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EXCHANGE FORCES IN THE QUARK BOUND STATES AND THE ASYMPTOTIC BEHAVIOUR OF FORM FACTORS

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A B S T R A C T

A Bethe-Salpeter type equation is proposed for taking into account the effects of particle exchanges between the quarks in the relativistic quark model of Feynman, Kislinger and Ravndal. The equation is explicitly soluble in the ladder approximation for zero mass exchange and it leads to asymptotically dipole-like electromagnetic form factors.

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The symmetric quark model of hadrons with relativistic harmonic interaction [1] gives a crude and simple but in many cases surprisingly adequate description of the observed hadron spectrum. The infinitely rising harmonic oscillator potential between the quarks acts drastically if the quarks are separated by large four-distances and it seems that this is required to reproduce the main features of the spectrum (linear trajectories, for instance). The harmonic interaction is, however, "soft" at small distances as it vanishes if the quarks are separated by a lightlike four-vector. Therefore the harmonic forces are certainly not dominating for small relative distances, or, which is the same, for large relative momenta. This can explain the failure of the oscillator quark model in producing exponentially decreasing form factors if $q^2 \to -\infty$. The experiments on the proton electromagnetic form factors show a definitely slower decrease and are well described by dipole fits behaving as $(q^2)^{-2}$ for large (spacelike) q^2 [2].

The asymptotic behaviour of the form factors can be understood, however, in a ladder Bethe-Salpeter model [3] of composite hadrons. This shows that the short range interaction between the constituents of hadrons is well approximated by an infinite ladder of (elementary) particle exchanges [*]. This suggests that exchange forces may play an important role also in the oscillator bound states at least as far as phenomena involving large relative momenta are concerned.

The purpose of the present paper is to show that adding a "gluon" mediated exchange interaction to the harmonic interaction leads to dipole-like asymptotic form factors, indeed [**]. For simplicity, here only the case of spinless quarks and mesons (two-quark bound states) will be considered. The "gluon" exchanged between the quarks is represented by a neutral,

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*) On the other hand the conventional Bethe-Salpeter models based on ordinary particle exchanges have troubles with reproducing the observed linearly rising trajectories. This can be easily understood having in mind that the exchange potentials decrease rapidly with increasing distances. A nearly linear spectrum can be obtained, of course, also in the conventional Bethe-Salpeter framework if the exchange potential is replaced artificially by some other type of potentials. See, e.g., Ref. 4).

**) An attempt to understand the asymptotics of form factors in the framework of the oscillator quark model was made in Ref. 5). The lesson drawn from this work was that field theoretic unitarity corrections represented asymptotically by Pomeron-exchange may essentially alter the asymptotics. The role of the short distance behaviour of the interaction was not realized, however.

scalar (or pseudoscalar) field of mass m. (In the actual calculation m will be chosen to be zero, too.) The extension to the cases with spins and baryons does not seem to involve any essential new difficulty besides the "spinology" encountered in every Bethe-Salpeter type model.

The bound state of two equal mass scalar quarks is described in the harmonic oscillator quark model by the bilocal field $\Psi_q(x_1,x_2)$, $x_{1,2}$ denoting the position four-vectors of the two quarks. The field $\Psi_q(x_1,x_2)$ obeys the equation [1),6),7)]

$$\left[2\Box_{x_1} + 2\Box_{x_2} - \frac{\omega^2}{16}(x_1-x_2)^2 + m_a^2\right]\Psi_q(x_1,x_2) = 0 \ . \tag{1}$$

Let us suppose that the quarks in the bound states are coupled to some neutral, scalar gluon [*)] field $\varphi(x)$ in such a way that the interaction Hamiltonian is given by

$$H_I = g\int d^4x_1\, d^4x_2\, \Psi_q(x_1,x_2)\Psi_q^+(x_1,x_2)\varphi(x_1)\varphi(x_2) \ . \tag{2}$$

In the interaction picture the field $\Psi_q(x_1,x_2)$ obeys the field equation (1) in the same way as $\varphi(x)$ does the free field equation. The perturbation expansion of the S operator is given by

$$S = \sum_{n=0}^{\infty}\frac{(-ig)^n}{n!}\int T\left\{\prod_{k=1}^{n}\Psi_q(y_{2k-1},y_{2k})\Psi_q^+(y_{2k-1},y_{2k})\varphi(y_{2k-1})\varphi(y_{2k})d^4y_{2k-1}\,d^4y_{2k}\right\} \tag{3}$$

The action of the time-ordering T on bilocal operators like $\Psi_q(x_1,x_2)$ is defined as if $\Psi_q(x_1,x_2)$ would belong to the centre-of-mass four-vector $x = \frac{1}{2}(x_1+x_2)$.

The Bethe-Salpeter equation [8)] for the <u>two bound state</u> Green function

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*) The "gluon" does not necessarily represent here some new kind of elementary particles rather it may be considered as an approximation for the description of an ordinary, composite particle. The pion, for instance, can be described for many purposes to a good accuracy by a local quantum field, even though it is considered as composite.

$$G(x_3,x_4;x_1,x_2) = \langle 0|T\{\Psi_{qH}(x_3,x_4)\Psi^{+}_{qH}(x_1,x_2)\}|0\rangle \tag{4}$$

can be derived in exactly the same way as the conventional Bethe-Salpeter equation is derived for the four point Green function. (Ψ_{qH} denotes the Heisenberg operator belonging to Ψ_q.) The equation has the form

$$G = G_0 + G_0 I G \quad , \tag{5}$$

where G_0 denotes the free two bound state function, i.e., the propagator of the oscillator bound state

$$G_0(x_3,x_4;x_1,x_2) = \langle 0|T\{\Psi_q(x_3,x_4)\Psi^{+}_q(x_1,x_2)\}|0\rangle =$$
$$= (x_3-x_4|\frac{i}{(2\pi)^4}\int d^4k \; \frac{e^{-ik[\frac{1}{2}(x_3+x_4)-\frac{1}{2}(x_1+x_2)]}}{k^2-M^2+i\varepsilon}|x_1-x_2) \; . \tag{6}$$

Here $M^2 = m_a^2 + 2\omega - \omega a^{+}\cdot a$ is the mass squared operator for the oscillator (a^{+}_{μ} and a_{μ} are the creation and annihilation operators of the internal oscillations) and $|\varphi)$ stands in general for a ket in the internal Hilbert space ($|\varphi>$ on the other hand denotes the state of the whole system, including also the centre-of-mass motion [6]). In Eq. (5), I stands for the sum of irreducible kernels which, in the ladder approximation, reduces simply to one gluon exchange between the two quarks in the bound state.

The wave function of the bound state $|p>$ with total momentum p is defined by

$$\chi_p(x_1,x_2) = \langle 0|\Psi_{qH}(x_1,x_2)|p\rangle \equiv e^{-ip\frac{1}{2}(x_1+x_2)} \psi_p(x_3-x_4) \; . \tag{7}$$

The equation for χ_p can be derived from Eq. (5) considering the residuum of the bound state pole. It is given by

$$\chi_p = G_0 I \chi_p \; . \tag{8}$$

We consider Eqs. (5), (8) in the ladder approximation which is represented graphically on Fig. 1. The integral equation for ψ_p in Eq. (7) is then

$$\psi_p(z) = \frac{g}{4\pi^2}\int d^4z'\,(z|\,\frac{1}{M^2+p_\varrho p_\varrho}\,|z')\,\frac{m}{\sqrt{z'_\varrho z'_\varrho}}\,K_1\left(m\sqrt{z'_\varrho z'_\varrho}\right)\psi_p(z')\,. \tag{9}$$

In this equation, a Wick rotation in the z_0 plane is already performed and therefore also $p_\varrho p_\varrho \equiv p_1p_1 + p_2p_2 + p_3p_3 + p_4p_4 = -p^2$. This is unavoidable in our case as the Wick rotation is inherent already in the pure harmonic oscillator model [*)] [6)]. It is obvious from Eq. (9) that it has an $O(4)$ symmetry for every value of the total momentum [**)]. Therefore it can be expanded according to the $O(4)$ harmonics $Y_{n\ell m}(\Omega_z)$ in the way

$$\psi_p(z) = \sum_{n\ell m} Y_{n\ell m}(\Omega_z)\,\psi_{p^2}(\zeta)_n \quad ,$$

where $\zeta \equiv \sqrt{z_\varrho z_\varrho}$. The equation for $\psi_{p^2}(\zeta)_n$ is :

$$\psi_{p^2}(\zeta)_n = \frac{g}{4\pi^2}\int_0^\infty d\zeta'\,\zeta'^3\,H_n(\zeta,\zeta')\,\frac{m}{\zeta'}\,K_1(m\zeta')\,\psi_{p^2}(\zeta')_n \quad ;$$

$$H_n(\zeta,\zeta') = \frac{1}{\omega(\zeta\zeta')^2}\,\frac{1}{n!}\,\Gamma\left(\frac{n\omega+m_a^2+\omega-p^2}{2\omega}\right). \tag{10}$$

$$\cdot\, M_{\frac{p^2-m_a^2}{2\omega},\,\frac{n}{2}}\left[\frac{\omega}{16}\left(\zeta^2+\zeta'^2-|\zeta^2-\zeta'^2|\right)\right] W_{\frac{p^2-m_a^2}{2\omega},\,\frac{n}{2}}\left[\frac{\omega}{16}\left(\zeta^2+\zeta'^2+|\zeta^2-\zeta'^2|\right)\right].$$

M and W denote the Whittaker functions [9)]. In the case of zero gluon mass the equation (10) simplifies a bit as then $(m/\zeta')K_1(m\zeta')$ is replaced by $(\zeta')^{-2}$.

The solution of Eq. (10) is facilitated if one goes over to a differential equation. It can be shown, namely, that the solution of Eq. (10) satisfies the differential equation

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*) This point is a delicate one and it is presumably connected also with the ghost problem but for the moment we cannot do anything better than accept the Wick rotated equation as a starting point.

**) Note that G in Eq. (4) is not a four-point function in the usual sense therefore this $O(4)$ symmetry has nothing to do with the $O(4)$ symmetry of the scattering amplitude at $p = 0$.

$$\left(\frac{d^2}{d\varsigma^2}+\frac{3}{\varsigma}\frac{d}{d\varsigma}+\frac{1-n^2}{\varsigma^2}+\frac{1}{4}(p^2-m_a^2)-\frac{\omega^2}{64}\varsigma^2\right)\psi_{p^2}(\varsigma)_n=\frac{g}{4\pi^2}\frac{m}{\varsigma}K_1(m\varsigma)\psi_{p^2}(\varsigma)_n . \tag{11}$$

This equation is explicitly soluble in the case of zero gluon mass when the solution less singular near the origin is [9] :

$$\psi_{p}(\varsigma)_n=\frac{1}{\varsigma}\left(\varsigma^2\frac{\omega}{8}\right)^{\frac{1}{2}\sqrt{n^2-\lambda}}e^{-\varsigma^2\frac{\omega}{16}}\phi\left(\frac{\omega+\omega\sqrt{n^2-\lambda}+m_a^2-p^2}{2\omega},1+\sqrt{n^2-\lambda};\varsigma^2\frac{\omega}{8}\right). \tag{12}$$

Here ϕ denotes the confluent hypergeometric function and λ is related to the quark-gluon coupling constant by $\lambda = (g/16\pi^2)$. In the harmonic oscillator model the bound state wave functions are square integrable in the Wick rotated z space [6], the eigenvalues of the mass squared operator being defined by the requirement of square integrability. In the same sense the solution of Eq. (12) is square integrable if

$$p^2=m_{Nn}^2=m_a^2+\omega\left(1+2N+\sqrt{n^2-\lambda}\right);\qquad N=0,1,2,\ldots \tag{13}$$

The confluent hypergeometric function is in this case reducing to a Laguerre polynomial, that is the (Wick rotated) wave function of the bound state with quantum numbers N, n, ℓ, m is

$$\psi(z)_{Nn\ell m}=\text{const.}\,Y_{n\ell m}(\Omega_z)\,\varsigma^{\sqrt{n^2-\lambda}-1}\,e^{-\varsigma^2\frac{\omega}{16}}\,L_N^{\sqrt{n^2-\lambda}}\left(\varsigma^2\frac{\omega}{8}\right), \tag{14}$$

$$N=0,1,2,\ldots\quad;\qquad n-1\geq\ell\geq|m| .$$

The O(4) trajectories corresponding to Eq. (13) are depicted on Fig. 2 in the case of attracting gluon interaction, that is for $\lambda > 0$. ($\lambda < 0$ corresponds, of course, to repulsive gluon interaction and $\lambda = 0$ to the pure harmonic oscillator model.) As it can be seen from Eq. (13), the ground state solution makes sense only if $\lambda \leq 1$ ($\lambda > 1$ would give complex mass squared for some of the states). The essential deviations are at small values of n (which means also small values of the spin ℓ). In general, the gluon interaction has the effect that it breaks the symmetry of the harmonic oscillator model. In the physical case of quarks with internal quantum numbers this leads also to the breaking of the SU(6,6) symmetry down to SU(3) or even to $SU(2)_I \times U(1)_Y$, depending on the actual form of the gluon coupling matrix in the space of internal quantum numbers.

The electromagnetic form factor is expressed by the Fourier transformed $\phi(k)$ of the wave function $\psi(z)$ in the following way [see, for instance, Ref. 5] :

$$F(q) = \frac{1}{(2\pi)^4} \int d^4k \; \phi(k)\, \phi(k \pm \tfrac{1}{2} q) \; . \tag{15}$$

The two signs correspond to the contributions of the two quarks, respectively. (The whole form factor is given actually by the sum of the two terms.) From the expression of the wave function Eq. (14), we get for the form factor of the ground state $(N = 0,\ n = 1,\ \ell = m = 0)$:

$$F(q^2) = \text{const.}\; \phi\left(2-2\Lambda, 2; \frac{q^2}{2\omega}\right) \; ; \qquad \Lambda \equiv \frac{1}{2}\left(1-\sqrt{1-\lambda}\right) . \tag{16}$$

The asymptotic behaviour of this function for $q^2 \to -\infty$ is [9] :

$$F(q^2) \sim \text{const.}\; 2^{-2\Lambda} \frac{1}{\Gamma(2\Lambda)} \left(-\frac{\omega}{q^2}\right)^{2-2\Lambda} . \tag{17}$$

It can be seen from Eq. (17) that the asymptotic behaviour of the form factor is essentially the same as that of a dipole. The deviation from the exact dipole depends on the quark gluon coupling constant Λ which is not too large if the original oscillator spectrum gives a good approximation for the physical spectrum. If $\Lambda = 0$ (no exchange forces) then the coefficient of the dipole term vanishes due to the pole of the gamma function. In that case the form factor $F(q^2)$ is a pure exponential as the confluent hypergeometric function degenerates to an exponential function. We note that in the timelike region $(q^2 > 0)$ the form factor $F(q^2)$ in Eq. (16) needs further corrections (in order to satisfy, for instance, unitarity) as it has no resonance poles and threshold branch points and behaves asymptotically (as $q^2 \to +\infty$) like an exponential.

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FIGURE CAPTIONS

Figure 1 Equations (5), (8) in the ladder approximation. The quarks are represented by the two edges of the shaded areas, the dashed line is the gluon.

a) Equation for G ;

b) Equation for χ_p.

Figure 2 The $O(4)$ trajectories for $\lambda = 0$ (empty circles : $\circ$) and $\lambda > 0$ (full circles : $\bullet$).

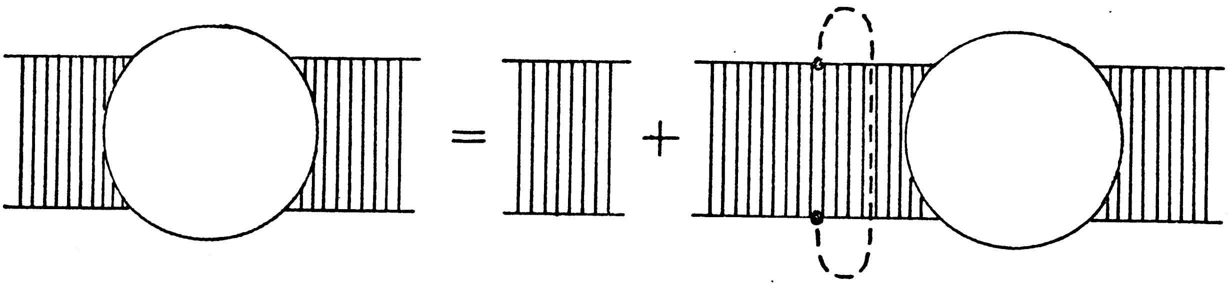

FIG 1a

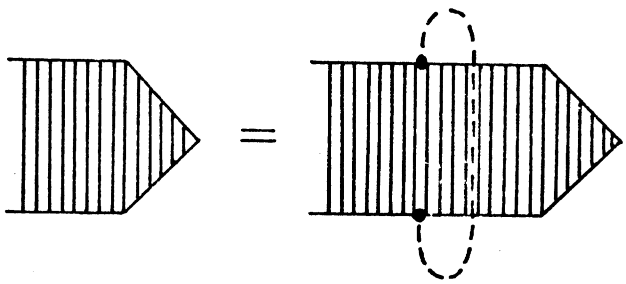

FIG 1b

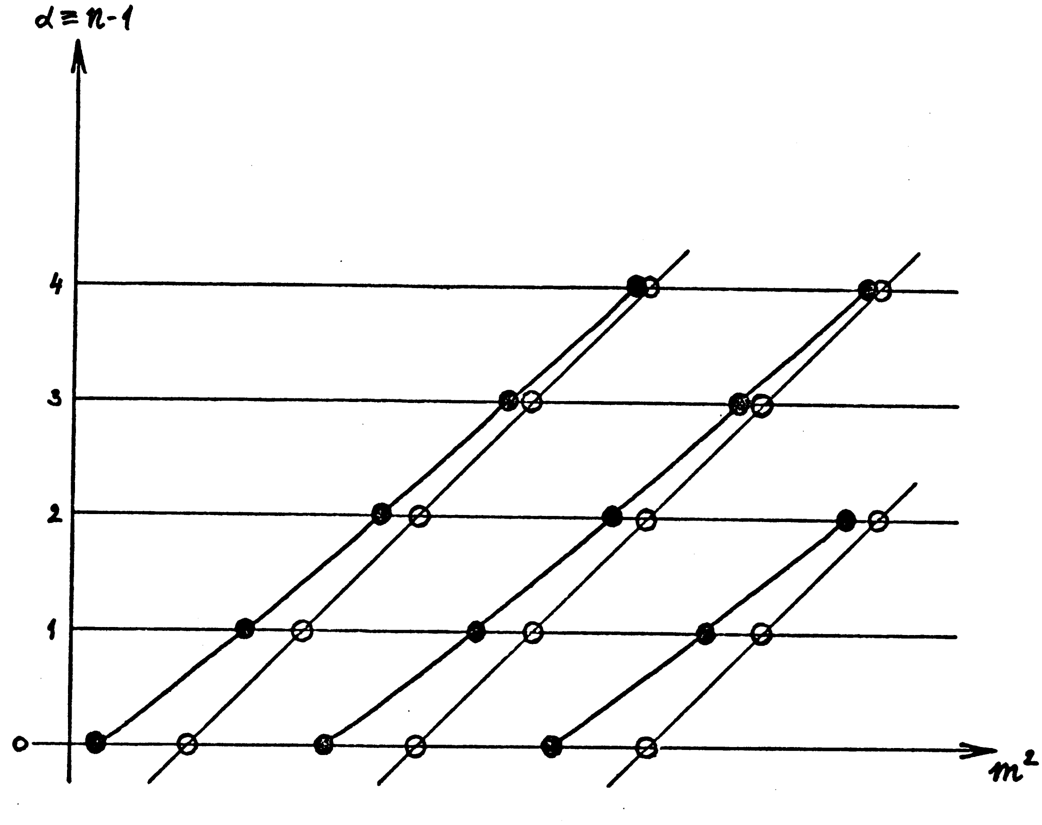

FIG 2