HIGH ENERGY BEHAVIOUR AND GAUGE SYMMETRY

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ABSTRACT

A wide class of theories is studied in which
massive self-interacting vector mesons are coupled
to fermions. Necessary and sufficient conditions
are derived which ensure that all four-point
functions respect unitarity bounds (up to log s)
in Born approximation. It is shown that a Yang-
Mills structure is necessary and that there must
exist scalar mesons with couplings intimately
connected with the masses of the particles with
which they interact. The coupling constants
satisfy relations which demonstrate the necessary
existence of a "hidden" representation of the
underlying symmetry group; this representation
has dimension equal to the number of scalar part-
icles plus the number of massive vector particles.
In simple cases there must be a "spontaneously
broken" gauge symmetry and the unique theory (in
the class studied) which respects unitarity is
of the Higgs type.

Intuition, based on dispersive calculations of Feynman diagrams, suggests that there is an intimate connection between the high energy behaviour of amplitudes and renormalizability. In the standard renormalizable field theories (QED, ϕ^3, etc.) it is obvious that the behaviour of the invariant amplitudes for N point tree diagrams is bounded by E^{4-N} when the energy $E \to \infty$ with the ratio of all invariants fixed; J.S. Bell has recently proved [1] that this is also true in spontaneously broken gauge theories. Whether or not this "good" high energy behaviour is necessary for renormalizability, it is perhaps reasonable to insist that the four-point functions do not grow as $E \to \infty$. Otherwise the corresponding partial wave amplitudes would exceed their (constant) unitarity limits above some energy, at which perturbation theory would necessarily fail [*].

In this note we study the necessary conditions that in Born approximation all four-point functions are "well behaved" (do not grow as $E \to \infty$) in theories in which massive vector mesons interact with fermions. We assume that there are no interactions characterized by coupling constants with inverse mass dimensions, as is the case in all known renormalizable theories [**]. Weinberg has stressed [2] that cancellations improve the high energy behaviour in the model invented by himself [3] and Salam [4]. Subsequently, several authors have "derived" various properties of simple models by insisting that certain terms cancel and it is known that, in general, the removal of the terms which are worst behaved in fermion-antifermion annihilation to two vector mesons ($F\bar{F} \to WW$) requires the F-W interaction to be that of a Yang-Mills theory [5],[6]. The constraints required to cancel the next to leading (but still growing) terms in $F\bar{F} \to WW$ and to ensure good behaviour in $WW \to WW$ had not been studied previously. After completing the work described here [***], however, I received a preprint from Cornwall, Levin and Tiktopoulos [8] who have independently addressed the same problem. Their work differs from mine mainly in that they also studied some five-point functions and deduced information about the self-interactions of their scalar fields, but they did not study fermions nor did they go so far in revealing the underlying symmetry in the general case (there are some other differences and some disagreements which will be noted below).

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[*] Even if the invariant amplitude is constant asymptotically the partial wave amplitude may grow logarithmically. This occurs in some renormalizable theories but the energy at which unitarity is violated [$s \sim M^2 \cdot \exp(1/g^2)$] is very large for a small coupling constant g.

[**] Without this assumption we can get nowhere; "bad" high energy behaviour can be cancelled in an ad hoc way by brutally adding new contact interactions to the Lagrangian.

[***] The results were described very briefly in very general terms in Ref. 7) and in somewhat more detail in lectures delivered at the Advanced School of Physics, Frascati, March 1973.

Let us represent all fundamental fermions (whose number is not specified) by a vector *)

$$\psi = \begin{pmatrix} \nu_e \\ e \\ \nu_\mu \\ \mu \\ \vdots \\ ? \end{pmatrix}$$

We suppose that there are g vector mesons represented by real fields W_μ^i ($i = 1,\ldots,g$) and write the $\overline{FF}$-W interaction in the form :

$$\mathcal{L}_1 = \overline{\psi}_\alpha \; \gamma^\mu \left[L_{\alpha\beta}^i \left(\frac{1-\gamma_5}{2} \right) + R_{\alpha\beta}^i \left(\frac{1+\gamma_5}{2} \right) \right] \psi_\beta \; W_\mu^i \tag{1}$$

where L and R are Hermitean matrices, a summation over repeated indices is implied (here and henceforth) and the notation is that employed by Bjorken and Drell in their book. Since one of the W's will eventually be identified with the photon, there must be a W self-interaction (e.g., a γW^+W^- vertex must exist). The most general form which satisfies our assumptions is :

$$\begin{aligned}
\mathcal{L}_2 = \; & \frac{1}{2} \, \mathcal{D}_{ij,k} \, W_\mu^k \left(W_\alpha^j \, \partial_\mu W_\alpha^i - W_\alpha^i \, \partial_\mu W_\alpha^j \right) \\
& + \frac{1}{2} \, \mathcal{D}_{ij,k}^+ \, W_\mu^k \left(W_\alpha^j \, \partial_\mu W_\alpha^i + W_\alpha^i \, \partial_\mu W_\alpha^j \right) \\
& + \frac{1}{2} \, G_{ij,k} \, \epsilon^{\alpha\beta\gamma\delta} \, W_\alpha^i \, W_\beta^j \, \partial_\gamma W_\delta^k \\
& + \frac{1}{4} \, C_{ij,k\ell} \, W_\mu^i \, W_\mu^j \, W_\nu^k \, W_\nu^\ell \\
& + \frac{1}{4} \, S_{ijk\ell} \, \epsilon^{\alpha\beta\gamma\delta} \, W_\alpha^i \, W_\beta^j \, W_\gamma^k \, W_\delta^\ell
\end{aligned} \tag{2}$$

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*) ν_μ, μ may be replaced by $\overline{\nu}_\mu$, $\overline{\mu}$ depending on what form of conservation laws we wish to embody in the theory. We do not consider the more general possibility of putting (e.g.) both μ^- and μ^+ in ψ, which would allow processes such as $W^{--} \to \mu^-\mu^-$.

where the couplings are all real and are supposed to have been fully (anti)
symmetrized in an appropriate way.

For the moment we suppose that all the W's are massive. Straight-
forward but tedious calculations show that the conditions

$$D^+ = S = G = 0$$

$$D_{ij,k} = -D_{ik,j} \ (\equiv D_{ijk})$$

$$2\,C_{ij,k\ell} = D_{i\ell p}D_{kjp} + D_{ikp}D_{\ell jp}$$

(3)

are both sufficient and necessary to remove all the badly behaved pieces of
the Born terms for $WW \to WW$, except for terms which grow like E^2 when all
four W's are longitudinal and like E when three are longitudinal and one
transverse (we return to these residual terms below). Equations (3) specify
a Yang-Mills theory.

Consider now $\bar{F}F \to WW$. Given that the 3W vertex has the Yang-Mills
form, the leading (E^2) pieces cancel if and only if the coupling constants
represent a Lie algebra [5],[6] :

$$[L^i, L^j] = i\,D_{ijk}\,L^k$$

$$[R^i, R^j] = i\,D_{ijk}\,R^k.$$

(4)

The relevant diagrams are shown in Fig. 1, where the origin of each term is
indicated for the case of left-handed leptons. Non-leading ($\sim E$) terms
necessarily remain unless either all fermions are massless or all fermions
in a given irreducible multiplet are degenerate and parity is conserved
[this can be inferred from Eq. (6) below], which would not be interesting
for physics. Additional particles must therefore be exchanged and, if we
wish to avoid the vicious problems associated with particles with spin $\geq \frac{3}{2}$,
they must have spin zero *).

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*) We introduce these scalar particles in connection with the process
$\bar{F}F \to WW$ where good high energy behaviour seems to be necessary to
ensure that the box diagram for $\bar{F}F \to \bar{F}F$ is renormalizable. We then
adjust their contributions to $WW \to WW$ to give good high energy beha-
viour there [but note that this is not necessary for renormalizability
at the one loop level, at which massive Yang-Mills theory is known to
be renormalizable [9]]. The rôle of scalar particles in cancelling
divergences in one-loop diagrams was (to my knowledge) first discussed
in detail in a model of the Higgs type in Ref. 10) ; the associated
improvement of high energy behaviour was noted explicitly in Ref. 11).

Using real scalar fields ϕ^i $(i = 1,\ldots,N)$ the most general F–ϕ–W interaction is :

$$\mathcal{L}_3 = \overline{\psi}_\alpha \left[C^b_{\alpha\beta} \left(\frac{1+\gamma_5}{2}\right) + C^{+b}_{\alpha\beta} \left(\frac{1-\gamma_5}{2}\right) \right] \psi_\beta \, \phi^b$$

$$+ \frac{1}{2} W^i_\mu W^j_\mu \, \phi^b \, K^b_{ij}$$

$$+ \frac{1}{2} T^i_{ba} \, W^i_\mu \, (\phi^a \partial_\mu \phi^b - \phi^b \partial_\mu \phi^a)$$

$$+ \frac{1}{4} M^{ij}_{ab} \, W^i_\mu W^j_\mu \, \phi^a \phi^b \tag{5}$$

except for a term $W^i_\mu(\phi^a \partial_\mu \phi^b + \phi^b \partial_\mu \phi^a)$ which we have omitted because it would necessarily give rise to bad behaviour in $\phi\phi \to \phi\phi$ (its absence ensures that $\phi\phi \to \phi W$ is well behaved)[*]. The couplings in Eq. (5) are real and are supposed to have been fully (anti) symmetrized. The additional conditions which are both necessary and sufficient to guarantee that $F\overline{F} \to WW$ and $WW \to WW$ are well behaved in all cases are [**]

$$2(m R^j - L^j m) R^i - 2 L^i (m R^j - L^j m)$$

$$= - C^b K^b_{ij} - i \frac{D_{ijk}}{M_k^2} (M_k^2 + M_j^2 - M_i^2)(m R^k - L^k m) \tag{6}$$

$$K^\alpha_{ab} K^\alpha_{cd} - K^\alpha_{ad} K^\alpha_{cb} + \frac{D_{kab}}{M_k} (M_k^2 + M_a^2 - M_b^2) \frac{D_{kcd}}{M_k} (M_k^2 + M_c^2 - M_d^2)$$

$$- \frac{D_{kad}}{M_k} (M_k^2 + M_a^2 - M_d^2) \frac{D_{kcb}}{M_k} (M_k^2 + M_c^2 - M_b^2) \tag{7}$$

$$= 2 D_{kbd} D_{kac} (M_a^2 + M_c^2 - M_k^2)$$

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[*] According to our general assumptions there are only ϕ^3 and ϕ^4 interactions - but we learn nothing about them (nor about the masses of these particles) from studying four-point functions in Born approximation.

[**] Our Eq. (7) differs slightly from the corresponding equation in Ref. 8) which we believe to be wrong. This, together with the retention of the $W^\mu(\phi^a \partial_\mu \phi^b + \phi^b \partial_\mu \phi^a)$ coupling, makes it impossible to recover the underlying symmetry from the equations in Ref. 8) in the general case.

where M_i are the vector meson masses and m is the diagonal matrix of fermion masses.

The necessary and sufficient conditions for good behaviour in $\emptyset\emptyset \to WW$, $\emptyset W \to WW$ and $F\bar{F} \to \emptyset W$ are :

$$\left[T^i, T^j\right]_{ba} - \frac{1}{4M_k^2}\left(K_{ik}^b K_{kj}^a - K_{ik}^a K_{kj}^b\right) = D_{kij} T_{ba}^k \tag{8}$$

$$\left\{T^i, T^j\right\}_{ba} - \frac{1}{4M_k^2}\left(K_{ik}^b K_{kj}^a + K_{ik}^a K_{kj}^b\right) = -M_{aL}^{ij} \tag{9}$$

$$T_{ak}^b K_{cd}^k - T_{ak}^d K_{bc}^k + \frac{K_{bk}^a D_{kdc}\left(M_k^2 + M_c^2 - M_d^2\right)}{2M_k^2}$$

$$- \frac{K_{dk}^a D_{kbc}\left(M_k^2 + M_c^2 - M_L^2\right)}{2M_k^2} = D_{kbd} K_{ck}^a \tag{10}$$

$$L^i C^a - C^a R^i = \frac{K_{ki}^a}{2M_k^2}\left(m R^k - L^k m\right) - i C^b T_{ab}^i \tag{11}$$

where $\left\{A, B\right\} = AB + BA$.

The diagrams which contribute to the "commutator-like" Eq. (8) are exhibited in Fig. 2, where the origin of each term is illustrated.

We now make various remarks on our results and especially on the "cancellation conditions" [Eqs. (3), (4), (6)-(11)] :

1. The only theories which have well behaved four-point Born amplitudes and satisfy our assumptions are those in which the W-W and W-F interactions have the Yang-Mills form.

2. There must be scalar particles coupled to fermions and W's in a way which is intimately connected with the masses. The conditions which the coupling constants must satisfy can be re-arranged in a compact form if we define

$$\mathcal{B}^a_{ci} = - \frac{K^i_{ac}}{2 M_c}$$

$$A^d_{tc} = \frac{\mathcal{D}_{tcd}}{2 M_t M_c} (M_d^2 - M_t^2 - M_c^2) \tag{12}$$

$$X^a_{ij} = \left(\frac{m_i \mathcal{R}^a_{ij} - \mathcal{L}^a_{ij} m_j}{M_a} \right) \quad (a \leq g)$$

$$= - i \, C^{a-g}_{ij} \quad (a = g+1, \cdots N)$$

Then, defining real antisymmetric $(N+g) \times (N+g)$ matrices P^a which may be written in partitioned form

$$\mathcal{P}^a = \left(\begin{array}{c|c} A^a & \mathcal{B}^a \\ \hline -\mathcal{B}^{aT} & T^a \end{array} \right) \begin{array}{c} \uparrow \\ g \\ \downarrow \\ \uparrow \\ N \\ \downarrow \end{array} \tag{13}$$
$$\leftarrow g \rightarrow \; \leftarrow N \rightarrow$$

Equations (7), (8) and (10) may be written

$$[\mathcal{P}^a, \mathcal{P}^b] = \mathcal{D}_{abc} \mathcal{P}^c \tag{14}$$

and Eqs. (6) and (11) may be written

$$\mathcal{L}^i X^a - X^a \mathcal{R}^i = i \, \mathcal{P}^i_{ca} X^c \tag{15}$$

i.e., there is a "hidden" representation of the Lie group with dimension equal to the number of scalars (N) plus the number of vectors (g) (still assumed to be all massive). Figure 2 "illustrates" part of Eq. (14), whose other pieces have a similar graphical interpretation.

3. Suppose there is one massless particle (the photon) and choose $W'_\mu = A^\gamma_\mu$. The discussion above holds, except that

$$\mathcal{D}_{1ab} = 0 \quad \text{unless} \quad M_a = M_b$$

$$T'_{ab} = 0 \quad \text{unless} \quad M_a = M_b \tag{16}$$

$$\mathcal{R}'_{ab} = \mathcal{L}'_{ab} = \delta_{ab} \lambda^a$$

(terms involving M_γ^{-2} can therefore never occur in the cancellation conditions). The cancellation conditions which are not satisfied identically are still summarized by Eqs. (12)-(15), except that we now have :

$$\mathcal{P}_{1b}^a = \mathcal{P}_{b1}^a = X_{ij}' = 0 \qquad (17)$$

i.e., the "hidden" representation of the Lie algebra has dimensions N+g-1 in this case.

4. Equations (12)-(15) exhibit many of the features of the "spontaneously broken" gauge theories of Englert and Brout [12] and Higgs [13]. There is a class of solutions to these equations in which there exists a vector η_j (j = 1,...,N) such that

$$T_{ij}^a \eta_j = 0 \qquad (18)$$

$$\eta_i K_{ab}^i = 2\delta_{ab} M_a^2$$

which implies

$$\delta_{ab} m_a = M_{ab} - \eta_i C_{ab}^i \qquad (19)$$

where

$$M_{ac} R_{cb}^i = L_{ac}^i M_{cb} .$$

Equations (12)-(19) define the general case of the non-Abelian Higgs model discussed by Kibble [14] *).

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*) Recall that, before spontaneous symmetry breaking occurs, there are (N+g) ∅'s which transform as an N+g dimensional representation of the Lie group; the ∅ interactions and the "bare" fermion mass term have the form :

$$\mathcal{L}' = \overline{\psi}_i \left[M_{ij} + X_{ij}^a \frac{(1+\gamma_5)}{2} + X_{ij}^{+a} \frac{(1-\gamma_5)}{2} \right] \psi_j \phi^a$$
$$+ \frac{1}{2} | \partial_\mu \phi_i + \mathcal{P}_{ij}^a W_\mu^a \phi_j |^2$$

where M is Hermitean, P is real and antisymmetric and

5. The question immediately arises whether there are solutions to the cancellation conditions other than those which specify a spontaneously broken gauge theory in the unitary gauge. The answer is yes; an example is provided by massive QED $(M_\gamma \neq 0)$ which is known to be a well behaved renormalizable theory. It might perhaps be that all solutions have the structure of generalized Higgs models except for certain gauge bosons corresponding to some Abelian subgroups, but I have been unable to find the complete answer to this question (I hope to discuss it further in a future publication containing full details of the work reported here). However, in models with only the known leptons and four vector mesons $(W^\pm, \gamma, Z^0)$ the unique (non-trivial) solution is the spontaneously broken gauge theory model of Weinberg[3] and Salam[4] if we take the minimal number (one) of scalar particles. [Cornwall, Levin and Tiktopoulos[8] studied the W–ø system in models with $W^\pm, \gamma, Z^0$ and ϕ^0 and found two non-trivial solutions – one corresponding to the Weinberg-Salam model and the other, in which the Z^0 decouples, to the Georgi-Glashow model[15]. Their study of five-point tree diagram uniquely fixes the ϕ^3 and ϕ^4 interactions in this case.]

ACKNOWLEDGEMENTS

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cont.

$$[P^a, P^b] = D_{abc} P^c$$
$$L^c X^b - X^b R^c = i P^c_{ab} X^a$$
$$L^a M = M R^a$$

where L and R are the $W\bar{F}F$ coupling matrices defined above. When the symmetry is completely "spontaneously broken" the Lagrangian in the "unitary gauge" is obtained by putting

$$\phi_i \to 0 \quad , i = 1 \cdots g .$$

where
$$\phi_i \to \eta_i + \phi_i \quad , i = g+1, \cdots g+N$$
$$P^a_{i+g, g+j} \eta_j = 0 \quad , (i = 1, \cdots N)$$

Choosing (without loss of generality) a representation in which the vector meson masses are diagonal and the fermion masses are diagonal and independent of γ_5, we see that the conditions in Eqs. (12)-(17) specify the theory in this "physical" gauge.

R E F E R E N C E S

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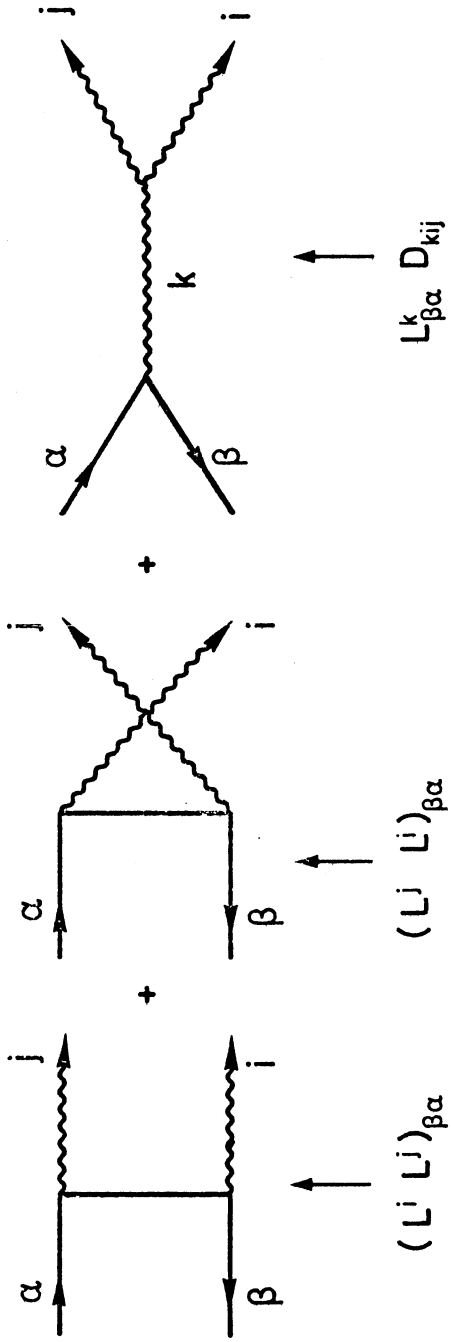15) H. Georgi and S.L. Glashow, Phys.Rev.Letters $\underline{28}$, 1494 (1972).
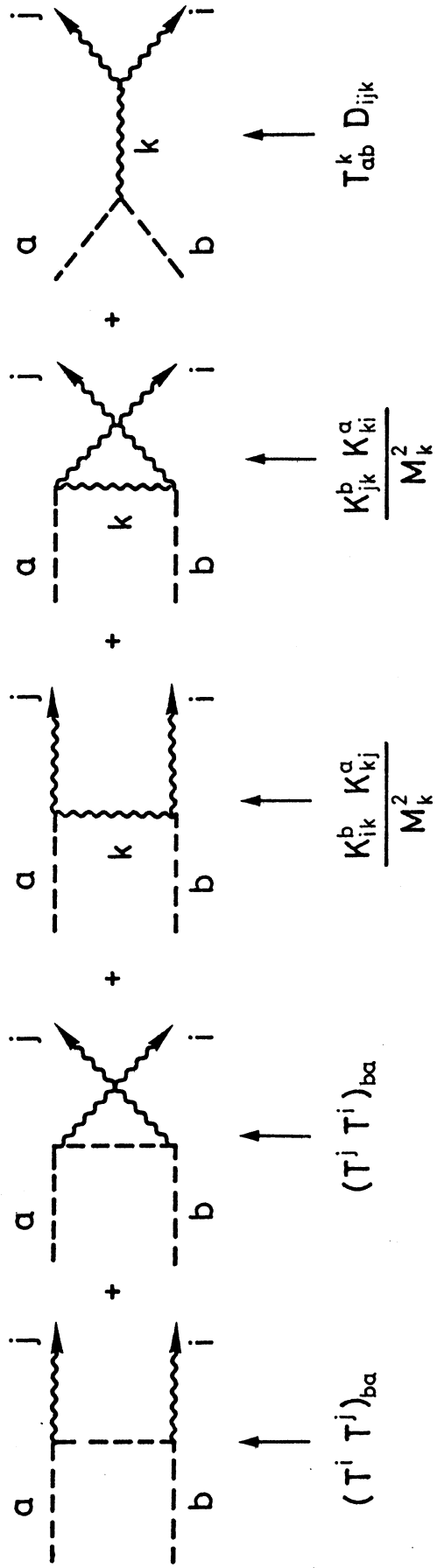
FIG.1

FIG.2

HIGH ENERGY BEHAVIOUR AND GAUGE THEORY

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ADDENDUM/ERRATUM

1. - The necessary and sufficient conditions in Eq. (3) should include the Jacobi identity

$$\mathcal{D}_{kac}\,\mathcal{D}_{kbd} - \mathcal{D}_{kab}\,\mathcal{D}_{kcd} - \mathcal{D}_{kad}\,\mathcal{D}_{kbc} = 0$$

[which also follows directly from Eq. (4)]. Equation (16) should include the obvious condition

$$K_{1j}^{b} = 0.$$

2. - The statement that a coupling $W^{\mu}(\phi^{a}\partial_{\mu}\phi^{b} + \phi^{b}\partial_{\mu}\phi^{a})$ necessarily leads to bad behaviour in $\phi\phi \to \phi\phi$ is false. In simple models it can be shown that this coupling must be absent but I have not yet succeeded in eliminating it in the general case. For the moment we simply assume that it is absent (noting that it is effectively equivalent to a contact interaction with a coupling with inverse mass dimensions, which we have already assumed to be absent).

3. - A partial answer to the final question (Remark 5) is that for semi-simple Lie algebras the vector η_{i} always exists and (apart from the ϕ mass terms and self-interactions) the unique Lagrangian with good high energy which satisfies our assumptions is of the generalized Higgs form. The question is still open for arbitrary non-semi-simple compact Lie algebras.