod_{\wp|\mathfrak{p}} B_{\mathfrak{p}}^\times))/(F^\times \cap \prod_{\mathfrak{q}\nmid\mathfrak{p}} \mathcal{O}_{K,\mathfrak{q}}^\times \prod_{\wp|\mathfrak{p}} F_{\mathfrak{p}}^\times) \subseteq \prod_{\wp|\mathfrak{p}} \mathrm{PGL}_2(F_{\wp})$$

is a discrete subgroup (acting on $\prod_{\wp|\mathfrak{p}} \overrightarrow{\mathcal{E}}_{\wp}$ via conjugation of Eichler orders of level $\mathfrak{p}$) and t satisfies $\widehat{B}^\times = (\widehat{F}^\times \prod_{\mathfrak{q}\nmid\mathfrak{p}} R_{\mathfrak{q}}^\times \prod_{\wp|\mathfrak{p}} \widehat{B}_{\wp}^\times B^\times) \times \{1, \dots, t\}$. Set up the following notations:

- $\theta_f : \mathbb{T}_{\mathfrak{n}} \to \mathcal{O}_{f,\pi}$: the morphism associated to f; for any integer $n \geq 1$, let $\theta_{f,n} : \mathbb{T}_{\mathfrak{n}} \to \mathcal{O}_{f,\pi}/(\pi^n)$ the composition of θ_f with the canonical projection;
- $\rho_f : G_F \to \mathrm{GL}_2(\mathcal{O}_{f,\pi})$: the π-adic representation associated to f; for any integer $n \geq 1$, let $\rho_{f,n} : G_F \to \mathrm{GL}_2(\mathcal{O}_{f,\pi}/(\pi^n))$ the composition of ρ_f with the canonical projection.

2.3 p-adic L-functions

2.3.1 Pairings

Same notations as in Section 2.2. Let K/F be as in Section 2.1. Since Assumption 1 is verified, all prime ideals dividing $\mathfrak{n}^-$ (respectively, $\mathfrak{n}^+$) are inert (respectively, are not ramified) in K/F. Define $\mathfrak{c} = \mathfrak{c}_{\mathfrak{p}} \subseteq \mathcal{O}_F$ by the following condition: for any prime $\mathfrak{q}$, $\mathfrak{q}^r \mid \mathfrak{c} \Leftrightarrow \mathfrak{q}^r \mid \mathfrak{n}^+/\mathfrak{p}$ and $\mathfrak{q}$ is inert in K/F (that is, $\mathfrak{c}$ is the maximal factor of $\mathfrak{n}^+/\mathfrak{p}$ which is divisible only by primes which are inert in K/F). (Note that $\mathfrak{c} = 1$ in [BD3].) By [Vi], under this assumption there exists an *optimal embedding* of $\mathcal{O}_{\mathfrak{c}}[1/\mathfrak{p}] := K \cap \prod_{\mathfrak{q}\nmid\mathfrak{p}} \mathcal{O}_{\mathfrak{c},\mathfrak{q}}$ into $R[1/\mathfrak{p}] := B \cap \prod_{\mathfrak{q}\nmid\mathfrak{p}} R_{\mathfrak{q}}$, that is, there is an embedding

$$\Psi : K \to B \quad \text{satisfying} \quad \Psi(\mathcal{O}_{\mathfrak{c}}[1/\mathfrak{p}]) = \Psi(K) \cap R[1/\mathfrak{p}].$$

By passing to the adelization, there is a map:

$$\widehat{\Psi} : \tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty} = (\prod_{\mathfrak{q}\nmid\mathfrak{p}} \mathcal{O}_{K,\mathfrak{q}}^\times)\widehat{F}^\times\backslash\widehat{K}^\times/K^\times \longrightarrow (\prod_{\mathfrak{q}\nmid\mathfrak{p}} R_\mathfrak{q}^\times)\widehat{F}^\times\backslash\widehat{B}^\times/B^\times,$$

where $\tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty}$ is defined in (2). By strong approximation again, there is an isomorphism:

$$\eta : \prod_{\mathfrak{q}\nmid\mathfrak{p}} R_\mathfrak{q}^\times \widehat{F}^\times\backslash\widehat{B}^\times/B^\times \simeq (\prod_{\wp|\mathfrak{p}} \mathrm{PGL}_2(F_\wp))/\Gamma_\mathfrak{p} \times \{1, \dots, t\}.$$

The natural action of $\mathrm{PGL}_2(F_\wp)$ on $\overrightarrow{\mathcal{E}}_\wp$ by isometries induces an action of $(\prod_{\wp|\mathfrak{p}} \mathrm{PGL}_2(F_\wp))/\Gamma_\mathfrak{p} \times \{1, \dots, t\}$ on $(\prod_{\wp|\mathfrak{p}} \overrightarrow{\mathcal{E}}_\wp)/\Gamma_\mathfrak{p} \times \{1, \dots, t\}$ which can be described as follows. Fix $(\bar{e}, i) \in (\prod_{\wp|\mathfrak{p}} \overrightarrow{\mathcal{E}}_\wp)/\Gamma_\mathfrak{p} \times \{1, \dots, t\}$ and let $(\bar{w}, j) \in (\prod_{\wp|\mathfrak{p}} \mathrm{PGL}_2(F_\wp))/\Gamma_\mathfrak{p} \times \{1, \dots, t\}$; then $(\bar{w}, j)(\bar{e}, i) = (\bar{w}(\bar{e}), j \cdot i)$ where:

- If $e = (e_\wp)_{\wp|\mathfrak{p}} \in \prod_{\wp|\mathfrak{p}} \overrightarrow{\mathcal{E}}_\wp$ is a representative of $\bar{e}$ (mod $\Gamma_\mathfrak{p}$) and $w = (w_\wp)_{\wp|\mathfrak{p}} \in \prod_{\wp|\mathfrak{p}} \mathrm{PGL}_2(F_\wp)$ is a representative of w (mod $\Gamma_\mathfrak{p}$), a representative of $\bar{w}(\bar{e})$ is $(w_\wp(e_\wp))_{\wp|\mathfrak{p}}$;

- If $\{g_1, \dots, g_t\} \subseteq \widehat{B}^\times$ is a set of representatives for the double quotient space $(\widehat{F}^\times \prod_{\mathfrak{q}\nmid\mathfrak{p}} R_\mathfrak{q}^\times \prod_{\wp|\mathfrak{p}} \widehat{B}_\wp^\times)\backslash\widehat{B}^\times/B^\times$, then $j \cdot i$ is defined by requiring that $g_{j \cdot i}$ is a representative of $g_j g_i$.

The modular form f yields a $\mathcal{O}_{f,\pi}$-valued pairing $[,]_\mathfrak{p}$ between the Galois group $\tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty}$ and $(\prod_{\wp|\mathfrak{p}} \overrightarrow{\mathcal{E}}_\wp)/\Gamma_\mathfrak{p} \times \{1, \dots, t\} \to \mathcal{O}_{f,\pi}$ by the rule:

$$[g, (e, i)]_\mathfrak{p} = f((\eta\widehat{\Psi}(g)(e, i)).$$

2.3.2 Gross points and measures

By [Vi], under the above assumptions, there exists an optimal embedding $\Psi_0 : K \to B$ of $\mathcal{O}_\mathfrak{c}$ into R_0, that is, $\Psi_0(\mathcal{O}_\mathfrak{c}) = \Psi(K) \cap R_0$, where $R_0 \supseteq R$ is an Eichler order of level $\mathfrak{n}^+/\mathfrak{p}$. Define the set $\mathrm{Gr}(\mathfrak{c})$ of *Gross points of conductor* $\mathfrak{c}$ to be the image of $\widehat{\mathcal{O}}_\mathfrak{c}\widehat{F}^\times\backslash\widehat{K}^\times/K^\times$ into $\widehat{R}_0^\times\widehat{F}^\times\backslash\widehat{B}^\times/B^\times$ via the adelization $\widehat{\Psi}_0$ of Ψ_0 (note that, since Ψ_0 is an optimal embedding, $\widehat{\Psi}_0$ is injective). By strong approximation, $\widehat{R}_0^\times\widehat{F}^\times\backslash\widehat{B}^\times/B^\times$ can be identified with $(\prod_{\wp|\mathfrak{p}} \mathcal{V}_\wp)/\Gamma_\mathfrak{p} \times \{1, \dots, t\}$, where $\mathcal{V}_\wp$ is the set of vertices of the Bruhat-Tits tree $\mathcal{T}_\wp$, so a point $P \in \mathrm{Gr}(\mathfrak{c})$ can be identified with a pair $(\bar{v}, j)$ (obvious notations). Furthermore, for any $m \geq 0$, there are optimal embeddings $\Psi_m : K \to B$ of $\mathcal{O}_{\mathfrak{c}\mathfrak{p}^m}$ into R, that is, as above, $\Psi_m(\mathcal{O}_{\mathfrak{c}\mathfrak{p}^m}) = R \cap \Psi_m(K)$.

Define the set $\mathrm{Gr}(\mathfrak{cp}^m)$ of *Gross points of conductor* $\mathfrak{cp}^m$ to be the image of $\widehat{\mathcal{O}}_{\mathfrak{cp}^m}\widehat{F}^\times\backslash\widehat{K}^\times/K^\times$ into $\widehat{R}^\times\widehat{F}^\times\backslash\widehat{B}^\times/B^\times$ via the adelization $\widehat{\Psi}_m$ of Ψ_m (as above, $\widehat{\Psi}_m$ is injective). So, a Gross point $P\in\mathrm{Gr}(\mathfrak{cp}^m)$ is a pair $(\bar{e}_{\mathfrak{cp}^m},j)$. Denote by $v\mapsto\bar{v}$ (respectively, $e\mapsto\bar{e}$) the canonical projection $\prod_{\wp|\mathfrak{p}}\mathcal{V}_\wp\to(\prod_{\wp|\mathfrak{p}}\mathcal{V}_\wp)/\Gamma_\mathfrak{p}$ (respectively, $\prod_{\wp|\mathfrak{p}}\overrightarrow{\mathcal{E}}_\wp\to(\prod_{\wp|\mathfrak{p}}\overrightarrow{\mathcal{E}}_\wp)/\Gamma_\mathfrak{p}$). Moreover, for any $e=(e_\wp)_{\wp|\mathfrak{p}}\in\prod_{\wp|\mathfrak{p}}\overrightarrow{\mathcal{E}}_\wp$, denote by s and t the source and target of e, so that $e=(s(e),t(e))=(s(e_\wp),t(e_\wp))_{\wp|\mathfrak{p}}$. Gross points enjoy the following properties:

- For any $e_{\mathfrak{cp}}\in\prod_{\wp|\mathfrak{p}}\overrightarrow{\mathcal{E}}_\mathfrak{p}$ so that, for some j, $P_{\mathfrak{cp}}:=(\bar{e}_{\mathfrak{cp}},j)\in\mathrm{Gr}(\mathfrak{cp})$, there exists $v_\mathfrak{c}\in\prod_{\wp|\mathfrak{p}}\mathcal{V}_\mathfrak{p}$ with $P_\mathfrak{c}:=(\bar{v}_\mathfrak{c},j)\in\mathrm{Gr}(\mathfrak{c})$ and $s(e_{\mathfrak{cp}^n})=v_\mathfrak{c}$. In this case, say that $P_{\mathfrak{cp}}$ and $P_\mathfrak{c}$ are *compatible.*

- For any $m\geq 1$ and any $e_{\mathfrak{cp}^{m+1}}\in\prod_{\wp|\mathfrak{p}}\overrightarrow{\mathcal{E}}_\mathfrak{p}$ so that, for some j, $P_{\mathfrak{cp}^{m+1}}:=(\bar{e}_{\mathfrak{cp}^{m+1}},j)\in\mathrm{Gr}(\mathfrak{cp}^{m+1})$, there exists $e_{\mathfrak{cp}^m}\in\prod_{\wp|\mathfrak{p}}\overrightarrow{\mathcal{E}}_\mathfrak{p}$ with $P_{\mathfrak{cp}^m}:=(\bar{e}_{\mathfrak{cp}^m},j)\in\mathrm{Gr}(\mathfrak{cp}^m)$ and $s(e_{\mathfrak{cp}^{m+1}})=t(e_{\mathfrak{cp}^m})$. In this case say that P_{m+1} and P_m are *compatible.*

Definition 2.2. *Define the set* $\mathrm{Gr}(\mathfrak{cp}^\infty)$ *of* Gross points of conductor $\mathfrak{cp}^\infty$ *to be the set of sequences* $P_{\mathfrak{cp}^\infty}=(P_{\mathfrak{cp}^m})_{m\geq 0}$ *where for every* m, $P_{\mathfrak{cp}^m}\in\mathrm{Gr}(\mathfrak{cp}^m)$ *and* $P_{\mathfrak{cp}^m}$ *is compatible with* $P_{\mathfrak{cp}^{m-1}}$.

For any $m\geq 2$, define $U_{\mathfrak{cp}^m}:=\mathrm{Ker}(\tilde{G}_{\mathfrak{cp}^\infty}\to\tilde{G}_{\mathfrak{cp}^m})$. Choose a Gross point $P_{\mathfrak{cp}^\infty}=(P_{\mathfrak{cp}^m})_{m\geq 0}\in\mathrm{Gr}(\mathfrak{cp}^\infty)$ such that, for $m\geq 2$, $\mathrm{Stab}_{\tilde{G}_{\mathfrak{cp}^\infty}}(P_{\mathfrak{cp}^m})=U_{\mathfrak{cp}^m}$. Since $\alpha_\mathfrak{p}:=\prod_{\wp|\mathfrak{p}}\alpha_\wp$ is a unit in $\mathcal{O}_{f,\pi}^\times$, the pairing $[,]_\mathfrak{p}$ can be used to define a $\mathcal{O}_{f,\pi}$-valued measure $\tilde{\nu}_\mathfrak{p}$ on $\tilde{G}_{\mathfrak{cp}^\infty}$ by the rule:

$$\tilde{\nu}_\mathfrak{p}(U):=\frac{[\mathcal{O}_\mathfrak{c}^\times:\mathcal{O}_{\mathfrak{cp}^m}^\times]}{\alpha_\mathfrak{p}^{m-2}}[g,P_{\mathfrak{cp}^m}]_\mathfrak{p}$$

for all compact open sets $U=gU_{\mathfrak{cp}^m}$ for some $g\in\tilde{G}_{\mathfrak{cp}^\infty}$. Note that, since $\cap_m\mathcal{O}_{\mathfrak{cp}^m}=\mathcal{O}_F$ and the index of $\mathcal{O}_F^\times$ in $\mathcal{O}_\mathfrak{c}^\times$ is finite because K/F is quadratic imaginary and F is totally real, the relation $[\mathcal{O}_\mathfrak{c}^\times:\mathcal{O}_{\mathfrak{cp}^m}^\times]=[\mathcal{O}_\mathfrak{c}^\times:\mathcal{O}_F^\times]$ holds for m sufficiently large. The distribution relation on $\tilde{\nu}_\mathfrak{p}$ can be obtained by observing that f is an eigenform for $\mathrm{U}_\mathfrak{p}:=\prod_{\wp|\mathfrak{p}}\mathrm{U}_\wp$ with eigenvalue $\alpha_\mathfrak{p}$ and the Galois group $\mathrm{Gal}(\tilde{K}_{\mathfrak{cp}^{m+1}}/\tilde{K}_{\mathfrak{cp}^m})$, whose order is $|\mathfrak{p}|/[\mathcal{O}_{\mathfrak{cp}^m}^\times:\mathcal{O}_{\mathfrak{cp}^{m+1}}^\times]$, acts on $\mathrm{Gr}(\mathfrak{cp}^{m+1})$ by permutation of the points which are compatible with $P_{\mathfrak{cp}^m}$.

2.3.3 p-adic L-functions

Define $\tilde{\Lambda}_{\mathfrak{cp}^\infty}$ to be the completed group ring:

$$\tilde{\Lambda}_{\mathfrak{cp}^\infty}=\mathcal{O}_{f,\pi}[\![\tilde{G}_{\mathfrak{cp}^\infty}]\!]:=\lim_{\leftarrow m}\mathcal{O}_{f,\pi}[\tilde{G}_{\mathfrak{cp}^m}],$$

where the inverse limit is taken with respect to the canonical restriction maps. Recall the (non canonical) isomorphism $\tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty} \simeq \mathrm{Gal}(G_{\mathfrak{p}^\infty}/K) \times \Delta_{\mathfrak{c}\mathfrak{p}^\infty}$, with $\mathrm{Gal}(G_{\mathfrak{p}^\infty}/K) \simeq \mathbb{Z}_p^{\deg(\mathfrak{p})}$ and the finite layers $\mathrm{Gal}(K_{\mathfrak{p}^m}/K) \simeq (\mathbb{Z}/p^m\mathbb{Z})^{\deg(\mathfrak{p})}$. Define the *Iwasawa algebra* $\Lambda_{\mathfrak{p}^\infty}$ to be completed group ring

$$\Lambda_{\mathfrak{p}^\infty} := \mathcal{O}_{f,\pi}[\![G_{\mathfrak{p}^\infty}]\!] = \lim_{\leftarrow m} \mathcal{O}_{f,\pi}[G_{\mathfrak{p}^m}] \simeq \mathcal{O}_{f,\pi}[\![T_1, \dots T_{\deg(\mathfrak{p})}]\!],$$

where $T_1, \dots T_{\deg(\mathfrak{p})}$ are variables. Write also $\Lambda_{\mathfrak{c}\mathfrak{p}^m}$ (respectively, $\Lambda_{\mathfrak{p}^m}$) for $\mathcal{O}_{f,\pi}[\tilde{G}_{\mathfrak{c}\mathfrak{p}^m}]$ (respectively, $\mathcal{O}_{f,\pi}[G_{\mathfrak{p}^m}]$). Note that $\Lambda_{\mathfrak{p}^m}$ and $\Lambda_{\mathfrak{p}^\infty}$ are independent of $\mathfrak{c}$.

Define for $m \geq 2$:

$$\tilde{\mathcal{L}}_{f,\mathfrak{c}\mathfrak{p}^m} := \sum_{g \in \tilde{G}_{\mathfrak{c}\mathfrak{p}^m}} \tilde{\nu}_{\mathfrak{p}}(gU_{\mathfrak{c}\mathfrak{p}^m}) \cdot g \in \mathcal{O}_{f,\pi}[\tilde{G}_{\mathfrak{c}\mathfrak{p}^m}].$$

The distribution relation satisfied by $\tilde{\nu}_{\mathfrak{p}}$ ensures that there exists an element:

$$\tilde{\mathcal{L}}_{f,\mathfrak{c}\mathfrak{p}^\infty} := \lim_{\leftarrow m} \tilde{\mathcal{L}}_{f,\mathfrak{c}\mathfrak{p}^m} \in \tilde{\Lambda}_{\mathfrak{c}\mathfrak{p}^\infty}.$$

Define $\mathcal{L}_{f,\mathfrak{c}\mathfrak{p}^\infty}$ to be the canonical image (via restriction) of $\tilde{\mathcal{L}}_{\mathfrak{c}\mathfrak{p}^\infty}$ in the Iwasawa algebra $\Lambda_{\mathfrak{p}^\infty}$. Denote by $\mathcal{L} \mapsto \mathcal{L}^*$ the canonical involution of $\Lambda_{\mathfrak{p}^\infty}$ defined to be the extension by $\mathcal{O}_{f,\pi}$-linearity of the involution $g \mapsto g^{-1}$ of $G_{\mathfrak{p}^\infty}$. Note that a different choice of $P_{\mathfrak{c}\mathfrak{p}^\infty}$ satisfying $\mathrm{Stab}_{\tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty}}(U_{\mathfrak{c}\mathfrak{p}^m}) = P_{\mathfrak{c}\mathfrak{p}^m}$ has the effect of multiplying $\tilde{\mathcal{L}}_{\mathfrak{c}\mathfrak{p}^\infty}$ (respectively, $\mathcal{L}_{\mathfrak{c}\mathfrak{p}^\infty}$) by an element of $\tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty}$ (respectively, $G_{\mathfrak{p}^\infty}$). So, the element $\tilde{L}_{\mathfrak{c}\mathfrak{p}^\infty}(f,K) := \tilde{\mathcal{L}}_{f,\mathfrak{c}\mathfrak{p}^\infty}\tilde{\mathcal{L}}^*_{f,\mathfrak{c}\mathfrak{p}^\infty} \in \tilde{\Lambda}_{\mathfrak{c}\mathfrak{p}^\infty}$ is well defined and the following definition is well-posed.

Definition 2.3. *The* anticyclotomic p-adic Rankin L-function *attached to f, $\mathfrak{p}$ and K is the element $L_{\mathfrak{p}}(f,K) \in \Lambda_{\mathfrak{p}^\infty}$ defined to be the image of $\tilde{L}_{\mathfrak{p}}(f,K)$ in $\Lambda_{\mathfrak{c}\mathfrak{p}^\infty}$.*

It follows immediately from Definition 2.3 that:

$$L_{\mathfrak{p}}(f,K) := \mathcal{L}_{f,\mathfrak{c}\mathfrak{p}^\infty}\mathcal{L}^*_{f,\mathfrak{c}\mathfrak{p}^\infty}.$$

Remark. Of course, the definition of $L_{\mathfrak{p}}(f,K)$ depends on $\mathfrak{c}$; but since $\mathfrak{c}$ itself depends only on f, $\mathfrak{p}$ and K, and the Iwasawa algebra $\Lambda_{\mathfrak{p}^\infty}$ does not depend on $\mathfrak{c}$, the subscript $\mathfrak{c}$ has been dropped from the notation of $L_{\mathfrak{p}}(f,K)$.

Fix a finite order character $\chi : \tilde{G}_{\mathfrak{c}\mathfrak{p}^\infty} \to \bar{\mathbb{Q}}_p$ and extend it by $\mathcal{O}_{f,\pi}$-linearity to $\tilde{\Lambda}_{\mathfrak{c}\mathfrak{p}^\infty} \to \bar{\mathbb{Q}}_p$ (this is possible because χ has finite order). Then the following interpolation formula holds (see [Zh2, Section 1.3]):

$$\chi(\tilde{L}_{\mathfrak{c}\mathfrak{p}^\infty}(f,K)) \doteq L_K(f,\chi,1),$$

where $L_K(f,\chi,1)$ is the L-function $L_K(f,s)$ twisted by the character χ, $\doteq$ means that the above equality holds up to an explicitly non-zero computable term and the values of χ are viewed as complex numbers by fixing an embedding $\mathbb{Q}_p \to \mathbb{C}$.

2.4 Relations between different p-adic L-functions

Assume that $\mathfrak{p}_1 \subseteq \mathcal{O}_F$ and $\mathfrak{p}_2 \subseteq \mathcal{O}_F$ are two divisors of p. Then there are two different p-adic L-functions $L_{\mathfrak{p}_1}(f_{\mathfrak{p}_1},K) \in \Lambda_{\mathfrak{c}_1\mathfrak{p}_1^\infty}$ and $L_{\mathfrak{p}_2}(f_{\mathfrak{p}_2},K) \in \Lambda_{\mathfrak{c}_2\mathfrak{p}_2^\infty}$, where $f_{\mathfrak{p}_i}$ and $\mathfrak{c}_i$ depend on $\mathfrak{p}_i$ (recall that $f_{\mathfrak{p}_i}$ is obtained by raising f_0 at some primes dividing $\mathfrak{p}_i$). Suppose that $\mathfrak{p}_1 \mid \mathfrak{p}_2$ and assume that $f_{\mathfrak{p}_1} = f_{\mathfrak{p}_2} =: f$ (that is, the primes $\wp \subseteq \mathcal{O}_F$ dividing $\mathfrak{p}_2/\mathfrak{p}_1$ are also divisors of $\mathfrak{n}_0$, the level of f_0). Since $\mathfrak{p}_1 \mid \mathfrak{p}_2$, it follows that $\mathfrak{c}_1 = \mathfrak{c}_2\mathfrak{p}$, where $\mathfrak{p} := \prod_{\wp\mid\mathfrak{p}_2,\wp\nmid\mathfrak{p}_1,(\frac{K}{\wp})=-1}\wp$ (recall the notations in Equation (1): $\mathfrak{p}$ is the product of the primes $\wp \mid \mathfrak{p}_2$, $\wp \nmid \mathfrak{p}_1$ which are inert in K/F). It follows that $\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m} \subseteq \tilde{K}_{\mathfrak{c}_2\mathfrak{p}_2^m}$ for all $m \geq 1$. So, there are canonical maps:

$$\tilde{\lambda}^m_{\mathfrak{p}_2/\mathfrak{p}_1} : \tilde{\Lambda}_{\mathfrak{c}_2\mathfrak{p}_2^m} \to \tilde{\Lambda}_{\mathfrak{c}_1\mathfrak{p}_1^m} \quad \text{for } m \geq 1, \quad \text{and} \quad \tilde{\lambda}^\infty_{\mathfrak{p}_2/\mathfrak{p}_1} : \tilde{\Lambda}_{\mathfrak{c}_2\mathfrak{p}_2} \to \tilde{\Lambda}_{\mathfrak{c}_1\mathfrak{p}_1^\infty}$$

induced by restriction maps.

Introduce these more general notations: For any ideal $\mathfrak{r} \subseteq \mathcal{O}_F$ define the extension $\tilde{K}_\mathfrak{r} \subseteq \mathrm{RCF}(\mathfrak{r})$ by requiring that $\mathrm{Gal}(\tilde{K}_\mathfrak{r}/K) \simeq \mathrm{Pic}(\mathcal{O}_\mathfrak{r})/\mathrm{Pic}(\mathcal{O}_F)$. Furthermore, for any ideal $\mathfrak{r} \subseteq \mathcal{O}_F$ so that there exists an optimal embedding $\Psi_\mathfrak{r} : K \to B$ of $\mathcal{O}_\mathfrak{r}$ into R, denote by $\mathrm{Gr}(\mathfrak{r})$ the image of $\widehat{\mathcal{O}}_\mathfrak{r}^\times \widehat{F}^\times \backslash \widehat{K}^\times/K^\times$ into $\widehat{R}^\times \widehat{F}^\times \backslash \widehat{B}^\times/B^\times$ via the adelization of $\Psi_\mathfrak{r}$. By definition:

$$\tilde{\lambda}^m_{\mathfrak{p}_2/\mathfrak{p}_1}(\tilde{\mathcal{L}}_{\mathfrak{c}_2\mathfrak{p}_2^m}) = \sum_{g\in\tilde{G}_{\mathfrak{c}_1\mathfrak{p}_1^m}} \frac{[\mathcal{O}_{\mathfrak{c}_2}^\times : \mathcal{O}_{\mathfrak{c}_2\mathfrak{p}_2^m}^\times]}{\alpha_{\mathfrak{p}_2}^{m-2}} \left(\sum_{\sigma\in\mathrm{Gal}(\tilde{K}_{\mathfrak{c}_2\mathfrak{p}_2^m}/\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m})} [\tilde{g}\sigma, P_{\mathfrak{c}_2\mathfrak{p}_2^m}]_{\mathfrak{p}_2} \right) \cdot g,$$

where $\tilde{g}$ is any element of $\tilde{G}_{\mathfrak{c}_2\mathfrak{p}_2^m}$ whose reduction is g. Write $\mathfrak{p}_* := \frac{\mathfrak{p}_2}{\mathfrak{p}_1}$; note that $\mathfrak{p} \mid \mathfrak{p}_*$. Since f is an eigenform for $\mathrm{U}_{\mathfrak{p}_*} := \prod_{\wp\mid\mathfrak{p}_*}\mathrm{U}_\wp$ with eigenvalue $\alpha_{\mathfrak{p}_*} := \prod_{\wp\mid\mathfrak{p}_*}\alpha_\wp$, the inner sum is equal to:

$$\sum_{\rho\in\mathrm{Gal}(\tilde{K}_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}}/\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m})} \left(\sum_{\tau\in\mathrm{Gal}(\tilde{K}_{\mathfrak{c}_2\mathfrak{p}_2^m}/\tilde{K}_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}})} [\tilde{g}\tilde{\rho}, \tau P_{\mathfrak{c}_2\mathfrak{p}_2^m}]_{\mathfrak{p}_2} \right) =$$

$$= \sum_{\rho\in\mathrm{Gal}(\tilde{K}_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}}/\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m})} \frac{\alpha_{\mathfrak{p}_*}}{[\mathcal{O}^\times_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}} : \mathcal{O}^\times_{\mathfrak{c}_2\mathfrak{p}_2^m}]} [\tilde{g}^{(m-1)}\rho, P_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}}]_{\mathfrak{p}_2};$$

here $\tilde{g}^{(m-1)}$ is any element of $\tilde{G}_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}}$ whose reduction coincide with g and $P_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}} \in \mathrm{Gr}(\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1})$. (Note that, if $P_{\mathfrak{c}_2\mathfrak{p}_2^m} = ((\bar{e}_\wp^{(m)})_\wp, j)$, then $P_{\mathfrak{c}_2\mathfrak{p}_1^m\mathfrak{p}_*^{m-1}}$ is represented by the product of edges $(\prod_{\wp|\mathfrak{p}_1} \bar{e}_\wp^{(m)} \prod_{\wp|\mathfrak{p}_*} \bar{e}_\wp^{(m-1)}, j)$, where, for $\wp \mid \mathfrak{p}_*$, the target of $e_\wp^{(m-1)}$ is equal to the source of $e_\wp^{(m)}$.) Define $\mathfrak{p}' := \mathfrak{p}_2/(\mathfrak{p}_1\mathfrak{p}) = \mathfrak{p}_*/\mathfrak{p}$. A recursive argument combined with the equality $\mathfrak{c}_1 = \mathfrak{c}_2\mathfrak{p}$ shows that the inner sum equals:

$$\sum_{\sigma \in \mathrm{Gal}(\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}/\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m})} \frac{\alpha_{\mathfrak{p}_*}^{m-1}}{[\mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}^\times : \mathcal{O}_{\mathfrak{c}_2\mathfrak{p}_2^m}^\times]} [\tilde{g}^{(1)}, P_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}]_{\mathfrak{p}_2}; \tag{3}$$

here $\tilde{g}^{(1)}$ is an element of $\tilde{G}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}$ whose reduction is g and $P_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}$ is a Gross point of $\mathrm{Gr}(\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}')$ which can be described by $(\prod_{\wp|\mathfrak{p}_1} \bar{e}_\wp^{(m)} \prod_{\wp|\mathfrak{p}'\mathfrak{p}} \bar{e}_\wp^{(1)}, j)$, where, for $\wp \mid \mathfrak{p}'\mathfrak{p}$, the distance between the target of $e_\wp^{(1)}$ and the source of $e_\wp^{(m)}$ is $m-2$. Note that:

$$\sum_{\sigma \in \mathrm{Gal}(\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}/\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m})} \sigma P_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'} = \frac{\mathrm{U}_{\mathfrak{p}'}(P_{\mathfrak{c}_1\mathfrak{p}_1^m})}{[\mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m}^\times : \mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}^\times]} - \sum_{\wp|\mathfrak{p}'} \frac{\delta_\wp(P_{\mathfrak{c}_1\mathfrak{p}_1^m})}{[\mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m}^\times : \mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}^\times]},$$

where $\delta_\wp$ is a suitable element in $\tilde{G}_{\mathfrak{c}_1\mathfrak{p}_1^m}$. (Recall that, by definition, if $\wp \mid \mathfrak{p}'$, then $\wp$ is split in K/F, so $\#\mathrm{Gal}(\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m\mathfrak{p}'}/\tilde{K}_{\mathfrak{c}_1\mathfrak{p}_1^m(\mathfrak{p}'/\wp)}) = |\wp| - 1$ and $\deg(\mathrm{U}_\wp) = |\wp|$; see also the analogous formula in [BD2, page 433, second case of p divides N]). Combining this formula with (3) yields the following expression for $\tilde{\lambda}_{\mathfrak{p}_2/\mathfrak{p}_1}(\tilde{\mathcal{L}}_{\mathfrak{c}_2\mathfrak{p}_2^m})$:

$$\sum_{g \in \tilde{G}_{\mathfrak{c}_1\mathfrak{p}_1^m}} \frac{\alpha_{\mathfrak{p}_*}^{m-1}\alpha_{\mathfrak{p}'}[\mathcal{O}_{\mathfrak{c}_2}^\times : \mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m}^\times]}{\alpha_{\mathfrak{p}_2}^{m-2}} \cdot [g, P_{\mathfrak{c}_1\mathfrak{p}_1^m}]_{\mathfrak{p}_1} \cdot g -$$

$$- \sum_{g \in \tilde{G}_{\mathfrak{c}_1\mathfrak{p}_1^m}} \frac{\alpha_{\mathfrak{p}_*}^{m-1}[\mathcal{O}_{\mathfrak{c}_2}^\times : \mathcal{O}_{\mathfrak{c}_1\mathfrak{p}_1^m}^\times]}{\alpha_{\mathfrak{p}_2}^{m-2}} [g\delta_\wp, P_{\mathfrak{c}_1\mathfrak{p}_1^m}]_{\mathfrak{p}_1} \cdot g.$$

Then:

$$\tilde{\lambda}_{\mathfrak{p}_2/\mathfrak{p}_1}^m(\tilde{\mathcal{L}}_{\mathfrak{c}_2\mathfrak{p}_2^m}) = \frac{\alpha_{\mathfrak{p}_2}\alpha_{\mathfrak{p}'}[\mathcal{O}_{\mathfrak{c}_1}^\times : \mathcal{O}_{\mathfrak{c}_2}^\times]}{\alpha_{\mathfrak{p}_1}} \tilde{\mathcal{L}}_{\mathfrak{c}_1\mathfrak{p}_1^m} - \sum_{\wp|\mathfrak{p}'} \frac{\alpha_{\mathfrak{p}_2}[\mathcal{O}_{\mathfrak{c}_1}^\times : \mathcal{O}_{\mathfrak{c}_2}^\times]}{\alpha_{\mathfrak{p}_1}} \delta_\wp^{-1} \tilde{\mathcal{L}}_{\mathfrak{c}_1\mathfrak{p}_1^m}. \tag{4}$$

The elements $\delta_\wp$ are independent of m (indeed, they depends only on the choice of $P_{\mathfrak{c}_2\mathfrak{p}_2^\infty}$) and they belong to the torsion subgroup $\Delta_{\mathfrak{c}_1\mathfrak{p}_1^\infty} \subseteq \tilde{G}_{\mathfrak{c}_1\mathfrak{p}_1^\infty}$ (this is because they act only on the $\wp$-component of $P_{\mathfrak{c}_1\mathfrak{p}^\infty}$).

Lemma 2.4. *Let $n_{\mathfrak{p}'} := \#\{\wp \mid \mathfrak{p}'\}$. Then:*

$$\tilde{\lambda}_{\mathfrak{p}_2/\mathfrak{p}_1}(\tilde{L}_{\mathfrak{c}_2\mathfrak{p}_2^\infty}(f,K)) = \left(\frac{\alpha_{\mathfrak{p}_2}[\mathcal{O}_{\mathfrak{c}_1}^\times : \mathcal{O}_{\mathfrak{c}_2}^\times]}{\alpha_{\mathfrak{p}_1}}\right)^2 l_{\mathfrak{p}_2/\mathfrak{p}_1}\tilde{L}_{\mathfrak{c}_1\mathfrak{p}_1^\infty}(f,K),$$

where $l_{\mathfrak{p}_2/\mathfrak{p}_1} := \alpha_{\mathfrak{p}'}^2 - \alpha_{\mathfrak{p}'}\sum_{\wp|\mathfrak{p}'}(\delta_\wp + \delta_\wp{}^{-1}) + n_{\mathfrak{p}'} + \sum_{\wp,\wp'|\mathfrak{p}'}\delta_\wp\delta_{\wp'}^{-1}$ is an element of $\mathcal{O}_{f,\pi}[\Delta_{\mathfrak{c}_1\mathfrak{p}_1^\infty}]$.

Proof. Combine equation (4) with the identity $(\delta_\wp{}^{-1}\tilde{\mathcal{L}}_{\mathfrak{c}_1\mathfrak{p}_1^\infty})^* = \delta_\wp\tilde{\mathcal{L}}_{\mathfrak{c}_1\mathfrak{p}_1^\infty}$. □

Define

$$\lambda_{\mathfrak{p}_2/\mathfrak{p}_1} : \Lambda_{\mathfrak{p}_2^\infty} \longrightarrow \Lambda_{\mathfrak{p}_1^\infty}$$

to be the natural map deduced by restriction on Galois groups.

Proposition 2.5. *Notations as in Lemma 2.4. Then*

$$\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(L_{\mathfrak{p}_2}(f,K)) = \left(\frac{\alpha_{\mathfrak{p}_2}[\mathcal{O}_{\mathfrak{c}_1}^\times : \mathcal{O}_{\mathfrak{c}_2}^\times]}{\alpha_{\mathfrak{p}_1}}\right)^2 (\alpha_{\mathfrak{p}'} - n_{\mathfrak{p}'})^2 L_{\mathfrak{p}_1}(f,K).$$

Proof. For any $\delta \in \Delta_{\mathfrak{c}_1\mathfrak{p}^\infty}$, the image of $\delta\tilde{L}_{\mathfrak{c}_1\mathfrak{p}_1^\infty}(f,K)$ in $\Lambda_{\mathfrak{p}^\infty}$ is the same as the image of $\tilde{L}_{\mathfrak{c}_1\mathfrak{p}_1^\infty}(f,K)$. The statement follows then from Lemma 2.4. □

Corollary 2.6. *Notations as in Lemma 2.4. If $\mathfrak{p} = (1)$ (that is, all primes dividing $\mathfrak{p}_2$ but not dividing $\mathfrak{p}_1$ are split in K/F) then $n_{\mathfrak{p}'}$ is the number of primes dividing $\mathfrak{p}_2/\mathfrak{p}_1$ and:*

$$\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(L_{\mathfrak{p}_2}(f,K)) = \frac{\alpha_{\mathfrak{p}_2}^2(\alpha_{\mathfrak{p}_2} - n_{\mathfrak{p}'}\alpha_{\mathfrak{p}_1})^2}{\alpha_{\mathfrak{p}_1}^4} L_{\mathfrak{p}_1}(f,K).$$

On the other hand, if $\mathfrak{p} = \mathfrak{p}_2/\mathfrak{p}_1$ (that is, all primes dividing $\mathfrak{p}_2$ but not dividing $\mathfrak{p}_1$ are inert in K/F) then $n_{\mathfrak{p}'} = 0$ and:

$$\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(L_{\mathfrak{p}_2}(f,K)) = (\alpha_{\mathfrak{p}_2}/\alpha_{\mathfrak{p}_1})^4 L_{\mathfrak{p}_1}(f,K).$$

Proof. Follows from Proposition 2.5. □

Fix an isomorphism $\tilde{\Lambda}_{\mathfrak{c}_1\mathfrak{p}_1^\infty} \simeq \mathcal{O}_{f,\pi}[\![T_1,\ldots,T_{\deg(\mathfrak{p}_1)}]\!]$. Then there is an isomorphism $\tilde{\Lambda}_{\mathfrak{c}_2\mathfrak{p}_2^\infty} \simeq \mathcal{O}_{f,\pi}[\![T_1,\ldots,T_{\deg(\mathfrak{p}_1)},T_{\deg(\mathfrak{p}_1)+1},\ldots,T_{\deg(\mathfrak{p}_2)}]\!]$ such that the kernel of the restriction map $\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}$ is the ideal $(T_{\deg(\mathfrak{p}_1)+1},\ldots,T_{\deg(\mathfrak{p}_2)})$. Fix also such an isomorphism. Then the above results simply assert that $L_{\mathfrak{p}_1}(f,K)$ is the *special value* of $L_{\mathfrak{p}_2}(f,K)$. More precisely: write $T_{\mathfrak{p}_1}$ (respectively, $T_{\mathfrak{p}_2}$) for the set of variables $T_1,\ldots,T_{\deg(\mathfrak{p}_1)}$ (respectively, for the set of variables $T_{\deg(\mathfrak{p}_1)+1},\ldots,T_{\mathfrak{p}_2}$). Then $L_{\mathfrak{p}_1}(f,K) = L_{\mathfrak{p}_1}(f,K)(T_{\mathfrak{p}_1})$ and $L_{\mathfrak{p}_2}(f,K) = L_{\mathfrak{p}_2}(f,K)(T_{\mathfrak{p}_1},T_{\mathfrak{p}_2})$.

Corollary 2.7. *Notations as in Lemma 2.4. Then*

$$L_{\mathfrak{p}_2}(f,K)(T_{\mathfrak{p}_1},0)=\left(\frac{\alpha_{\mathfrak{p}_2}[\mathcal{O}_{\mathfrak{c}_1}^\times:\mathcal{O}_{\mathfrak{c}_2}^\times]}{\alpha_{\mathfrak{p}_1}}\right)^2(\alpha_{\mathfrak{p}'}-n_{\mathfrak{p}'})^2L_{\mathfrak{p}_1}(f,K)(T_{\mathfrak{p}_1}).$$

Proof. The isomorphism between an Iwasawa algebra $\mathbb{Z}_p[\![G]\!]$ (where $G\simeq\mathbb{Z}_p$ is the Galois group of a $\mathbb{Z}_p$-extension) and the formal power series ring $\mathbb{Z}_p[\![X]\!]$ is defined by sending a topological generator γ of G to $1-X$. The result follows. $\square$

Example. Assume that $F/\mathbb{Q}$ is real quadratic, that $p=\wp_1\wp_2$ with $\wp_i\subseteq\mathcal{O}_F$ prime ideals, $\wp_1\neq\wp_2$. Define $\mathfrak{p}_1=\wp_1$ and $\mathfrak{p}_2=p=\wp_1\wp_2$. If $\wp_2$ is inert in F/K then $\mathfrak{p}'=(1)$ and $L_p(f,K)(T_{\wp_1},0)=\left(\alpha_{\wp_2}[\mathcal{O}_{\mathfrak{c}_1}^\times:\mathcal{O}_{\mathfrak{c}_2}^\times]\right)^2L_{\wp_1}(f,K)(T_{\wp_1})$. On the other hand, if $\wp_2$ is split in K/F, then $\mathfrak{p}=(1)$ and $L_p(f,K)(T_{\wp_1},0)=\frac{\alpha_p^2(\alpha_p-\alpha_{\wp_1})^2}{\alpha_{\wp_1}^4}L_{\wp_1}(f,K)(T_{\wp_1})$. If $\alpha_{\wp_1}=\alpha_p$, that is, $\alpha_{\wp_2}=1$, then the above formula implies that $L_p(f,K)(T,0)=0$. This phenomenon can be regarded as a sort of *extra-zero* of the restriction of $L_p(f,K)$ to the first variable.

3 Iwasawa's Main Conjecture

3.1 Selmer groups attached to modular forms

Notations as in the above sections. Let $T_f=T_{f,\pi}$ be the G_F-module, free of rank 2 over $\mathcal{O}_{f,\pi}$, associated to the representation ρ_f; define $V_f=V_{f,\pi}:=T_f\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $A_f=A_{f,\pi}=V_f/T_f$ (hence, $A_f\simeq(K_{f,\pi}/\mathcal{O}_{f,\pi})^2$ as $\mathcal{O}_{f,\pi}$-modules). Moreover, define $T_{f,n}:=T_f/\pi^nT_f$ and $A_{f,n}=A_f[\pi^n]$, so that there both $T_{f,n}$ and $A_{f,n}$ are $\mathcal{O}_{f,\pi}/(\pi^n)$-modules free of rank 2 and there is an isomorphism of G_F-modules $T_{f,n}\simeq A_{f,n}$. Furthermore, $A_f\simeq\lim_{\rightarrow n}A_{f,n}$ with respect to the inclusion maps and $T_f\simeq\lim_{\leftarrow n}T_{f,n}$ with respect to the multiplication by π maps. Taking the direct (respectively, inverse) limit as $m\to\infty$ of the groups $H^1(K_{\mathfrak{p}^m},A_{f,n})$ with respect to the restriction (respectively, corestriction maps) yields the groups:

$$H^1(K_{\mathfrak{p}^\infty},A_{f,n}):=\lim_{\rightarrow m}H^1(K_{\mathfrak{p}^m},A_{f,n}),$$

$$\hat{H}^1(K_{\mathfrak{p}^\infty},T_{f,n}):=\lim_{\leftarrow m}H^1(K_{\mathfrak{p}^m},T_{f,n}),$$

where the notations are the same as in Section 2.1 and 2.2 (in particular, $\mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)\simeq\mathbb{Z}_p^{\deg(\mathfrak{p})}$).

Denote by $\epsilon:G_F\to\mathbb{Z}_p^\times$ the cyclotomic character giving the action of G_F on the p-power roots of unity μ_{p^∞}. Assume that the representation $\rho_{f,\pi}$

satisfies Assumption 2 so that for any prime ideal $\mathfrak{q} \subseteq \mathcal{O}_F$ dividing exactly $\mathfrak{n}$ and not dividing p there is an exact sequence of G_q-modules: $0 \to A_{f,1}^{(\mathfrak{q})} \to A_{f,1} \to A_{f,1}^{(1)} \to 0$ such that $G_{F_\mathfrak{q}}$ acts on the one-dimensional $\mathcal{O}_{f,\pi}/(\pi)$-vector space $A_{f,1}^{(\mathfrak{q})}$ as multiplication by ϵ or $-\epsilon$. Recall that, by Assumption 1, f_0 is assumed to be π-ordinary at p. This condition implies that for any $n \geq 1$ and any $\wp \mid p$ there is an exact sequence of $I_\wp$-modules: $0 \to A_{f,1}^{(\wp)} \to A_{f,n} \to A_{f,n}^{(1)} \to 0$ where $I_\wp$ acts on the free of rank one $\mathcal{O}_{f,\pi}/(\pi^n)$-module $A_{f,n}^{(\wp)}$ by ϵ while it acts trivially on the quotient $A_{f,n}^{(1)}$ (which is also free of rank one over $\mathcal{O}_{f,\pi}/(\pi^n)$).

Define the following finite/singular and finite/ordinary structure, where $M = A_{f,n}$ or $M = T_{f,n}$:

Let $\mathfrak{q} \subseteq \mathcal{O}_F$ be a prime ideal such that $\mathfrak{q} \nmid \mathfrak{n}p$. Denote by $I_\mathfrak{q}$ the direct sum of the inertia subgroups $I_{\mathfrak{q}'} \subseteq G_{\mathfrak{q}'}$, where $\mathfrak{q}'$ ranges over the set of primes of $\mathcal{O}_{K_{\mathfrak{p}^m}}$ dividing $\mathfrak{q}$ and $G_{\mathfrak{q}'} \subseteq G_{K_{\mathfrak{p}^m}}$ is the choice of a decomposition subgroup. The *singular part* of $H^1(K_{\mathfrak{p}^m,\mathfrak{q}}, M)$ is $H^1_{\mathrm{sing}}(K_{\mathfrak{p}^m,\mathfrak{q}}, M) := H^1(I_\mathfrak{q}, M)^{\mathrm{Gal}(K^{\mathrm{unr}}_{\mathfrak{p}^m,\mathfrak{q}}/K_{\mathfrak{p}^m,\mathfrak{q}})} := \oplus_{\mathfrak{q}'|\mathfrak{q}} H^1(I_{\mathfrak{q}'}, M)^{\mathrm{Gal}(K^{\mathrm{unr}}_{\mathfrak{p}^m,\mathfrak{q}'}/K_{\mathfrak{p}^m,\mathfrak{q}'})}$. The kernel of the *residue map* $\partial_\mathfrak{q} : H^1(K_{\mathfrak{p}^m,\mathfrak{q}}, M) \to H^1_{\mathrm{sing}}(K_{\mathfrak{p}^m,\mathfrak{q}}, M)$ is the *finite part* of $H^1(K_{\mathfrak{p}^m,\mathfrak{q}}, M)$ and is denoted by $H^1_{\mathrm{fin}}(K_{\mathfrak{p}^m,\mathfrak{q}}, M)$. Taking direct limits for $m \to \infty$ yields:

$$H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n}) := \lim_{\to m} H^1_{\mathrm{fin}}(K_{\mathfrak{p}^m,\mathfrak{q}}, A_{f,n}),$$

$$H^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n}) := \lim_{\to m} H^1_{\mathrm{sing}}(K_{\mathfrak{p}^m,\mathfrak{q}}, A_{f,n}),$$

while taking inverse limits yields:

$$\hat{H}^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n}) := \lim_{\leftarrow m} H^1_{\mathrm{fin}}(K_{\mathfrak{p}^m,\mathfrak{q}}, T_{f,n}),$$

$$\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n}) := \lim_{\leftarrow m} H^1_{\mathrm{sing}}(K_{\mathfrak{p}^m,\mathfrak{q}}, T_{f,n}).$$

The groups $H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$ and $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n})$ are annihilators of each other under the local Tate pairing $\langle , \rangle_\mathfrak{q}$. Moreover, as in [BD3, Lemma 2.4 and Lemma 2.5]:

- If $\mathfrak{q}$ is split in K/F, $H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n}) = 0$ and $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n}) = 0$.
- If $\mathfrak{q}$ is inert in K/F, $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n}) \simeq H^1_{\mathrm{sing}}(K_\mathfrak{q}, T_{f,n}) \otimes \Lambda_{\mathfrak{p}^\infty}$ and $H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n}) \simeq \mathrm{Hom}(H^1_{\mathrm{sing}}(K_\mathfrak{q}, T_{f,n}) \otimes \Lambda_{\mathfrak{p}^\infty}, \mathbb{Q}_p/\mathbb{Z}_p)$ (notations for the Iwasawa algebra $\Lambda_{\mathfrak{p}^\infty}$ as in Section 2.3).

Let $\mathfrak{q} \subseteq \mathcal{O}_F$ be a prime ideal such that $\mathfrak{q} \mid \mathfrak{n}$ exactly and the residue characteristic of $\mathfrak{q}$ is not p. Recall the module $A_{f,n}^{(\mathfrak{q})}$ defined by the above exact sequence for $A_{f,n}$ at $\mathfrak{q}$. The *ordinary part* of $H^1(K_{\mathfrak{p}^m,\mathfrak{q}}, A_{f,n})$ is defined as $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\mathfrak{q}}, A_{f,n}) := H^1(K_{\mathfrak{p}^m,\mathfrak{q}}, A_{f,n}^{(\mathfrak{q})})$. Taking direct limits:

$$H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n}) := \lim_{\rightarrow m} H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\mathfrak{q}}, A_{f,n}).$$

Use the isomorphism $A_{f,n} \simeq T_{f,n}$ to define $T_{f,n}^{(\mathfrak{q})} \simeq A_{f,n}^{(\mathfrak{q})}$ and $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\mathfrak{q}}, T_{f,n})$ $:= H^1(K_{\mathfrak{p}^m,\mathfrak{q}}, T_{f,n}^{(\mathfrak{q})})$. Taking inverse limits:

$$\hat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n}) := \lim_{\leftarrow m} H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\mathfrak{q}}, T_{f,n}).$$

The groups $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$ and $\hat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n})$ are annihilators of each other under the local Tate pairing $\langle , \rangle_{\mathfrak{q}}$.

Let $\wp \subseteq \mathcal{O}_F$ be a prime ideal such that $\wp \mid p$. Recall the module $A_{f,n}^{(\wp)}$ defined by the above exact sequence for $A_{f,n}$ at $\wp$. The *ordinary part* of $H^1(K_{\mathfrak{p}^m,\wp}, A_{f,n})$ is $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\wp}, A_{f,n}) := \mathrm{res}_\wp^{-1}(H^1(I_\wp, A_{f,n}^{(\wp)}))$, where the following notations are used: $I_\wp$ is the (as above) sum of the inertia subgroups $I_{\wp'} \subseteq G_{\wp'}$, where $\wp'$ ranges over the set of primes of $\mathcal{O}_{K_{\mathfrak{p}^m}}$ dividing $\wp$ and $G_{\wp'} \subseteq G_{K_{\mathfrak{p}^m}}$ is the choice of a decomposition subgroup; $\mathrm{res}_\wp : H^1(K_{\mathfrak{p}^m,\wp}, A_{f,n}) \to H^1(I_\wp, A_{f,n})$ is the restriction map. Taking direct limits:

$$H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\wp}, A_{f,n}) := \lim_{\rightarrow m} H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\wp}, A_{f,n}).$$

Again, use the isomorphism $A_{f,n} \simeq T_{f,n}$ to define $T_{f,n}^{(\wp)}$ and $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\wp}, T_{f,n})$ $:= \mathrm{res}_\wp^{-1}(H^1(I_\wp, T_{f,n}^{(\wp)}))$. Taking inverse limits:

$$\hat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\wp}, T_{f,n}) := \lim_{\leftarrow m} H^1_{\mathrm{ord}}(K_{\mathfrak{p}^m,\wp}, T_{f,n}).$$

The groups $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\wp}, A_{f,n})$ and $\hat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\wp}, T_{f,n})$ are annihilators of each other under the local Tate pairing $\langle , \rangle_{\mathfrak{q}}$.

For any prime ideal $\mathfrak{q} \subseteq \mathcal{O}_F$, let $\mathrm{res}_{\mathfrak{q}} : H^1(K_{\mathfrak{p}^\infty}, A_{f,n}) \to H^1(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$ be the restriction map. Furthermore, for a prime $\mathfrak{q} \subseteq \mathcal{O}_F$ not dividing $\mathfrak{n}p$ let $\partial_{\mathfrak{q}}$ denote (by an abuse of notations) the maps $\partial_{\mathfrak{q}} : H^1(K_{\mathfrak{p}^\infty}, A_{f,n}) \to H^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$ and $\partial_{\mathfrak{q}} : H^1(K_{\mathfrak{p}^\infty}, T_{f,n}) \to \hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n})$ resulting by composing $\mathrm{res}_{\mathfrak{q}}$ with the residue maps $H^1(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n}) \to H^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$ and $H^1(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n}) \to \hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, T_{f,n})$. Moreover, if $\partial_{\mathfrak{q}}(\kappa) = 0$, denote by $v_{\mathfrak{q}}(\kappa)$ the image of $\mathrm{res}_{\mathfrak{q}}(\kappa)$ in the kernel of the residue maps.

Definition 3.1. *The* Selmer group $\mathrm{Sel}_{f,n}(K_{\mathfrak{p}^\infty})$ *attached to f, n and $K_{\mathfrak{p}^\infty}$ is the group of elements $s \in H^1(K_{\mathfrak{p}^\infty}, A_{f,n})$ satisfying:*

- *For primes* $\mathfrak{q} \subseteq \mathcal{O}_F$ *which do not divide* $\mathfrak{n}p$, $\partial_{\mathfrak{q}}(s) = 0$.
- *For primes* $\mathfrak{q} \subseteq \mathcal{O}_F$ *dividing* $\mathfrak{n}^-$ *exactly,* $\mathrm{res}_{\mathfrak{q}}(s) \in H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$.
- *For primes* $\wp \subseteq \mathcal{O}_F$ *dividing* p, $\mathrm{res}_{\mathfrak{q}}(s) \in H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$.
- *For primes* $\mathfrak{q} \subseteq \mathcal{O}_F$ *dividing* $\mathfrak{n}^+$ *and not dividing* p, $\mathrm{res}_{\mathfrak{q}}(s) = 0$.

Definition 3.2. *Let* $\mathfrak{s} \subseteq \mathcal{O}_F$ *be a square free ideal prime to* $\mathfrak{n}$. *The* compactified Selmer group $\hat{H}^1_{\mathfrak{s}}(K_{\mathfrak{p}^\infty}, T_{f,n})$ *attached to* f, n *and* $K_{\mathfrak{p}^\infty}$ *is the groups of elements* $\kappa \in \hat{H}^1(K_{\mathfrak{p}^\infty}, T_{f,n})$ *satisfying:*

- *For primes* $\mathfrak{q} \subseteq \mathcal{O}_F$ *which do not divide* $\mathfrak{n}\mathfrak{s}p$, $\partial_{\mathfrak{q}}(s) = 0$.
- *For primes* $\mathfrak{q} \subseteq \mathcal{O}_F$ *dividing* $\mathfrak{n}^-$ *exactly,* $\mathrm{res}_{\mathfrak{q}}(s) \in H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$.
- *For primes* $\wp \subseteq \mathcal{O}_F$ *dividing* p, $\mathrm{res}_{\mathfrak{q}}(s) \in H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{f,n})$.
- *For primes* $\mathfrak{q} \subseteq \mathcal{O}_F$ *dividing* $\mathfrak{n}^+\mathfrak{s}$ *and not dividing* p, $\mathrm{res}_{\mathfrak{q}}(s)$ *is arbitrary.*

The *global reciprocity law of class field theory* implies that for any $s \in \mathrm{Sel}_{f,n}(K_{\mathfrak{p}^\infty})$ and any $\kappa \in \hat{H}^1_{\mathfrak{s}}(K_{\mathfrak{p}^\infty}, T_{f,n})$:

$$\sum_{\mathfrak{q}|\mathfrak{s}} \langle \partial_{\mathfrak{q}}(\kappa), v_{\mathfrak{q}}(s) \rangle_{\mathfrak{q}} = 0. \tag{5}$$

Define $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty}) := \lim_{\to n} \mathrm{Sel}_{f,n}(K_{\mathfrak{p}^\infty})$ (direct limits with respect to the inclusion maps). The Selmer group $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})$ has a natural structure of $\Lambda_{\mathfrak{p}^\infty}$-module. For any $\Lambda_{\mathfrak{p}^\infty}$-module M, denote by $M^\vee$ its Pontryagin dual. Then $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})^\vee$ has a characteristic power series which will be denoted by $\mathrm{Char}_{\mathfrak{p}}(f, K)$. Recall the map $\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}$ of Section 2.4, where $\mathfrak{p}_1 \mid \mathfrak{p}_2 \mid p$. The following conjecture is motivated by Corollary 2.7.

Conjecture 3.3. *The following relation holds:*

$$\lambda_{\mathfrak{p}_2/\mathfrak{p}_1}(\mathrm{Char}_{\mathfrak{p}_2}(f, K)) = u[\mathcal{O}^\times_{\mathfrak{c}_1} : \mathcal{O}^\times_{\mathfrak{c}_2}]^2(\alpha_{\mathfrak{p}'} - n_{\mathfrak{p}'})^2 \mathrm{Char}_{\mathfrak{p}_1}(f, K),$$

where $u \in \Lambda^\times_{\mathfrak{p}_1^\infty}$.

Remark. Of course, this conjecture is implied by Corollary 2.7 and Iwasawa's Main Conjecture. Note also that this conjecture does not really imply directly a relation between the Selmer groups $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}_1^\infty})$ and $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}_2^\infty})$. Indeed, $\mathrm{Char}_{\mathfrak{p}}(f, K)$ carries only a part of the information on $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})$, since it does not view its submodules and quotients whose support contains primes of height greatest that 2.

3.2 Iwasawa's Main Conjecture

The main result which will by proved in Section 4 is the following:

Theorem 3.4. *Assume that f_0 satisfies Assumption 1 and f satisfies Assumptions 2 and 3. The characteristic power series* $\mathrm{Char}_{\mathfrak{p}}(f,K)$ *of the Pontryagin dual* $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})^\vee$ *divides the p-adic L-function $L_{\mathfrak{p}}(f,K)$.*

The proof of this result is based on a generalization of the argument in [BD3], which carries over, mutatis mutandis, in the totally real case. So, it will be presented in Section 4 only the sketch of the argument where the necessary adaptations to the totally real case will be pointed out.

3.3 Modular abelian varieties

Let A/F be an abelian variety of GL_2-type, that is, $[\mathrm{End}_{\mathbb{Q}}(A):\mathbb{Q}]=\dim(A)$, where $\mathrm{End}_{\mathbb{Q}}(A):=\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$. Set $E:=\mathrm{End}_{\mathbb{Q}}(A)$ and assume moreover that $\mathrm{End}(A)\simeq\mathcal{O}_E$. For any prime ideal $\pi\subseteq\mathcal{O}_E$, denote by $A[\pi^n]$ the π^n-torsion in A and by $T_\pi(A)$ its π-adic Tate module. Denote by $\rho_{A,\pi}:G_F\rightarrow\mathrm{Aut}(T_\pi(A))\simeq\mathrm{GL}_2(\mathcal{O}_{E,\pi})$ the associated representation. Finally, denote by $\mathfrak{n}_0\subseteq\mathcal{O}_F$ the arithmetic conductor of A/F.

Definition 3.5. *Say that A/F is* modular *if there exists f_0 as in Section 2.2 such that $E=K_{f_0}$ and there exists a prime $\pi\subseteq\mathcal{O}_{f_0}$ such that $\rho_{A,\pi}\simeq\rho_{f_0,\pi}$, where $\rho_{f_0,\pi}$ is the π-adic representation associated to f_0.*

Say that a modular abelian variety A/F is π-*ordinary* at p if the same is true for f_0. This is equivalent to require that A has ordinary reduction at any prime ideal $\wp\subseteq\mathcal{O}_F$ above p.

Let A/F be modular (denote always by π the prime attached to A in Definition 3.5) and p-ordinary. The Selmer group $\mathrm{Sel}_{A,n}(K_{\mathfrak{p}^\infty})$ defined by the exact sequence

$$0\longrightarrow\mathrm{Sel}_{A,n}(K_{\mathfrak{p}^\infty})\longrightarrow H^1(K_{\mathfrak{p}^\infty},A[\pi^n])\longrightarrow\prod_{\mathfrak{q}}\delta_{\mathfrak{q}}(A(K_{\mathfrak{p}^\infty,\mathfrak{q}})/(\pi^n)),$$

where $\delta_{\mathfrak{q}}$ is the local Kummer map and $\mathfrak{q}\subseteq\mathcal{O}_F$ ranges over all prime ideals. Define $\mathrm{Sel}_{A,\infty}(K_{\mathfrak{p}^\infty}):=\lim_{\rightarrow n}\mathrm{Sel}_{A,n}(K_{\mathfrak{p}^\infty})$ (direct limits with respect to the inclusion maps). Then $\mathrm{Sel}_{A,\infty}(K_{\mathfrak{p}^\infty})$ acquires a structure of $\Lambda_{\mathfrak{p}^\infty}$-module. Moreover, $\mathrm{Sel}_{A,\infty}(K_{\mathfrak{p}^\infty})$ contains $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})$ with finite index, so that there is a pseudo-isomorphism of $\Lambda_{\mathfrak{p}^\infty}$-modules:

$$\mathrm{Sel}_{A,\infty}(K_{\mathfrak{p}^\infty})\sim\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty}).$$

In particular, the characteristic power series of their Pontriagin duals are the same. Define $L_{\mathfrak{p}}(A,K):=L_{\mathfrak{p}}(f,K)$. Then:

Corollary 3.6. *Same hypotheses as in Theorem 3.4. The characteristic power series of* $\mathrm{Sel}_{A,\infty}(K_{\mathfrak{p}^\infty})^\vee$ *divides the p-adic L-function* $L_{\mathfrak{p}}(A,K)$.

Remark. Suppose that the same conditions as above hold for all prime ideals $\pi \subseteq \mathcal{O}_E = \mathcal{O}_{f_0}$ dividing p. Since $T_p(A) \simeq \oplus_{\pi|p} T_\pi(A)$, then there is a similar statement as in Corollary 3.6 for the Selmer group attached to the p-torsion of the abelian variety A.

4 The proof

4.1 Admissible primes and rigid pairs

A prime ideal $\ell \subseteq \mathcal{O}_F$ is said to be *n-admissible* if:

1. ℓ does not divide $\mathfrak{n}p$;
2. ℓ is inert in K/F;
3. π does not divide $|\ell|^2 - 1$;
4. π^n divides $|\ell| + 1 + \theta_f(\mathrm{T}_\ell)$ or $|\ell| + 1 - \theta_f(\mathrm{T}_\ell)$.

As in [BD3, Lemma 2.6, 2.7], it is possible to show that $H^1_{\mathrm{sing}}(K_\ell, T_{f,n})$ and $H^1_{\mathrm{fin}}(K_\ell, T_{f,n})$ are both isomorphic to $\mathcal{O}_{f,\pi}/(\pi^n)$ and that $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell}, T_{f,n})$ and $\hat{H}^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell}, T_{f,n})$ are both free of rank one over $\Lambda_{\mathfrak{p}^\infty}/\pi^n\Lambda_{\mathfrak{p}^\infty}$.

Proposition 4.1. *Let* $s \in H^1(K, A_{f,1})$ *be a non-zero element. Then there exist infinitely many admissible primes* ℓ *such that* $\partial_\ell(s) = 0$ *and* $v_\ell(s) \neq 0$.

Proof. Easy generalization of [BD3, Theorem 3.2]. □

Denote by ad^0_f the $k := \mathcal{O}_{f,\pi}/(\pi)$-vector space of trace-zero endomorphisms in $\mathrm{Hom}(A_{f,1}, A_{f,1})$. Let G_F acts on ad^0_f by conjugation of endomorphisms.

Recall the notations of Section 3.1. For all prime ideals $\wp \subseteq \mathcal{O}_F$ dividing p, define $\mathrm{ad}^{0(\wp)}_f$ to be the k-space of trace zero endomorphisms in $\mathrm{Hom}(A^{(1)}_{f,1}, A^{(\wp)}_{f,1})$ and denote by $H^1_{\mathrm{ord}}(F_\wp, \mathrm{ad}^0_f)$ the k-vector space consisting of those classes whose restriction to $H^1(I_\wp, \mathrm{ad}^0_f)$ belongs to $H^1(I_\wp, \mathrm{ad}^{0(\wp)}_f)$, where $I_\wp \subseteq G_{F_\wp}$ is the inertia subgroup. Moreover, if $\mathfrak{q} \subseteq \mathcal{O}_F$ is a prime ideal dividing $\mathfrak{n}$ exactly of residual characteristic different from p, then define $\mathrm{ad}^{0(\mathfrak{q})}_f$ to be the k-space of trace zero endomorphisms in $\mathrm{Hom}(A^{(1)}_{f,1}, A^{(\mathfrak{q})}_{f,1})$ and set $H^1_{\mathrm{ord}}(F_\mathfrak{q}, \mathrm{ad}^0_f) := H^1(F_\mathfrak{q}, \mathrm{ad}^{0(\mathfrak{q})}_f)$. Finally, if ℓ is a 1-admissible prime, denote by $\mathrm{ad}_f^{0(\ell)}$ the unique one dimensional k-vector subspace of ad_f^0 on which $\mathrm{Frob}_F(\ell)$ acts with eigenvalue $|\ell|$ (the existence and uniqueness of ad_f^0 follows because $|\ell|^2 \neq 1$ in k). Then define $H^1_{\mathrm{ord}}(F_\ell, \mathrm{ad}_f^0) := H^1(F_\ell, \mathrm{ad}_f^{0(\ell)})$. For any prime ideal $\mathfrak{q} \nmid \mathfrak{n}p$ define $H^1_{\mathrm{fin}}(F_\mathfrak{q}, \mathrm{ad}_f^0) := H^1(F_\mathfrak{q}^{\mathrm{unr}}/F_\mathfrak{q}, \mathrm{ad}_f^0)$. Then if ℓ is 1-admissible, there is a decomposition in one-dimensional k-vector spaces: $H^1(F_\ell, \mathrm{ad}_f^0) = H^1_{\mathrm{fin}}(F_\ell, \mathrm{ad}_f^0) \oplus H^1_{\mathrm{sing}}(F_\ell, \mathrm{ad}_f^0)$.

Let $\mathfrak{s}$ be a square-free product of 1-admissible primes. Define the $\mathfrak{s}$-*Selmer group* $\mathrm{Sel}_\mathfrak{s}(F, \mathrm{ad}_f^0)$ attached to ad_f^0 to be the k-vector space consisting of those classes $\xi \in H^1(F, \mathrm{ad}_f^0)$ such that:

1. For $\mathfrak{q} \nmid p\mathfrak{n}\mathfrak{s}$, $\mathrm{res}_\mathfrak{q}(\xi) \in H^1_{\mathrm{fin}}(F_\ell, \mathrm{ad}_f^0)$;
2. For $\mathfrak{q} \mid \mathfrak{n}\mathfrak{s}p$ exactly, $\mathrm{res}_\mathfrak{q}(\xi) \in H^1_{\mathrm{ord}}(F_\mathfrak{q}, \mathrm{ad}_f^0)$;
3. For $\mathfrak{q}^2 \mid \mathfrak{n}$ of residual characteristic different from p, the image of $\mathrm{res}_\mathfrak{q}(\xi)$ in $H^1(I_\mathfrak{q}, \mathrm{ad}_f^0)$ is trivial (where $I_\mathfrak{q} \subseteq G_{F_\mathfrak{q}}$ is the inertia subgroup).

Define a pair (ℓ_1, ℓ_2) of n-admissible primes a *rigid pair* if $\mathrm{Sel}_{\ell_1\ell_2}(F, \mathrm{ad}_f^0) = 0$.

Proposition 4.2. *Assume that f is π-isolated. Let ℓ_1 be an n-admissible prime and $s \in H^1(K, A_{f,1})$ be a non-zero class. Then there exists infinitely many admissible primes ℓ_2 such that:*

1. *$\partial_{\ell_2}(s) = 0$ and $v_{\ell_2}(s) \neq 0$;*
2. *Either (ℓ_1, ℓ_2) is a rigid pair or $\mathrm{Sel}_{\ell_2}(F, \mathrm{ad}_f^0)$ is one dimensional over k.*

Moreover, if $\mathrm{Sel}_{\ell_1}(F, \mathrm{ad}_f^0)$ is one-dimensional over k, then there are infinitely many n-admissible primes such that:

1. *$\partial_{\ell_2}(s) = 0$ and $v_{\ell_2}(s) \neq 0$;*
2. *(ℓ_1, ℓ_2) is a rigid pair.*

Proof. This is a generalization of [BD3, Theorems 3.10 and 3.11]. It can be easily performed by replacing:

- The result of Wiles [W2] on universal deformation rings and Hecke algebras with the following result: Denote by R the universal deformation ring attached to deformation of ρ of the representation $G_F \to \mathrm{Aut}(A_{f,1})$ satisfying:

1. The determinant of ρ is the cyclotomic character ϵ describing the action of G_F on the p-power roots of unity;

2. ρ is unramified outside $\mathfrak{n}p$;

3. The restriction of ρ to the inertia $I_\wp \subseteq G_{F_\wp}$ for primes $\wp \mid p$ is of the form $\begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$;

4. For $\mathfrak{q} \mid n$ exactly, the restriction of ρ to a decomposition group at $\mathfrak{q}$ is of the form $\begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$.

Let $\mathfrak{m}_{f,\pi} := \mathrm{Ker}(\mathbb{T}_\mathfrak{n} \xrightarrow{\theta_{f,1}} k)$ and denote (as in Assumption 3) by $\mathbb{T}_f$ the completion of $\mathbb{T}_\mathfrak{n}$ at $\mathfrak{m}_{f,\pi}$. Then R is isomorphic to $\mathbb{T}_f$. This result has been obtained in an unpublished work of Fujwara [Fu]. For precise references and a proof when $[F : \mathbb{Q}]$ is even, see [JM].

• The computations on Selmer groups in [DDT, Section 2] (when $F = \mathbb{Q}$) with the analogous computations which can be found, for example, in [SW] or [JM]. $\square$

4.2 Congruences between modular forms and the Euler system

Fix an n-admissible prime ℓ. By [W1], [Ta] and [Ra], it is known that there exists an Hilbert modular form $f_\ell \in S_2(\mathfrak{n}\ell)$ which is new at ℓ and such that:

- For primes $\mathfrak{q} \nmid \mathfrak{n}\ell$, $\mathrm{T}_\mathfrak{q}(f_\ell) \equiv \theta_f(\mathrm{T}_\mathfrak{q}) f_\ell \pmod{\pi^n}$;
- For primes $\mathfrak{q} \mid \mathfrak{n}$, $\mathrm{U}_\mathfrak{q}(f_\ell) \equiv \theta_f(\mathrm{U}_\mathfrak{q}) f_\ell \pmod{\pi^n}$;
- $\mathrm{U}_\ell(f_\ell) \equiv \epsilon f_\ell \pmod{\pi^n}$, where π^n divides $|\ell| + 1 - \epsilon\theta_f(\mathrm{T}_\mathfrak{q})$.

Denote by $X^{(\ell)}$ the Shimura curve (defined over F) whose complex points are given by $X^{(\ell)}(\mathbb{C}) = \widehat{\mathcal{R}}^\times \widehat{F}^\times \backslash \mathcal{H}^\pm \times \widehat{\mathcal{B}}^\times / \mathcal{B}^\times$, where $\mathcal{H}^\pm := \mathbb{C} - \mathbb{R}$, $\mathcal{B}/F$ is a quaternion algebra of discriminant $\mathfrak{n}^-\ell$ which is ramified in exactly one of the archimedean places and $\mathcal{R} \subseteq \mathcal{B}$ is an Eichler order of level $\mathfrak{n}^+$. Let $J^{(\ell)}$ be the jacobian variety (defined over F) associated to $X^{(\ell)}$. Denote by $T_p(J^{(\ell)})$ the p-adic Tate module of $J^{(\ell)}$ and by Φ_ℓ the group of connected components of the fiber at ℓ of the Néron model of $J^{(\ell)}$ over $\mathcal{O}_K$. Denote by $\mathcal{I}_{f_\ell}$ the kernel of the map $\mathbb{T}_{\mathfrak{n}\ell} \to \mathcal{O}_{f,\pi}/(\pi^n)$ associated to the modular form f_ℓ. The results contained in [L2, Theorem 4.13] when f has rational coefficients can be easily extended to this general case (see [L1, Section 4.8]) proving that there exists a canonical submodule $\mathcal{D}_\ell \subseteq T_p(J^{(\ell)})/\mathcal{I}_{f_\ell}$, such that $\mathcal{D}_\ell \simeq T_{f,n}$ (as Galois modules); moreover, $T_p(J^{(\ell)})$ decomposes (as Galois module) in a direct sum $\mathcal{D}_\ell \oplus \mathcal{D}'_\ell$.

For any $m \geq 0$ fix an Heegner point $\tilde{P}_m \in X^{(\ell)}(\tilde{K}_{\mathfrak{p}^\infty})$ (see [Zh1, Section 2] for the precise definitions: in the notations of [Zh1], $\tilde{P}_m = (1, z)$ where $\mathrm{End}(z) \simeq \mathcal{O}_{\mathfrak{cp}^m}$). Since $\mathcal{I}_{f_\ell}$ is not Eisenstein, there is an isomorphism $J^{(\ell)}(\tilde{K}_{\mathfrak{cp}^m})/\mathcal{I}_{f_\ell} \to \mathrm{Pic}(X^{(\ell)})(\tilde{K}_{\mathfrak{cp}^m})/\mathcal{I}_{f_\ell}$; denote again by $\tilde{P}_m$ the natural image. For all $\wp$, let $\alpha_\wp$ be the unit root of Frob at $\wp$; set $\alpha_{\mathfrak{p}} := \prod_{\wp|\mathfrak{p}} \alpha_\wp$ and $\tilde{P}^*_m := \alpha_\wp^{-m}[\mathcal{O}^\times_{\mathfrak{c}} : \mathcal{O}^\times_{\mathfrak{cp}^m}]\tilde{P}^*_m$. The points $\tilde{P}^*_m$ are norm-compatible. Then their images under Kummer map followed by projection:

$$J^{(\ell)}(\tilde{K}_{\mathfrak{cp}^m})/\mathcal{I}_{f_\ell} \to H^1(\tilde{K}_{\mathfrak{cp}^m}, T_p(J^{(\ell)})/\mathcal{I}_{f_\ell}) \to H^1(\tilde{K}_{\mathfrak{cp}^m}, \mathcal{D}_\ell) \simeq H^1(\tilde{K}_{\mathfrak{cp}^m}, T_{f,n})$$

yields a sequence of cohomology classes, $\tilde{\kappa}_m(\ell)$, which are compatible under corestriction. So, taking limit defines a class $\tilde{\kappa}(\ell) \in \hat{H}^1(\tilde{K}_{\mathfrak{cp}^\infty}, T_{f,n})$. Define the class $\kappa(\ell) \in \hat{H}^1(K_{\mathfrak{p}^\infty}, T_{f,n})$ to be the corestriction of $\tilde{\kappa}(\ell)$ from $\tilde{K}_{\mathfrak{cp}^\infty}$ to $K_{\mathfrak{p}^\infty}$.

Choose distinct n-admissible primes $\ell_1 \nmid \mathfrak{c}$ and $\ell_2 \nmid \mathfrak{c}$ so that p^n divides both $|\ell_1| + 1 - \epsilon_1 a_{\ell_1}(f)$ and $|\ell_2| + 1 - \epsilon_2 a_{\ell_2}(f)$, with ϵ_1, ϵ_2 equal to ± 1. Let $\mathbb{T}_{\ell_1}$ be the Hecke algebra acting on the Shimura curve $X^{(\ell_1)}$ (notations as above). Assume that f is p-isolated. The map arising from Kummer theory composed with the canonical projection as above yields a map:

$$J^{(\ell_1)}(K_{\ell_2})/\mathcal{I}_{f_{\ell_1}} \to H^1(K_{\ell_2}, T_p(J^{(\ell_1)})/\mathcal{I}_{f_{\ell_1}}) \to H^1(K_{\ell_2}, \mathcal{D}_{\ell_1}) \simeq H^1(K_{\ell_2}, T_{f,n})$$

whose image is equal to $H^1_{\mathrm{fin}}(K_{\ell_2}, T_{f,n})$ because both $T_p(J^{(\ell_1)})$ and $T_{f,n}$ are unramified at ℓ_2. For the same reason and the fact that $\ell_2 \nmid p$, the map induced by reduction $\pmod{\ell_2}$: $J^{(\ell_1)}(K_{\ell_2})/\mathcal{I}_{f_{\ell_1}} \to J^{(\ell_1)}(\mathbb{F}_{\ell_2^2})/\mathcal{I}_{f_{\ell_1}}$ is an isomorphism, where $\mathbb{F}_{\ell_2^2}$ is the residue field of the ring of integers of K_{ℓ_2}. The identification $H^1_{\mathrm{fin}}(K_{\ell_2}, T_{f,n}) \simeq \mathcal{O}_{f,\pi}/(\pi^n)$ and the inverse of the above map yield a surjective map:

$$J^{(\ell_1)}(\mathbb{F}_{\ell_2^2})/\mathcal{I}_{f_{\ell_1}} \to \mathcal{O}_{f,\pi}/(\pi^n). \tag{6}$$

Let $\mathcal{S}_{\ell_2} \subseteq X^{(\ell_1)}(\mathbb{F}_{\ell_2^2})$ be the set of supersingular points of $X^{(\ell_1)}$ in characteristic ℓ_2 and let $\mathrm{Div}(\mathcal{S}_{\ell_2})$ and $\mathrm{Div}^0(\mathcal{S}_{\ell_2})$ be the set of formal divisors and the set of formal degree zero divisors with $\mathbb{Z}$-coefficients supported on $\mathcal{S}_{\ell_2}$. Let the Hecke algebra $\mathbb{T}_{\ell_1}$ act on $\mathrm{Div}(\mathcal{S}_{\ell_2})$ and $\mathrm{Div}^0(\mathcal{S}_{\ell_2})$ via Albanese functoriality (it makes no difference if the Picard functoriality were chosen: see the discussion in [BD3, Section 9]). Since $\mathcal{I}_{f_{\ell_1}}$ is not Eisenstein, there is an identification $\mathrm{Div}(\mathcal{S}_{\ell_2})/\mathcal{I}_{f_{\ell_1}} \simeq \mathrm{Div}^0(\mathcal{S}_{\ell_2})/\mathcal{I}_{f_{\ell_1}}$, so there is a map:

$$\gamma : \mathrm{Div}(\mathcal{S}_{\ell_2}) \to \mathcal{O}_{f,\pi}/(\pi^n).$$

Write $\bar{\mathrm{T}}$ for the image of $\mathrm{T} \in \mathbb{T}_{\ell_1}$ into $\mathbb{T}_{\ell_1}/\mathcal{I}_{f_{\ell_1}}$, so that for primes $\mathfrak{q} \nmid \mathfrak{n}\ell_1$, $\bar{\mathrm{T}}_{\mathfrak{q}} \equiv \theta_f(\mathrm{T}_{\mathfrak{q}}) \pmod{\pi^n}$, for primes $\mathfrak{q} \mid \mathfrak{n}$, $\bar{\mathrm{U}}_{\mathfrak{q}} \equiv \theta_f(\mathrm{U}_{\mathfrak{q}}) \pmod{\pi^n}$ and $\bar{\mathrm{U}}_{\ell_1} \equiv \epsilon_1 \pmod{\pi^n}$. An easy generalization of [BD3, Lemma 9.1] shows that, for $x \in \mathrm{Div}(\mathcal{S}_{\ell_2})$ the following relations hold: for $\mathfrak{q} \nmid \mathfrak{n}\ell_1$, $\gamma(\mathrm{T}_{\mathfrak{q}}x) = \bar{\mathrm{T}}_{\mathfrak{q}}\gamma(x)$; for $\mathfrak{q} \mid \mathfrak{n}\ell_1$, $\gamma(\mathrm{U}_{\mathfrak{q}}x) = \bar{\mathrm{U}}_{\mathfrak{q}}\gamma(x)$; $\gamma(\mathrm{T}_{\ell_2}x) = \bar{\mathrm{T}}_{\ell_2}\gamma(x)$; $\gamma(\mathrm{Frob}_{\ell_2}x) = \epsilon_2\gamma(x)$.

Proposition 4.3. *γ is surjective.*

Proof. The proof requires slight modification with respect to the proof of [BD3, Proposition 9.2].

- Instead of considering the subgroup of norm one elements $\Gamma^{(\ell_2)}$ contained in $\mathcal{R}[1/\ell_2]^\times/\{\pm 1\}$ (notations as in [BD3, Proposition 9.2]), consider the product $\Gamma^{(\ell_2)} := \prod_{j=1}^{l} \Gamma_j^{(\ell_1)}$, where $\Gamma_j^{(\ell_2)}$ is the group of norm one elements in $\mathcal{R}_j[1/\ell_2]^\times/\mathcal{O}_F^\times$ (here $\mathcal{R} \subseteq \mathcal{B}$ is the Eichler order of level $\mathfrak{n}^+$ defining $X^{(\ell_1)}$ and $\mathcal{R}_j$ is defined by $\mathcal{R}_j := \mathcal{B} \cap g_j^{-1}\widehat{\mathcal{R}}g_j$, where as usual $\{g_j\}_{j=1,\dots,l}$ is a set of representatives of $\widehat{\mathcal{R}}^\times\backslash\widehat{\mathcal{B}}^\times/\mathcal{B}^\times$).

- The Shimura curve $\tilde{X}$ (notations as in [BD3, Proposition 9.2]) can be defined here in the same way by imposing an extra $\Gamma_1(\mathfrak{p})$-level structure (recall that $\mathfrak{p} \mid \mathfrak{n}^+$). As a consequence, the subgroup $\tilde{\Gamma}^{(\ell_2)}$ (notations as in [BD3, Proposition 9.2]) is defined to be the finite index subgroup of $\Gamma^{(\ell_2)} := \prod_{j=1}^{l} \Gamma_j^{(\ell_1)}$ defined by taking the product of the subgroup of $\mathcal{R}_j[1/\ell_2]^\times/\mathcal{O}_F^\times$ consisting of those elements which are congruent to the standard unipotent matrices (fix isomorphisms $\mathcal{R}_{j,\wp} \simeq \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\wp}\right\}$ for all $\wp \mid \mathfrak{p}$ and all $j = 1, \dots, t$).

- The crucial ingredient in the proof of [BD3, Proposition 9.2] is the analogue in the context of Shimura curves of Ihara's Lemma [Ih]. This result is provided over $\mathbb{Q}$ by [DT, Theorem 2], which establishes that the action of $G_{\mathbb{Q}}$ on a certain module factors through $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. The analogue when $F \neq \mathbb{Q}$ still hold: see [Ja, Section 6].

- The cokernel of the natural map $\tilde{J}^{(\ell_1)}(\mathbb{F}_{\ell_2^2}) \to J^{(\ell_1)}(\mathbb{F}_{\ell_2^2})$ (notations as in [BD3, Proposition 9.2]) has order dividing $p-1$ (hence, prime to p) because it can be identified with the Cartier dual of $\Sigma := \mathrm{Ker}(\tilde{J}^{(\ell_1)} \to J^{(\ell_1)})$ (see for example [Co, Section 7]), which is known to have order $\varphi(p)$ by [Li]. When $F \neq \mathbb{Q}$, the only difference is the analogue of [Li], which can be obtained as follows. Write $X = \coprod \Gamma_{j=1}^{h^+}\backslash\mathcal{H}^+$ and $\tilde{X} = \coprod_{j=1}^{h^+} \tilde{\Gamma}_j\backslash\mathcal{H}^+$, where h^+ is the narrow class number of F and denote by J and $\tilde{J}$ the jacobian varieties of X and $\tilde{X}$. There is a canonical injection: $0 \to \Sigma \to \oplus_{j=1}^{h^+}\mathrm{Hom}(\Gamma_j/\tilde{\Gamma}_j, U)$ where $U = \{x \in \mathbb{C} : |z| = 1\}$. Consider the injection $\iota : \Gamma_j \hookrightarrow \prod_{\wp|\mathfrak{p}} \Gamma_0(\wp)$, where $\Gamma_0(\wp)$ is the subgroup of $\pmod{\wp}$ upper triangular matrices in $\mathrm{GL}_2(\mathcal{O}_{F,\wp})$; write $\iota(\gamma) = \begin{pmatrix} a_\wp & b_\wp \\ c_\wp & d_\wp \end{pmatrix}_\wp$ and define $a_\mathfrak{p} := \prod_{\wp|\mathfrak{p}} a_\wp$, $b_\mathfrak{p} := \prod_{\wp|\mathfrak{p}} b_\wp$. Then there is an exact sequence: $0 \to \tilde{\Gamma}_j \to \Gamma_j \xrightarrow{\phi} (\mathcal{O}_F/\mathfrak{p})^\times \times (\mathcal{O}_F/\mathfrak{p})^\times$ where $\phi(\gamma) = (a_\mathfrak{p}, b_\mathfrak{p})$. It follows that $\#\Sigma \mid \varphi(\mathfrak{p})^2$. $\square$

Let B' be the quaternion algebra of discriminant $\mathrm{Disc}(B') = \mathrm{Disc}(B)\ell_1\ell_2$ and let R' be an Eichler $\mathcal{O}_F[1/\mathfrak{p}]$-order of B' of level $\mathfrak{n}^+$.

Proposition 4.4. *There exists $g \in S_2^{B'}(\mathfrak{n}^+, \mathcal{O}_{f,\pi}/(\pi^n))$ such that:*

- *For prime ideals $\mathfrak{q} \nmid \mathfrak{n}\ell_1\ell_2$, $\mathrm{T}_\mathfrak{q}(g) \equiv \theta_f(\mathrm{T}_\mathfrak{q})g \pmod{\pi^n}$;*
- *For prime ideals $\mathfrak{q} \mid$, $\mathrm{U}_\mathfrak{q}(g) \equiv \theta_f(\mathrm{U}_\mathfrak{q})g \pmod{\pi^n}$;*
- $\mathrm{U}_{\ell_1} g \equiv \epsilon_1 g \pmod{\pi^n}$ *and* $\mathrm{U}_{\ell_2} g \equiv \epsilon_2 g \pmod{\pi^n}$.

Furthermore, if (ℓ_1, ℓ_2) is a rigid pair, then g can be lifted to a π-isolated form in $S_2^{B'}(\mathfrak{n}^+, \mathcal{O}_{f,\pi})$.

Proof. Provided Proposition 4.3 and the results of [Zh2, Section 5] on the description of the set of supersingular points in terms of double coset spaces, the proof is the same as [BD3, Theorem 9.3, Corollary 9.3 and Proposition 3.12]. $\square$

4.3 Explicit reciprocity laws

The two following theorems explore the relations between the classes $\kappa(\ell)$ constructed in Section 4.2 and the p-adic L-functions of Section 2. Assume from now on that $\ell \nmid \mathfrak{c}$ (where $\mathfrak{c}$ is defined as in Section 2.1). Recall the notations for ∂_ℓ and v_ℓ before Definition 3.1.

Theorem 4.5. $v_\ell(\kappa(\ell)) = 0$ *and the equality*

$$\partial_\ell(\kappa(\ell)) \equiv \mathcal{L}_{f,\mathfrak{c}\mathfrak{p}^\infty} \pmod{\pi^n}$$

holds in $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell}, T_{f,n}) \simeq \Lambda_{\mathfrak{p}^\infty}/\pi^n\Lambda_{\mathfrak{p}^\infty}$ up to multiplication by elements in $\mathcal{O}_{f,\pi}^\times$ and G_∞.

Proof. Denote by $\tilde{\partial}_\ell$ the residue map $\hat{H}^1(\tilde{K}_{\mathfrak{cp}^\infty}, T_{f,n}) \to \hat{H}^1_{\mathrm{sing}}(\tilde{K}_{\mathfrak{cp}^\infty,\ell}, T_{f,n})$, the cohomology groups being defined in the obvious way. In is enough to show that $\tilde{\partial}_\ell(\tilde{P}^*_m) \equiv \tilde{\mathcal{L}}_{f,\mathfrak{cp}^\infty}$. The Čerednik-Drinfeld description of the special fiber $X^{(\ell)}_\ell$ at ℓ of the integral model of the Shimura curve $X^{(\ell)}$ (which is recalled in [L2, Sections 4.2, 4.3] or [Zh2, Section 5]) combined with the ℓ-adic description of the image of $\tilde{P}_m$ in $X^{(\ell)}_\ell$ (see [L2, Section 5.2]) imply that $\tilde{P}_m$ can be identified with a pair $(g, z) \in \widehat{R}[1/\ell]^\times \widehat{F}^\times \backslash (\widehat{B}^\times \times \mathcal{H}_\ell)/B^\times \simeq X^{(\ell)}_\ell(\mathbb{C}_\ell)$, where:

- B/F is, as in Section 2.2, the quaternion algebra which is ramified at archimedean places and whose discriminant is $\mathrm{Disc}(B) = \mathfrak{n}^-$;
- $R \subseteq B$ is, as in Section 2.2, an Eichler order of level $\mathfrak{n}^+$ and $\widehat{R}[1/\ell] := \prod_{\mathfrak{q} \neq \ell} R_{\mathfrak{q}} \times B_\ell$;
- $\mathcal{H}_\ell := \mathbb{C}_\ell - F_\ell$ is the ℓ-adic upper half plane, where $\mathbb{C}_\ell$ is the completion of an algebraic closure of F_ℓ;
- z is one of the two fixed points of $\Psi(K^\times)$ acting on $\mathcal{H}_\ell$, where $\Psi \in \mathrm{Hom}(K, B)$ is deduced by reduction of endomorphisms $\mathrm{End}(\tilde{P}_m) \otimes \mathbb{Q} \to \mathrm{End}(\overline{\tilde{P}_m}) \otimes \mathbb{Q}$ (recall that $\tilde{P}_m$ can be described in terms of a certain polarized abelian variety defined over K_ℓ by [Zh1, Section 1]; then $\overline{\tilde{P}_m}$ represents the reduced abelian variety over the residue field of $\mathcal{O}_{K_\ell}$; moreover, the choice of z can be normalized imposing that the action of $\Psi \otimes_F F_\ell$ on the tangent space at P is via the character $z \mapsto z/\tau(z)$, where $\tau \in \mathrm{Gal}(K_\ell/F_\ell)$ is the non-trivial automorphism);
- g satisfies $\mathrm{End}(\overline{\tilde{P}_m}) \otimes_{\mathcal{O}_F} \mathcal{O}_F[1/\ell] \simeq R_g[1/\ell]$ and $R_g[1/\ell] := g^{-1}\widehat{R}[1/\ell]g \cap B$.

Write $X^{(\ell)}_\ell(\mathbb{C}_\ell) \simeq \coprod_{j=1}^s \mathcal{H}_\ell/\Gamma_{\ell,j}$, where $\{g_1, \dots, g_s\}$ is a set of representatives of $\widehat{R}[1/\ell]^\times \widehat{F}^\times \backslash \widehat{B}^\times / B^\times$ and $\Gamma_{\ell,j} := \widehat{g}_j^{-1} R[1/\ell]^\times g_j \cap B$. Note that $P_m(\ell) := (g, z) \in X^{(\ell)}_\ell(K_\ell)$ (this integrality property can be deduced by recalling that, since ℓ is inert in K/F, then it splits completely in anticyclotomic extensions of conductor prime to ℓ (see [Iw])). Consider the natural reduction map: $r_\ell : \mathcal{H}_\ell \to \mathcal{E}_\ell \cup \mathcal{V}_\ell$, where $\mathcal{E}_\ell$ (respectively, $\mathcal{V}_\ell$) is the set of unoriented edges (respectively, the set of vertices) of the Bruhat-Tits tree $\mathcal{T}_\ell$ of $\mathrm{PGL}_2(F_\ell)$. By the Tate-Honda Theorem, the image of z corresponds to a vertex, say v_z (that is, the reduction of z does not correspond to a singular point in the special fiber of $X^{(\ell)}_\ell$). It follows that $r_\ell(P_m(\ell))$ can be identified with a pair $(v_m(\ell), j)$ with $v_m(\ell) \in \mathcal{V}_\ell/\Gamma_\ell$. By the strong approximation theorem, there is an identification between $(\prod_{\wp|p} \overrightarrow{\mathcal{E}}_{\wp})/\Gamma_{\mathfrak{p}} \times \{1,\dots,t\} \simeq \mathcal{V}_\ell/\Gamma_\ell \times \{1,\dots,s\}$ (notations as in Section 2.2 for the first set). Summing up, $r_\ell(P_m(\ell))$ can be identified with an edge $(e_m, j) \in (\prod_{\wp|p} \overrightarrow{\mathcal{E}}_{\wp})/\Gamma_{\mathfrak{p}} \times \{1,\dots,t\}$. By [L2, Section 5.2], the Galois action of $\tilde{G}_{\mathfrak{cp}^\infty}$ on $P_m(\ell)$ is compatible with the action of $\tilde{G}_{\mathfrak{cp}^\infty}$ on the edges (e_m, j) defined in Section 2.3.1. Fix a prime ℓ_∞ of $\tilde{K}_{\mathfrak{cp}^\infty}$ dividing ℓ and set $\ell_m := \ell_\infty \cap K_{\mathfrak{cp}^m}$. Let Φ_ℓ (respectively, $\Phi_{\mathfrak{l}}$) be the group of connected components of the fiber at ℓ (respectively, at $\mathfrak{l}$, where $\mathfrak{l} \mid \ell$ is a prime of some $\tilde{K}_{\mathfrak{cp}^\infty}$) of the Néron model of $J^{(\ell)}$ over $\mathcal{O}_K$ (respectively, over $\mathcal{O}_{\tilde{K}_{\mathfrak{cp}^m}}$). Since ℓ splits completely in $\tilde{K}_{\mathfrak{cp}^\infty}/K$, the choice of ℓ_∞ yields an identification:

$$\Phi_{\ell,m} := \oplus_{\mathfrak{l}|\ell}\Phi_{\mathfrak{l}} \simeq \Phi_\ell[\tilde{G}_{\mathfrak{cp}^m}].$$

By [L2, Propositions 4.10, 4.11], the image of $\tilde{P}_m$ is contained in a canonical component $\mathcal{C}_{\ell_m} \subseteq \Phi_{\ell_m}/\mathcal{I}_{f_\ell}$, such that $\mathcal{C}_{\ell_m} \simeq H^1_{\mathrm{sing}}(K_\ell, T_{f,n})$ corresponds to the singular part of $H^1(K_\ell, \mathcal{D}_\ell)$. For $\sigma \in \tilde{G}_{\mathfrak{cp}^\infty}$, write $\partial_{\ell_m}(\sigma(\tilde{P}_m))$ for the image of $\sigma(\tilde{P}_m)$ in $\mathcal{C}_{\ell_m} \simeq \mathcal{O}_{f,\pi}/(\pi^n)$. Then ∂_{ℓ_m} can be viewed as a map:

$$(\prod_{\wp|p} \overrightarrow{\mathcal{E}}_{\wp})/\Gamma_{\mathfrak{p}} \times \{1,\dots,t\} \to \mathcal{O}_{f,\pi}/(\pi^n).$$

By multiplicity one, this map is equal $\pmod{\pi^n}$ to the modular form f (up to multiplication by an element of $(\mathcal{O}_{f,\pi}/(\pi^n))^\times$). The equality $\partial_{\ell_m}(\sigma(P_m(\ell))) \equiv [\sigma, e_m]_{\mathfrak{p}} \pmod{\pi^n}$ holds for a suitable choice of ℓ_∞ (note that the different choices of ℓ_∞ are permuted by the multiplication by an element of $\tilde{G}_{\mathfrak{cp}^\infty}$, and the same dependence holds for the definition of $\tilde{\mathcal{L}}_{\mathfrak{cp}^\infty}$). The result now follows from the definition of $\tilde{P}^*_m$ and $\tilde{\mathcal{L}}_{\mathfrak{cp}^\infty}$. □

Theorem 4.6. *Let g be as in Proposition 4.4. The equality*

$$v_{\ell_2}(\kappa(\ell_1)) = \mathcal{L}_{g,\mathfrak{cp}^\infty}$$

holds in $\hat{H}^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_2}, T_{f,n}) \simeq \Lambda_{\mathfrak{p}^\infty}/\pi^n\Lambda_{\mathfrak{p}^\infty}$ up to multiplication by elements in $\mathcal{O}^\times_{f,\pi}$ and $G_{\mathfrak{cp}^\infty}$.

Proof. Consider the sequence $\{\tilde{P}_m\}_m$ of Heegner points. Fix (as in the proof of the above theorem) a prime $\ell_{2,\infty}$ of $\tilde{K}_{\mathfrak{cp}^\infty}$ above ℓ_2 and let $\ell_{2,m} := \ell_{2,\infty} \cap \tilde{K}_{\mathfrak{cp}^m}$. Since ℓ_2 is inert in K, the points P_m reduce modulo $\ell_{2,\infty}$ to supersingular points $\bar{P}_m \in X^{(\ell_1)}(\mathbb{F}_{\ell_{2,m}})$, where $\mathbb{F}_{\ell_{2,m}}$ is the reside field of $\tilde{K}_{\mathfrak{cp}^m}$ at $\ell_{2,m}$. Identify $\mathbb{F}_{\ell_{2,m}}$ with $\mathbb{F}_{\ell_2^2}$ for all m. Then $\bar{P}_m$ can be viewed as a point in $\mathcal{S}_{\ell_2}$, and hence, by strong approximation, as a sequence of consecutive edges in $\overrightarrow{\mathcal{E}}_{\ell_2}/\Gamma'$ ($\Gamma' := R'[1/\mathfrak{p}]^\times/\mathcal{O}_F[1/\mathfrak{p}]^\times$, R' being defined before Proposition 4.4). Reduction modulo $\ell_{2,m}$ of endomorphism yields by extension of scalars an embedding $\Psi : K \to B'$, which is independent of m (B'/F is the quaternion algebra defined before Proposition 4.4). The natural Galois action of $\tilde{G}_{\mathfrak{cp}^\infty}$ on $\tilde{P}_m$ is compatible with the action of $\tilde{G}_{\mathfrak{cp}^\infty}$ on the e_m via Ψ. Write:

$$\tilde{\mathcal{L}}_{g,\mathfrak{cp}^m} := \alpha_{\mathfrak{p}}^{-m} \sum_{\sigma \in \tilde{G}_{\mathfrak{cp}^m}} g(\overline{\sigma P_m}) \cdot \sigma \in \mathcal{O}_{f,\pi}/(\pi^n)[\tilde{G}_{\mathfrak{cp}^m}];$$

the sequence $\{\tilde{\mathcal{L}}_{g,m}\}$ is compatible under norms and defines an element $\tilde{\mathcal{L}}_{g,\mathfrak{cp}^\infty} \in \mathcal{O}_{f,\pi}/(\pi^n)[\![\tilde{G}_\infty]\!]$. Define the cohomology groups

$$H^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^m,\ell_2}, T_{f,n}) := \oplus_{\ell_{2,m}|\ell_2} H^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^m,\ell_{2,m}}, T_{f,n}),$$

$$\hat{H}^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^\infty,\ell_{2,\infty}}, T_{f,n}) := \varprojlim_m H^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^m,\ell_{2,m}}, T_{f,n})$$

(inverse limit with respect to the corestriction maps). The choice of $\ell_{2,\infty}$ together with the isomorphism $H^1_{\mathrm{fin}}(K_{\ell_2}, T_{f,n}) \simeq \mathcal{O}_{f,\pi}/(\pi^n)$ yields identification:

$$H^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^m,\ell_2}, T_{f,n}) = \mathcal{O}_{f,\pi}/(\pi^n)[\tilde{G}_{\mathfrak{cp}^m}],$$
$$\hat{H}^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^\infty,\ell_{2,\infty}}, T_{f,n}) = \mathcal{O}_{f,\pi}/(\pi^n)[\![\tilde{G}_{\mathfrak{cp}^\infty}]\!].$$

By the definition of γ, the image of $\tilde{P}^*_m$ in $H^1_{\mathrm{fin}}(\tilde{K}_{\mathfrak{cp}^m,\ell_{2,m}}, T_{f,n})$ corresponds to $\tilde{\mathcal{L}}_{g,\mathfrak{cp}^m}$ (mod π^n) and so the image of the compatible sequence $\{\tilde{P}^*_m\}$ corresponds to $\tilde{\mathcal{L}}_{g,\mathfrak{cp}^\infty}$. Define the class $\tilde{\kappa}(\ell_1)$ to be the image of $\{\tilde{P}^*_m\}_m$ in $\hat{H}^1(\tilde{K}_{\mathfrak{cp}^\infty}, T_{f,n})$. Then $v_{\ell_2}(\tilde{\kappa}(\ell_1)) \in \hat{H}^1_{\mathrm{fin}}(\tilde{K}_{\infty,\ell_2}, T_{f,n})$ is equal to the image of $\{\tilde{P}^*_m\}_m$, and hence to $\tilde{\mathcal{L}}_{g,\mathfrak{cp}^\infty}$ (mod π^n). Since $\kappa(\ell_1)$ is the corestriction of $\tilde{\kappa}(\ell_1)$ from $\tilde{K}_{\mathfrak{cp}^\infty}$ to $K_{\mathfrak{p}^\infty}$, the result follows. □

Corollary 4.7. *The equality*

$$v_{\ell_1}(\kappa(\ell_2)) \equiv v_{\ell_2}(\kappa(\ell_1)) \pmod{\pi^n}$$

holds in $\Lambda_{\mathfrak{p}^\infty}/\pi^n\Lambda_{\mathfrak{p}^\infty}$ *up to multiplication by elements in* $\mathcal{O}^\times_{f,\pi}$ *and* $G_{\mathfrak{p}^\infty}$.

Proof. Since the definition of g is symmetric in ℓ_1 and ℓ_2, this is obvious. □

4.4 The argument

Let $\Lambda := A[\![T_1, \ldots, T_l]\!]$ be a ring of formal power series in $l \geq 1$ variables, where A is a discrete valuation ring. Let X be a finitely generated Λ-module and denote by r its Λ-rank. By [Bo, §4, 4, Théorèmes 4, 5], there exists an exact sequence of Λ-modules: $0 \to A \to X \to \oplus_{i=1}^s \Lambda/(g_i) \times \Lambda^r \to B \to 0$, where A and B are Λ-torsion modules whose support contains only ideals whose height is ≥ 2 and $g_i \in \Lambda$, all $i = 1, \ldots s$. By definition the characteristic power series attached to the Λ-module X is $\mathrm{Char}_\Lambda(X) = g := \prod_{i=1}^s g_i$. If M is a finitely presented module over a ring R, denote by $\mathrm{Fitt}_R(X)$ its Fitting ideal over R.

Proposition 4.8. *Let X be a finitely generated Λ-module and $\mathcal{L} \in \Lambda$. Suppose that $\varphi(\mathcal{L})$ belongs to $\mathrm{Fitt}_{\mathcal{O}}(X \otimes_\varphi \mathcal{O})$ for all homomorphisms $\varphi : \Lambda \to \mathcal{O}$, where $\mathcal{O}$ is a discrete valuation ring. Then $\mathcal{L}$ belongs to $\mathrm{Char}(X)$.*

Proof. If X is not Λ-torsion, then the result follows easily as in [BD3, Proposition 3.1]. So, in the following, assume that X is Λ-torsion. Note that

$$\bigcap_{\mathfrak{P}} \mathrm{Fitt}(X)_{\mathfrak{P}} = (g)$$

in Λ, where $\mathfrak{P}$ ranges over the set of prime ideals of Λ of height ≤ 1 (see [HT, page 101]). Since $\Lambda_{\mathfrak{P}}$ is a discrete valuation ring, by assumption $\mathcal{L} \in \mathrm{Fitt}(X)_{\mathfrak{P}}$ for all $\mathfrak{P}$; it follows that $g \mid \mathcal{L}$. $\square$

The rest of the section is devoted to sketch the proof of Theorem 3.4. Being $\mathfrak{p}$ fixed, denote $\mathrm{Sel}_{f,\infty}(K_{\mathfrak{p}^\infty})$ (respectively, $\mathrm{Sel}_{f,n}(K_{\mathfrak{p}^\infty})$) simply by $\mathrm{Sel}_{f,\infty}$ (respectively, $\mathrm{Sel}_{f,n}$). By Proposition 4.8, it is enough to show that $\varphi(\mathcal{L}_f)^2$ belongs to $\mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,\infty}^\vee \otimes_\varphi \mathcal{O})$ for all $\varphi \in \mathrm{Hom}(\Lambda, \mathcal{O})$ where $\mathcal{O}$ is a discrete valuation ring. For this, it is enough to show that $\varphi(\mathcal{L}_f)^2$ belongs to $\mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,n}^\vee \otimes_\varphi \mathcal{O})$ for all $n \geq 1$. Fix $\mathcal{O}$ and φ as above. Write ν for an uniformizer of $\mathcal{O}$. Set

$$t_f := \mathrm{ord}_\nu(\varphi(\mathcal{L}_f)).$$

If $\varphi(\mathcal{L}_f) = 0$, then $\varphi(\mathcal{L}_f)^2$ belongs trivially to $\mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,n}^\vee \otimes_\varphi \mathcal{O})$ for all $n \geq 1$, so assume $\varphi(\mathcal{L}_f) \neq 0$. Moreover, if $\mathrm{Sel}_{f,\infty}^\vee \otimes_\varphi \mathcal{O}$ is trivial, then its Fitting ideal is equal to $\mathcal{O}$ and, again, $\varphi(\mathcal{L}_f)^2$ belongs trivially to $\mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,n}^\vee \otimes_\varphi \mathcal{O})$ for all $n \geq 1$, so assume that $\mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,n}^\vee \otimes_\varphi \mathcal{O}) \neq 0$. The theorem is proved now by induction on t_f.

Step I: Construction of $\kappa_\varphi(\ell)$. Let ℓ be any $(n + t_f)$-admissible prime and enlarge $\{\ell\}$ to a $(n + t_f)$-admissible set S: such a set consists of distinct $(n + t_f)$-admissible primes such that the map

$$\mathrm{Sel}_{f,n+t_f}(K) \to \oplus_{\ell \in S} H^1_{\mathrm{fin}}(K_\ell, A_{f,n+t_f})$$

is injective (Proposition 4.1 shows that such a set exists). Denote by $\mathfrak{s}$ the square-free product of the primes in S and let $\kappa(\ell) \in \hat{H}^1_\ell(K_{\mathfrak{p}^\infty}, T_{f,n+t_f}) \subseteq \hat{H}^1_{\mathfrak{s}}(K_{\mathfrak{p}^\infty}, T_{f,n+t_f})$ be the cohomology class attached to ℓ. Let $\kappa_\varphi(\ell)$ be the natural image of this class in $\mathcal{M} := \hat{H}^1_{\mathfrak{s}}(K_\infty, T_{f,n+t_f}) \otimes_\varphi \mathcal{O}$. By Theorem 4.5,

$$\mathrm{ord}_\nu(\kappa_\varphi(\ell)) \leq \mathrm{ord}_\nu(\partial_\ell(\kappa_\varphi(\ell))) = \mathrm{ord}_\nu(\varphi(\mathcal{L}_f)). \tag{7}$$

Choose an element $\tilde{\kappa}_\varphi(\ell) \in \mathcal{M}$ so that $\nu^t \tilde{\kappa}_\varphi(\ell) = \kappa_\varphi(\ell)$. This element is well defined modulo the ν^t-torsion subgroup of $\mathcal{M}$; to remove this ambiguity, denote by $\kappa'_\varphi(\ell)$ the natural image of $\tilde{\kappa}_\varphi(\ell)$ in $H^1_{\mathfrak{s}}(K_{\mathfrak{p}^\infty}, T_{f,n}) \otimes_\varphi \mathcal{O}$. The following properties of $\kappa'_\varphi(\ell)$ holds:

1. $\mathrm{ord}_\nu(\kappa'_\varphi(\ell)) = 0$ (because $\mathrm{ord}_\nu(\kappa_\varphi(\ell)) = t \leq t_f$);
2. $\partial_{\mathfrak{q}}(\kappa'_\varphi(\ell)) = 0$ for all $\mathfrak{q} \nmid \ell\mathfrak{n}^-$ (because $\kappa(\ell) \in \hat{H}^1_{\mathfrak{s}}(K_{\mathfrak{p}^\infty}, T_{f,n+t_f})$);
3. $v_\ell(\kappa'_\varphi(\ell)) = 0$ (by Theorem 4.5);
4. $\partial_\ell(\kappa'_\varphi(\ell)) = t_f - t$ (by Theorem 4.5 and formula (7)). Moreover, the element $\partial_\ell(\kappa'_\varphi(\ell))$ belongs to the kernel of the natural homomorphism:

$$\eta_\ell : \hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell}, T_{f,n}) \otimes_\varphi \mathcal{O} \to \mathrm{Sel}^\vee_{f,n} \otimes_\varphi \mathcal{O}. \tag{8}$$

To prove this statement use the global reciprocity law of class field theory (5) and the definition of $\kappa'_\varphi(\ell)$ (for details, see [BD3, Lemma 4.6]).

Step II: Case of $t_f = 0$. This is the basis for the induction argument. If $t_f = 0$, that is, $\mathcal{L}_f$ is a unit, then $\mathrm{Sel}^\vee_{f,n} = 0$. To prove this, note that, for all n-admissible primes ℓ, Theorem 4.5 implies that $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell}, T_{f,n}) \otimes \mathcal{O}$ is generated by $\partial_\ell(\kappa_\varphi(\ell))$ (as $\mathcal{O}$-module) and that the map η_ℓ in (8) is trivial. Assume now that $\mathrm{Sel}^\vee_{f,n}$ is not trivial. Then Nakayama's lemma implies that the group $\mathrm{Sel}^\vee_{f,n}/\mathfrak{m} = (\mathrm{Sel}_{f,n}[\mathfrak{m}])^\vee$ is not trivial, where $\mathfrak{m}$ is the maximal ideal of $\Lambda_{\mathfrak{p}^\infty}$. Let now $s \in \mathrm{Sel}_{f,n}[\mathfrak{m}]$ be a non trivial element. By Assumption 3, ρ_f is residually irreducible. This can be used to show that there is an isomorphism $H^1(K, A_{f,1}) \to H^1(K_{\mathfrak{p}^\infty}, A_{f,n})[\mathfrak{m}]$ (see [BD3, Theorem 3.4] for details), which allows to look at s as an element of $H^1(K, A_{f,1})$. Invoke Proposition 4.1 to choose an n-admissible prime $\ell \nmid \mathfrak{c}$ so that $\partial_\ell(s) = 0$ and $v_\ell(s) \neq 0$. Then the non degeneracy of the local Tate pairing implies that η_ℓ is trivial, which is a contradiction.

Step III: The minimality property. Let now Π be the set of primes of $\mathcal{O}_F$ so that:

- ℓ is $n + t_f$-admissible;
- The number $t = \mathrm{ord}_\nu(\kappa_\varphi(\ell))$ is minimal among the set of $n + t_f$-admissible primes.

By Proposition 4.1, $\Pi \neq \emptyset$. Let t be the common value of $\mathrm{ord}_\nu(\kappa_\varphi(\ell))$ for $\ell \in \Pi$. Then $t < t_f$. This assertion can be proved by an argument, similar to that used in Step II, which combines Proposition 4.1, the properties of the map η_ℓ and the non degeneracy of the local Tate pairing (for details, see [BD3, Lemma 4.8]).

Step IV: Rigid pairs with the minimality property. This step is devoted to the proof that there exist primes $\ell_1, \ell_2 \in \Pi$ so that (ℓ_1, ℓ_2) is a rigid pair. To prove this, start by choosing any prime $\ell_1 \in \Pi$ and denote by s the natural image of $\kappa'_\varphi(\ell_1)$ in $(\hat{H}^1_{\mathfrak{s}}(K_\infty, T_f[\pi^n])/\mathfrak{m}) \otimes_\varphi \mathcal{O}/(\nu)$, where $\mathfrak{m}$ is the maximal ideal of $\Lambda_{\mathfrak{p}^\infty}$. The argument in [BD1, Theorem 3.2], generalized to this situation as suggested in [BD3, Proposition 3.3, Theorem 3.4], allows to view s as a non-zero element in $H^1(K, T_{f,1}) \otimes_\varphi \mathcal{O}/(\nu)$. Note that $\partial_{\mathfrak{q}}(s) = 0$ for all $\mathfrak{q} \nmid \ell_1\mathfrak{n}$. By Proposition 4.2 choose a $n + t_f$ admissible prime $\ell_2 \nmid \mathfrak{c}$ so that $\partial_{\ell_2}(s) = 0$, $v_{\ell_2}(s) \neq 0$ and either (ℓ_1, ℓ_2) is a rigid pair or $\mathrm{Sel}_{\ell_2}(F, W_f)$ is one-dimensional. The following relation hold:

$$t = \mathrm{ord}_\nu(\kappa_\varphi(\ell_1)) \leq \mathrm{ord}_\nu(\kappa_\varphi(\ell_2)) \leq \mathrm{ord}_\nu(v_{\ell_1}(\kappa_\varphi(\ell_2))). \tag{9}$$

The first inequality follows from the minimality property using that $\ell_1 \in \Pi$ and that ℓ_2 is a $n + t_f$-admissible prime using the minimality assumption on t, while the second is clear. By the choice of ℓ_2 and Corollary 4.7, $\mathrm{ord}_\nu(v_{\ell_1}(\kappa_\varphi(\ell_2))) = \mathrm{ord}_\nu(v_{\ell_2}(\kappa_\varphi(\ell_1)))$. Now note that $\mathrm{ord}_\nu(v_{\ell_2}(\kappa_\varphi(\ell_1))) \geq \mathrm{ord}_\nu(\kappa_\varphi(\ell_1))$ and that the strict inequality holds if and only if $v_{\ell_2}(s) = 0$, so, since $v_{\ell_2}(s) \neq 0$, $\mathrm{ord}_\nu(v_{\ell_1}(\kappa_\varphi(\ell_2))) = \mathrm{ord}_\nu(\kappa_\varphi(\ell_1))$. Combining this with the inequalities in formula (9) shows that:

$$t = \mathrm{ord}_\nu(\kappa_\varphi(\ell_1)) = \mathrm{ord}_\nu(\kappa_\varphi(\ell_2)). \tag{10}$$

It follows that $\ell_2 \in \Pi$. If (ℓ_1, ℓ_2) is not a rigid pair, then $\mathrm{Sel}_{\ell_2}(F, W_f)$ is one dimensional. In this case, by Proposition 4.2, choose a $n + t_f$ admissible prime $\ell_2 \nmid \mathfrak{c}$ so that $\partial_{\ell_2}(s) = 0$, $v_{\ell_2}(s) \neq 0$ and (ℓ_2, ℓ_3) is a rigid pair. Repeat the argument above with ℓ_2 replacing ℓ_1 and ℓ_3 replacing ℓ_2 to show that $\ell_3 \in \Pi$. In any case then, either (ℓ_1, ℓ_2) or (ℓ_2, ℓ_3) is a rigid pair and the claim at the beginning of Step IV follows.

Step V: The congruence argument. Choose by the above considerations a rigid pair (ℓ_1, ℓ_2) with $\ell_1, \ell_2 \in \Pi$. Note that, by Proposition 4.2,

$$t = t_g = \mathrm{ord}_\nu(\mathcal{L}_g) \tag{11}$$

(here g is the congruent modular form attached to (ℓ_1, ℓ_2) by Proposition 4.4). There is an exact sequence of Λ-modules:

$$0 \to \mathrm{Sel}^f_{\ell_1\ell_2} \to \mathrm{Sel}^\vee_{f,n} \to \mathrm{Sel}^\vee_{[\ell_1,\ell_2]} \to 0, \tag{12}$$

where $\mathrm{Sel}_{[\ell_1,\ell_2]} \subseteq \mathrm{Sel}_{f,n}$ is defined by the condition that the restriction at the primes ℓ_1 and ℓ_2 must be trivial and $\mathrm{Sel}^f_{\ell_1\ell_2}$ is the kernel of the natural surjection of duals. There is a natural inclusion:

$$(\mathrm{Sel}^f_{\ell_1\ell_2})^\vee \subseteq H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \oplus H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_2}, A_{f,n}).$$

The dual of $H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \oplus H^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_2}, A_{f,n})$, by the non-degeneracy of the local Tate pairing, is $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \oplus \hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell_2}, A_{f,n})$, so the above inclusion leads to a surjection:

$$\eta_f : \hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \oplus \hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \to \mathrm{Sel}^f_{\ell_1\ell_2}.$$

Recall that, since ℓ_1 is n-admissible, $\hat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \simeq \Lambda_{\mathfrak{p}^\infty}/(\pi^n)$. Let η_f^φ be the map induced by η_f after tensoring by $\mathcal{O}$ via φ. Then the domain of η_f^φ is isomorphic to $(\mathcal{O}/\varphi(\pi)^n)^2$. By property 4 above enjoyed by the classes $\kappa'_\varphi(\ell_1)$ and $\kappa'_\varphi(\ell_2)$, the kernel of η_f^φ contains $(\partial_{\ell_1}\kappa'_\varphi(\ell_1), 0)$ and $(0, \partial_{\ell_2}\kappa'_\varphi(\ell_2))$. The same property combined with equations (10) and (11) yields:

$$t_f - t_g = \mathrm{ord}_\nu(\partial_{\ell_1}\kappa'_\varphi(\ell_1)) = \mathrm{ord}_\nu(\partial_{\ell_2}(\kappa'_\varphi(\ell_2)).$$

It follows that:

$$\nu^{2(t_f - t_g)} \quad \text{belongs to the Fitting ideal of} \quad \mathrm{Sel}^f_{\ell_1\ell_2} \otimes_\varphi \mathcal{O}. \tag{13}$$

Repeat now the argument with the modular form g: there is an exact sequence:

$$0 \to \mathrm{Sel}^g_{\ell_1\ell_2} \to \mathrm{Sel}^\vee_{g,n} \to \mathrm{Sel}^\vee_{[\ell_1,\ell_2]} \to 0,$$

and a natural surjection:

$$\eta_g : \hat{H}^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \oplus \hat{H}^1_{\mathrm{fin}}(K_{\mathfrak{p}^\infty,\ell_1}, A_{f,n}) \to \mathrm{Sel}^g_{\ell_1\ell_2}.$$

Let η_g^φ be the map induced by η_g after tensoring by $\mathcal{O}$ via φ. By the global reciprocity law of class field theory, the kernel of η_g^φ contains the elements

$$(v_{\ell_1}(\kappa'_\varphi(\ell_1)), v_{\ell_2}(\kappa'_\varphi(\ell_1))) = (v_{\ell_1}(\kappa'_\varphi(\ell_1)), 0),$$
$$(v_{\ell_1}(\kappa'_\varphi(\ell_2)), v_{\ell_2}(\kappa'_\varphi(\ell_2))) = (0, v_{\ell_2}(\kappa'_\varphi(\ell_2))),$$

where the equalities follow from property 3 above enjoyed by the classes $\kappa'_\varphi(\ell_1)$ and $\kappa'_\varphi(\ell_2)$. Note that $\mathrm{ord}_\nu(v_{\ell_2}\kappa'_\varphi(\ell_1)) = \mathrm{ord}_\nu(v_{\ell_1}\kappa'_\varphi(\ell_2)) = t_g - t = 0$. From this it follows that the module $\mathrm{Sel}^g_{\ell_1\ell_2}$ is trivial, so, the natural surjection

$$\mathrm{Sel}^\vee_{g,n} \otimes_\varphi \mathcal{O} \to \mathrm{Sel}^\vee_{[\ell_1\ell_2]} \otimes_\varphi \mathcal{O} \quad \text{is an isomorphism.} \tag{14}$$

Step VI: The inductive argument. Now assume that the theorem is true for all $t' < t_f$ and prove that it is true for t_f. Recall that $t = t_g < t_f$. Moreover, since (ℓ_1, ℓ_2) is a rigid pair, the modular form g satisfy the assumptions in the theorem, so, by inductive hypothesis,

$$\varphi(\mathcal{L}_g) \text{ belongs to the Fitting ideal of } \mathrm{Sel}_{g,n}^{\vee} \otimes_{\varphi} \mathcal{O}. \tag{15}$$

Now use the theory of Fitting ideals:

$$\begin{aligned} \nu^{2t_f} &= \nu^{2(t_f - t_g)} \nu^{2t_g} \\ &\in \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{\ell_1\ell_2}^{f} \otimes_{\varphi} \mathcal{O}) \cdot \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{g,n}^{\vee} \otimes_{\varphi} \mathcal{O}), \text{ by (13) and (15)} \\ &= \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{\ell_1\ell_2}^{f} \otimes_{\varphi} \mathcal{O}) \cdot \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{[\ell_1\ell_2]}^{\vee} \otimes_{\varphi} \mathcal{O}), \text{ by (14)} \\ &\subseteq \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,n}^{\vee} \otimes_{\varphi} \mathcal{O}), \text{ by (12).} \end{aligned}$$

Since by definition $\mathrm{ord}(\mathcal{L}_f) = t_f$, it follows that $\varphi(\mathcal{L}_f)^2 \in \mathrm{Fitt}_{\mathcal{O}}(\mathrm{Sel}_{f,n}^{\vee} \otimes_{\varphi} \mathcal{O})$, proving the result.

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