[redacted]GANIZATION FOR NUCLEAR RESEARCH

CERN/EP/DHR 76-9
19 July 1976

PARAMETRIZATION OF THE LAMP W75 MAGNETIC FIELD

D. Newton, A.M. Osborne and G.F. Pearce

CERN, Geneva

Inside a bounded rectangular volume ($|x| \leqslant a$, $|y| \leqslant b$, $|z| \leqslant c$) the main component (B_y) of the W75 magnetic field has been measured [1]. By parameterizing this component only, enough information has been obtained to calculate the smaller components (B_x and B_z) in a manner which guarantees that Maxwell's equations are satisfied.

A program (SUPER) to fit the main field component as described here, is available in 'PHYSICS PROGRAMS AND ALGORITHMS'(*).

METHOD

The principle of the method, lies in the fact that the three components of the magnetic field (B_x,B_y,B_z) must be derivable from a potential $\phi_m(x,y,z)$ which satisfies the Laplace equation $\nabla^2\phi_m(x,y,z) = 0$.

One possible form for such a potential would be:

$$\phi_m(x,y,z) = \exp(\alpha x + \gamma y + \beta z) \quad \text{where } \alpha^2 + \beta^2 + \gamma^2 = 0$$

or, generalising this slightly in terms of trigonometric and hyperbolic functions,

$$\phi_m(x,y,z) = (a\cos(\alpha x) + b\sin(\alpha x))(\cosh(\gamma y) + c\sinh(\gamma y))(d\cos(\beta z) + e\sin(\beta z)) + Ay,$$

where a,b,c,d and e are constants and where now $\gamma^2 = \alpha^2 + \beta^2$. The term Ay contributes a constant term in the fit to By(**).

(*) As we will see Ay represents the constant term in the fit to the main component By. It will later be absorbed into the series.

(**) CERN software pool.

AMO/mk

This will hold for any value of the parameters α and β, and thus for any linear combination of them. To form one such linear combination, we write

$$\alpha = \frac{\pi j}{x_o} \text{ and } \beta = \frac{\pi i}{z_o} \quad \text{for } i,j = 0,1,2,3 \ldots\ldots N,$$

where x_o and z_o are parameters to be used as scaling factors.

Hence, in terms of the set coefficients $\{a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}\}$ we may construct the following solution to Laplace's equation:

$$\phi_m(x,y,z) = \sum_{i=o}^{N_i} \sum_{j=o}^{N_j} \left\{a_{ij}\cos\left(\frac{i\pi z}{z_o}\right)\cos\left(\frac{j\pi x}{x_o}\right) + b_{ij}\cos\left(\frac{i\pi z}{z_o}\right)\sin\left(\frac{j\pi x}{x_o}\right)\right.$$

$$\left. + c_{ij}\sin\left(\frac{i\pi z}{z_o}\right)\cos\left(\frac{j\pi x}{z_o}\right) + d_{ij}\sin\left(\frac{i\pi z}{z_o}\right)\sin\left(\frac{i\pi x}{x_o}\right)\right\} x \left\{\sinh\left[\pi y\sqrt{\left(\frac{i}{z_o}\right)^2 + \left(\frac{j}{x_o}\right)^2}\right]\right.$$

$$\left. + e_{ij}\cosh\left[\pi y\sqrt{\left(\frac{i}{z_o}\right)^2 + \left(\frac{j}{x_o}\right)^2}\right]\right\} + Ay \ldots \quad . \qquad (1)$$

Applying to this the identities

$$B_x = -\frac{\partial\phi_m}{\partial x}, \; B_y = -\frac{\partial\phi_m}{\partial y}, \; B_z = -\frac{\partial\phi_m}{\partial z}$$

the following expressions for the three components of the magnetic field can be obtained:

$$B_x = \sum_{i,j} J \left\{a_{ij}\cos(Iz)\sin(Jx) - b_{ij}\cos(Iz)\cos(Jx) + c_{ij}\sin(Iz)\sin(Jx)\right.$$

$$\left. - d_{ij}\sin(Iz)\cos(Jx)\right\} \cdot \left\{\sinh(y\sqrt{I^2 + J^2}) + e_{ij}\cosh(y\sqrt{I^2 + J^2})\right\}$$

$$B_y = -\sum_{i,j} (I^2 + J^2)^{\frac{1}{2}}\left\{a_{ij}\cos(Iz)\cos(Jx) + b_{ij}\cos(Iz)\sin(Jx)\right.$$

$$+ c_{ij}\sin(Iz)\cos(Jx)$$

$$\left. + d_{ij}\sin(Iz)\sin(Jx)\right\} \cdot \left\{\cosh(y\sqrt{I^2 + J^2}) + e_{ij}\sinh(y\sqrt{I^2 + J^2}\right\} + A.$$

Note that all other terms other than A have an integrated (over Z) average of zero.

$$B_z = \sum_{i,j} I\left\{a_{ij}\sin(Iz)\cos(Jx) + b_{ij}\sin(Iz)\sin(Jx) - c_{ij}\cos(Iz)\cos(Jx) - d_{ij}\cos(Iz)\sin(Jx)\right\}\left\{\sinh(y\sqrt{I^2 + J^2}) + e_{ij}\cosh(y\sqrt{I^2 + J^2})\right\} \ldots (2)$$

where $I = \frac{\pi}{z_o} i$, $J = \frac{\pi}{x_o} j$.

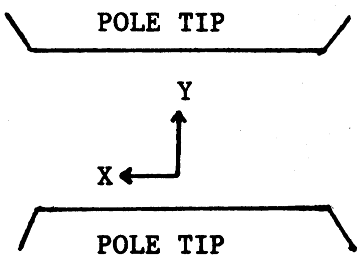
Note that a knowledge of B_y throughout a volume would determine the coefficients in (2) except for

a_{oo}, b_{io}, c_{oj}, d_{io}, d_{oj} and e_{oo}.

However, these coefficients are not required in the expressions for B_x and B_z, and correspond to a constant term in the potential eq. (1).

SYMMETRIES IN THE LAMP W75 MAGNET

Using the coordinate system illustrated below,



in which the z-axis is defined along the direction of the beam, two symmetries of the W75 field (both of which have been observed) may be described.

(a) $B_y(+y) = + B_y(-y)$ (i.e. both B_x and B_z are zero on the plane y = 0). This implies that only even functions in y may exist in the expression for B_y, so that the coefficients e_{ij} in eq. (2) must all vanish.

(b) $B_y(+x) = + B_y(-x)$ (i.e. B_x must vanish on the plane x = 0).
This implies that only even functions in x may exist in the expression for B_y, so that the coefficients B_{ij}, d_{ij} in eq. (2) must all vanish. Thus eq. (2) reduce to

$$B_x = \sum_{i=0}^{N_i} \sum_{j=0} J \left\{ a_{ij}\cos(Iz) + c_{ij}\sin(Iz) \right\} \sin(Jx)\sinh(y\sqrt{I^2 + J^2})/R_{ij}$$

$$B_y = - \sum_{i=0}^{N_i} \sum_{j=0}^{N_j} \left\{ a_{ij}\cos(Iz) + c_{ij}\sin(Iz) \right\} \cos(Jx)\cosh(y\sqrt{I^2+J^2}) \qquad (3)$$

$$B_z = \sum_{i=0}^{N_i} \sum_{j=0}^{N_j} I \left\{ a_{ij}\sin(Iz) - c_{ij}\cos(Iz) \right\} \cos(Jx)\sinh(y\sqrt{I^2 + J^2})/R_{ij} ,$$

where $I = \frac{\pi}{z_o} . i$, $J = \frac{\pi}{z_o} . j$ and $R_{ij} = \sqrt{I^2+J^2}$.

Note that the only coefficients that cannot be computed from measurements on B_y (i.e. a_{oo} and c_{oj}) do not contribute to any of the three expressions. We have absorbed the term $\sqrt{I^2+J^2}$ into the definition of the coefficients and at the same time included the term A as the first term of the series for B_y. B_x and B_z have no contribution from the term with i = j = 0.

DETERMINATION OF THE COEFFICIENTS

The coefficients $\{a_{ij}, c_{ij}\}$ have been determined by fitting the expression for B_y in eq. (3) to the field measurements using the least squares method; i.e. by finding that set of coefficients which minimises the sum

$$S = \sum_{a,b,c} \left[\frac{\varepsilon_{abc}}{\sigma_{abc}} \right]^2 ,$$

where

$$\varepsilon_{abc} = (B^{MEAS}_{abc} - B^{CALC}_{abc}) = B_{abc} + \sum_{k,\ell} \left\{ a_{k\ell}\cos(Kz_{abc}) \right.$$

$$\left. + c_{k\ell}\sin(Kz_{abc}) \right\} \cos(Lx_a)\cosh(y_{ab}\sqrt{K^2 + L^2})$$

for $K = \left(\frac{\pi}{z_o}\right) k$

$L = \left(\frac{\pi}{z_o}\right) \ell$

x_a = a^{th} x-position at which B_y is measured

y_{ab} = b^{th} y-position at $x = x_a$

z_{abc} = c^{th} z-position at $x = x_a$, $y = y_{ab}$

B_{abc} = measurement of B_y at the point (x_a, y_{ab}, z_{abc})

σ_{abc} = error on the measurement B_{abc} which for simplicity can be set to 1.0 as the error is generally independent of a,b,c. In fact the method described here relies on this independence.

These coefficients will then be determined by solving the two equations

$$\frac{\partial S}{\partial a_{ij}} = \frac{\partial S}{\partial c_{ij}} = 0 \quad \text{for each i and j.}$$

Explicitly writing down these equations yields

$$\sum_{abc} \frac{B_{abc}}{\sigma^2_{abc}} \cos\,(Iz_{abc} \cos(Jx_a) \cosh(y_{ab}\sqrt{I^2 + J^2}$$

$$= - \sum_{k\ell} \left\{ \sum_a \cos(Lx_a) \cos(Jx_a) \sum_a \cosh(y_{ab}\sqrt{K^2 + L^2}) \cosh(y_{ab}\sqrt{I^2 + J^2} \right.$$

$$\left. \sum_c \frac{[a_{k\ell}\cos(Kz_{abc}) + c_{k\ell}\sin(Kz_{abc})]\cos(Iz_{abc})}{\sigma^2_{abc}} \right\}$$

and

$$\sum_{abc} \frac{B_{abc}}{\sigma^2_{abc}} \sin(Iz_{abc}) \cos(Jz_a) \cosh(y_{ab}\sqrt{I^2 + J^2}$$

$$= - \sum_{k\ell} \left\{ \sum_a \cos(Lx_a) \cos(Jx_a) \sum_b \cos(y_{ab}\sqrt{K^2 + L^2}) \cosh(y_{ab}\sqrt{I^2 + J^2} \right.$$

$$\left. \sum_c \frac{[a_{k\ell}\cos(Kz_{abc}) + c_{k\ell}\sin(Kz_{abc})]\sin(Iz_{abc})}{\sigma^2_{abc}} \right\} \qquad (4)$$

for all i and j.

Now, providing the field measurements have been taken at regular intervals in z (which we have done), these equations may be simplified by choosing the scale factor z_o so that $\frac{\pi z}{z_o}$ covers the full range $\pm\pi$ uniformly and applying the orthogonality properties of sine and cosine functions. With a suitable re-definition of the error σ_{abc} (see Appendix) they may be reduced to the following two uncoupled sets of equations

$$\sum_{abc} \frac{B_{abc}}{\sigma^2_{abc}} \cos(Iz_{abc})\cos(Jx_a)\cosh(y_{ab}\sqrt{I^2+J^2})$$

$$= - \sum_{\ell} a_{i\ell} \left\{ \sum_a \cos(Lx_a)\cos(Jx_a) \sum_b \cosh(y_{ab}\sqrt{I^2+L^2})\cosh(y_{ab}\sqrt{I^2+J^2}) \sum_c \frac{\cos^2(Iz_{abc})}{\sigma^2_{abc}} \right\}$$

and

$$\sum_{abc} \frac{B_{abc}}{\sigma^2_{abc}} \sin(Iz_{abc})\cos(Jx_a)\cosh(y_{ab}\sqrt{I^2+J^2})$$

$$= -\sum_{\ell} c_{i\ell} \left\{ \sum_a \cos(Lx_a)\cos(Jx_a) \sum_b \cosh(y_{ab}\sqrt{I^2+L^2})\cosh(y_{ab}\sqrt{I^2+J^2}) \sum_c \frac{\sin^2(Iz_{abc})}{\sigma^2_{abc}} \right\}$$

$\forall\ i,j$

Introducing the definitions

$$G^{ij} = \sum_{abc} \frac{B_{abc}}{\sigma^2_{abc}} \cos(Iz_{abc})\cos(Jx_a)\cosh(y_{ab}\sqrt{I^2+J^2})$$

$$H^{ij}_{\ell} = \sum_a \cos(Lx_a)\cos(Jx_a) \sum_b \cosh(y_{ab}\sqrt{I^2+L^2})\cosh(y_{ab}\sqrt{I^2+J^2}) \sum_c \frac{\cos^2(Iz_{abc})}{\sigma^2_{abc}}$$

the first of these equations may be written

$$\begin{bmatrix} G^{io} \\ G^{i1} \\ G^{i2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ G^{iN_j} \end{bmatrix} = \begin{bmatrix} H_o^{io} & H_1^{io} & \cdots\cdots & H_{N_j}^{io} \\ H_o^{i1} & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H_o^{iN_j} & & & H_{N_j}^{iN_j} \end{bmatrix} \begin{bmatrix} a_{io} \\ a_{i1} \\ a_{i2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{iN_j} \end{bmatrix} \quad Vi \qquad (5)$$

and similarly for the coefficients c_{ij}.

Applying orthogonality properties to the z-dimension, then, not only allows an independent solution to each of the two sets of coefficients, a_{ij} and c_{ij}, but also separates the $N_i \times N_j$ simultaneous equations involved into N_i independent sets of j equations for both sets of coefficients.

If ~~we now define~~ our other free parameter, x_o, such that we may apply orthogonality properties to the cosine terms in x in eq. (4) (i.e. $x_o = x_{max}$) in the same way that we have done with z, the H-matrix can, in fact, be diagonalised to give the solution

$$a_{ij} = - \frac{\sum_{abc} \frac{B_{abc}}{\sigma^2_{abc}} \cos(Iz_{abc})\cos(Jx_a)\cosh(y_{ab}\sqrt{I^2+J^2})}{\sum_a \cos^2(Jx_a)\sum_b \cosh^2 y_{ab}\sqrt{I^2+J^2})\sum_c \frac{\cos^2(Iz_{abc})}{\sigma^2_{abc}}} \qquad (6)$$

and similarly for the coefficients c_{ij}, where, again, the error matrix σ_{abc} must be slightly re-defined to allow exact cancellation in the sum over x terms as well as z.

One should note that defining $X_o = X_{max}$ such that $x' = \frac{Jx\pi}{X_o}$ ranges from $0 \rightarrow \pi$ will cause a problem in that many more terms will be required in the fit - if indeed a fit is now possible. The problem stems from the fact that cos nX have a zero derivatives as $X \to \pi$ and so it will be difficult to fit a field that varies a lot as $x \to x_{max}$ (as will be the case if Xmax corresponds to the pole piece). Thus one should choose a value of $x_o > x_{max}$.

Similarly since the Z dependence of the fit has zero derivatives at $Z = \pm$ Zmax (for cosine terms), one should not attempt to fit a subset (in Z) of the field map - but rather fit the whole field volume down to $|B| = 0$.

RESULTS OF THE FITTING

Eq. (5) has been solved using the main field component measurements of the LAMP W75 magnet for all combinations of the maximum number of terms up to $N_i = 40$, $N_j = 8$. In each case the field was reconstructed at all data points using eq. (3), and the overall accuracy of the fit determined by computing the rms error, σ_{rms}, defined as

$$\sigma_{rms} = \sqrt{\sum_{abc}^{N} (B_{abc}^{CALC} - B_{abc}^{DATA})^2 / N} \quad \text{DVM units},$$

where N = total number of data points and 1 DVM unit = 4 Gauss.

This error is plotted as a function of the total number of terms used for I = 2500A in fig. 1. This illustrates a good plateau in σ_{rms} starting at roughly $N_i = 25$, $N_j = 4$ (*). Histogram of the individual errors, $(B_{CALC} - B_{DATA})$ are shown in fig. 2. These exhibit the expected symmetric distribution peaked about zero. A more detailed scrutiny of these error distributions shows that the larger deviations from the measured field occur predominantly, in those regions of the magnet near the coils and pole tips.

(*) Footnote to LAMP group

Unfortunately extrapolations of the fit to regions of large x ($56 \leq x \leq 65$ cms) are required to fill in unmeasured regions of the W75 field [3]. For this reason high powers in x, and hence the number of terms in j, should be restricted to avoid strong oscillations in the extrapolated field; the number of terms in j used in the fit must therefore be the smallest consistent with a good fit to the data (i.e. $N_j = 4$).

APPENDIX A

The orthogonality properties of sine and cosine functions, i.e. $\int_{-\pi}^{+\pi} \cos iz \cos jz = \delta_{ij}$ etc., cannot be directly applied in this case as we are dealing, not with an infinite sum, but with a sum over a finite number of terms; this will always leave us with non-cancelling contributions from end point terms of the sum to violate the supposed orthogonality. However, if we introduce the function

$$D(z) = \begin{cases} 1 \text{ for } z \neq \pm\pi \\ 2 \text{ for } z = \pm\pi \end{cases}$$

these end terms can be forced to cancel [2], leaving the three identities

$$\sum_{-\pi}^{\pi} \frac{\cos iz \cos jz}{D(z)} = \left[\sum_{-\pi}^{\pi} \frac{\cos^2 iz}{D(z)}\right] \delta_{ij}$$

$$\sum_{-\pi}^{\pi} \frac{\sin iz \cos jz}{D(z)} = 0 \ \forall i,j$$

and

$$\sum_{-\pi}^{\pi} \frac{\sin iz \sin jz}{D(z)} = \left[\sum_{-\pi}^{\pi} \frac{\sin^2 iz}{D(z)}\right] \delta_{ij}.$$

To apply this to eq. (4) we need only make the transformation

$$\sigma(z) \rightarrow \begin{cases} \sigma(z) & \text{for } z \neq \pm\pi \\ \sqrt{2}\,\sigma(z) & \text{for } z = \pm\pi. \end{cases}$$

REFERENCES

[1] Note to the LAMP group entitled 'W75 Magnetic Field Measurement' by A.M. Osborne, 17/1/74.

[2] C. Lanczos - Applied Analysis p.233.

[3] Note to the LAMP group entitled 'W75 Magnetic Field' by A.M. Osborne, 16/5/74.

FIGURE CAPTIONS

Fig. 1 rms error for fit to the measured y-component of the field at a current setting of 2500 amps as a function of the number of terms used in the fit.

Fig. 2 Distribution of errors for fit to field at a current setting of 2500 amps.

Fig. 3 Plots of B_y and B_z as a function of z for 7000 amps obtained from the fit. Field units are arbitary.

2500 AMPS

FIG 1

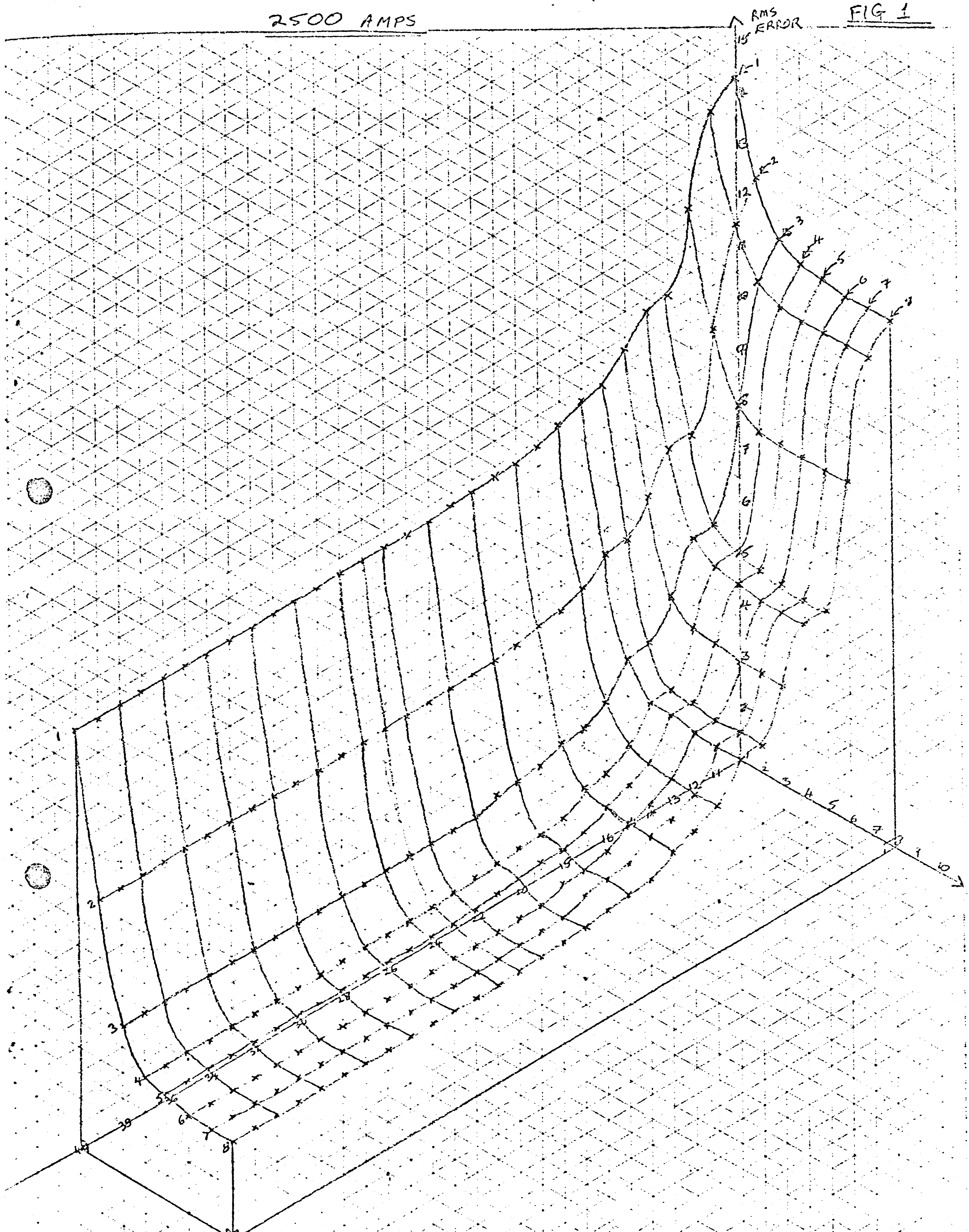
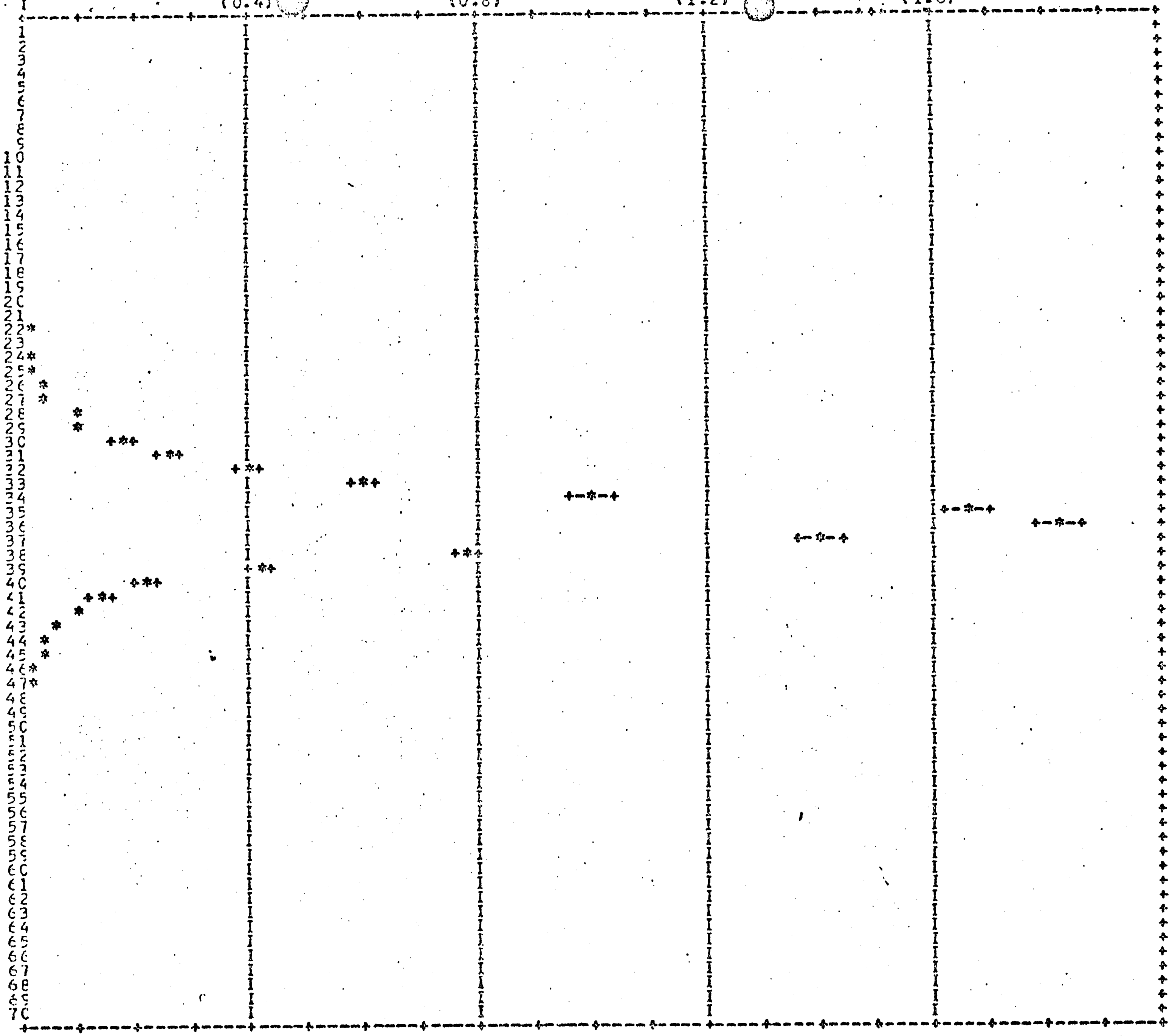


FIG 2

FULL SCALE= 0.20E+04 HIST CUT BY 0 MIN EVENTS



N(I)	X(I)	I
0	-35.0000	1
0	-34.0000	2
0	-33.0000	3
0	-32.0000	4
0	-31.0000	5
0	-30.0000	6
0	-29.0000	7
0	-28.0000	8
0	-27.0000	9
1	-26.0000	10
0	-25.0000	11
0	-24.0000	12
0	-23.0000	13
2	-22.0000	14
0	-21.0000	15
0	-20.0000	16
0	-19.0000	17
2	-18.0000	18
0	-17.0000	19
2	-16.0000	20
2	-15.0000	21
12	-14.0000	22
8	-13.0000	23
25	-12.0000	24
21	-11.0000	25
45	-10.0000	26
47	-9.0000	27
91	-8.0000	28
100	-7.0000	29
183	-6.0000	30
264	-5.0000	31
405	-4.0000	32
592	-3.0000	33
1003	-2.0000	34
1664	-1.0000	35
1825	0.0000	36
1403	1.0000	37
771	2.0000	38
416	3.0000	39
220	4.0000	40
139	5.0000	41
91	6.0000	42
53	7.0000	43
46	8.0000	44
37	9.0000	45
20	10.0000	46
13	11.0000	47
6	12.0000	48
5	13.0000	49
1	14.0000	50
2	15.0000	51
2	16.0000	52
1	17.0000	53
0	18.0000	54
0	19.0000	55
0	20.0000	56
0	21.0000	57
0	22.0000	58
0	23.0000	59
0	24.0000	60
0	25.0000	61
0	26.0000	62
0	27.0000	63
0	28.0000	64
0	29.0000	65
0	30.0000	66
0	31.0000	67
0	32.0000	68
0	33.0000	69
0	34.0000	70

9520 (TOTAL) SCALE FACTOR = 0.10E+01 0 TOO LOW 0 TOO HIGH

MEAN= -0.3887E-02 STANDARD DEVIATION= 0.3145E+01 X(I) DEFINED AT MINIMUM OF BIN

PLOT OF FITTED FIELD

FIG. B

7000AMPSBZ NSS 7000AMPSBZ NSS

VARYING Y X= 63.0

SS MAX=3899