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Geometry Reconstruction using Muon Tracks in DATCHA CERN

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Abstract

The geometry of the muon chambers is reconstructed using straight muon tracks. The cross-plates are taken as the basic units to be fitted, with three parameters each (Δz,Δy,$\Delta\alpha$). This gives 7 angular and 12 positional parameters. The fit reproduces known input parameters in a Monte Carlo run within a few sigma. With 50k tracks the cross-plate positions are determined in the sagitta direction with a precision of 10-20 μm. The multiple scattering of 1 mrad is the dominant contribution to the error. Applied to 10 real data runs, the corrected sagitta is consistent with zero all over the BML chamber. The angles between pairs of track segments are also consistent with zero. Using the geometry reconstructed from the muons, the 16 RASNIK systems are calibrated 10 times. The error on this calibration in the sagitta direction is 40-60 μm per run of 50k tracks; the r.m.s. over the 10 runs is 30-160 μm.

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1 Introduction

One of the challenges of DATCHA is to test the feasibility of the absolute calibration of the RASNIKs in-situ using straight muon tracks. This can be done in three steps. The first step is to reconstruct the detector geometry from the (straight) muon tracks. The second step is to predict the RASNIK readings from the new positions of the alignment components and to compare these to the RASNIK readings measured. This gives the offsets of the RASNIK coordinates, i.e. the calibration. The calibration is tested in the third step by doing track-by-track sagitta correcction using the RASNIK measurements and then to check whether the average sagitta is zero all over the chamber. This notes deals in detail with the first step: an algorithm to reconstruct the detector geometry from the straight muon tracks. It also gives the RASNIK calibration values from 10 different runs.
Similar work has been published in ATLAS Muon Note 246 [1], where a different algorithm is used to reconstuct the geometry from the muon tracks.

2 Choice of Parameters

The tracks that are found depend only on the wire positions, so it suffices to know where the wires are. The wire positions are determined by the wire locators and the gravitional sag in between them. The wire locators are positioned by the cross-plates, and the gravitational sag is known when the wire tension is known. These gravitational sags are already taken into account in the DATCHA muon track reconstruction. Hence the basic units to be fitted are the cross-plates, since they determine the wire positions.

The coordinate system is defined as follows:

x: along the wire,
y: vertical,
z: along the cross-plate,
with rotations α, β, γ around the x, y, z axis respectively.

This gives a priory six degrees of freedom per cross-plate (neglecting cross-plate sag and expansion), or 9x6 =54 degrees of freedom for the total of nine cross-plates. Per cross-plate only three of the six degrees of freedom (y,z,α) influence (to first order) the wire positions and are thus seen by the tracks, and they are the only ones that can be fitted. This reduces the number of degrees of freedom by a factor of two to 27. The BIL middle cross-plate is irrelevant, since BIL does not have a central wire locator. So we are left with 27-3=24 degrees of freedom. For the angles α one cross-plate has to be fixed and for the positions y and z two cross-plates have to be fixed. The number of parameters to be fitted is thus 24-1-4=19.

We use upper index '-' for the cross-plate at $x < 0$, '0' for the one at x=0, and '+' for the one at $x > 0$; lower index 'i' for BIL, 'm' for BML and 'o' for BOL. We denote the displacement and rotation parameters of a cross-plate as Δz, Δy and $\Delta\alpha$. (see figure 1).We take the BML middle cross-plate as a reference for the angles, and the two outer BIL cross-plates as a reference for the positions:

$$\Delta\alpha_m^0 = \Delta z_i^- = \Delta y_i^- = \Delta z_i^+ = \Delta y_i^+ = 0\,.$$

and the 19 parameters are:

BIL: $\Delta\alpha_i^-$, $\Delta\alpha_i^+$ (2 parameters),

BML: $\Delta\alpha_m^-$, $\Delta\alpha_m^+$, Δz_m^-, Δy_m^-, Δz_m^0, Δy_m^0, Δz_m^+, Δy_m^+ (8 parameters),

BOL: $\Delta\alpha_o^-$, $\Delta\alpha_o^0$, $\Delta\alpha_o^+$, Δz_o^-, Δy_o^-, Δz_o^0, Δy_o^0, Δz_o^+, Δy_o^+ (9 parameters).

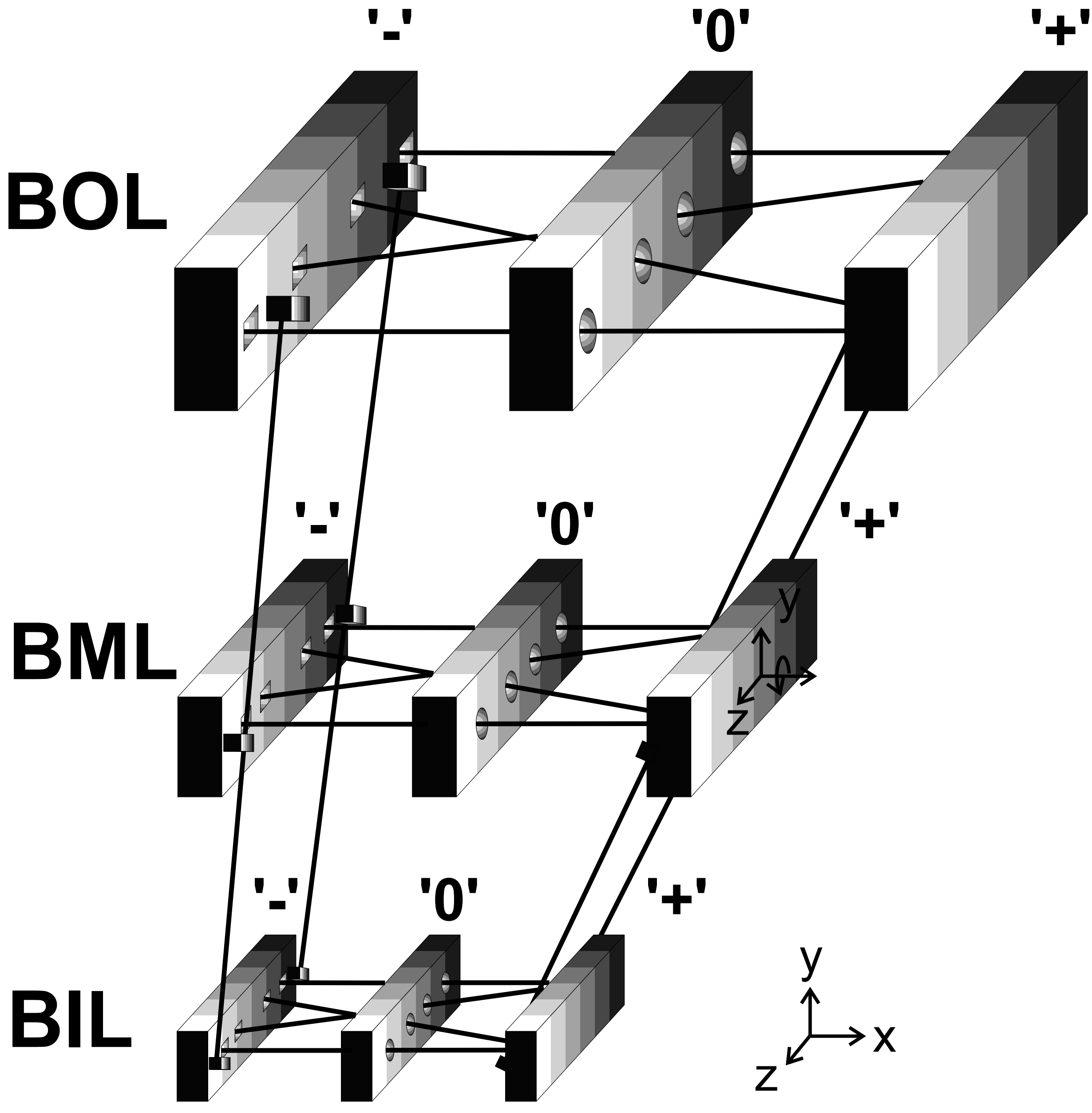

FIGURE 1. Choice of parameters and designation of cross-plates.

3 Fitting the parameters

The parameters are fitted by minimising the χ^2, which is made up of the information contained in the track segments in each of the three chambers. Denoting the track segments, which are just vectors, by $\vec{l}$ (see figure 2), the χ^2 is given by

$$\chi^2 = \left\|\vec{l}_i - \vec{l}_m\right\|^2 + \left\|\vec{l}_o - \vec{l}_m\right\|^2 + \left\|\vec{l}_i - \vec{l}_o\right\|^2$$

The crucial question is how to define $\left\|\vec{l} - \vec{l}'\right\|$?

To make life easier, we separate the fit of the angles from the fit of the positions, and correct the one for the other in between the two fitting stages.

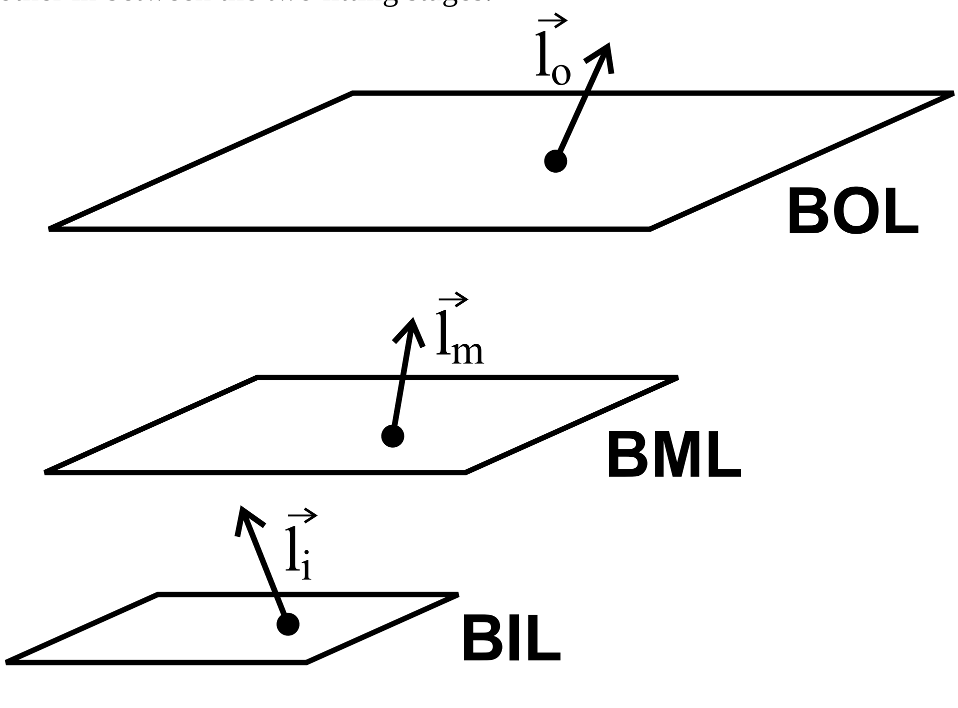

FIGURE 2. The three track segments in the chambers.

The local displacement (Δz,Δy) of a *wire* is obtained by linear interpolation in *x* between the two cross-plates concerned. This is because the angle between the wires and the gravitational field hardly changes. We define $\hat{x}$ as being the fractional position in *x* of the track in a chamber:

$$\hat{x} = 2\frac{x - x^0}{x^+ - x^-} = \frac{2x}{x^+ - x^-} \text{ so that } -1 < \hat{x} < +1 .$$

The total χ^2 is the sum of the χ^2's of the track segments in all three possible pairs of chambers:

$$\chi^2 = \chi^2_{i,m} + \chi^2_{o,m} + \chi^2_{i,o}$$

This χ^2 consists of 8 parts, depending on the chambers concerned and where the track traverses the chambers. $\chi^2_{i,m}$ and $\chi^2_{i,o}$ each have 2 parts, and $\chi^2_{m,o}$ has 4 parts.

4 Fitting the angles

The real angle of a track is the measured angle plus a local (i.e. at a certain x) track angle correction. The local rotation of the chamber is a simple linear interpolation between the rotation of the neighbouring cross-plates:

$$\alpha^{real} = \alpha^{meas} + \Delta\alpha(\hat{x})$$

For the three chambers $\Delta\alpha$ is given by

$$\Delta\alpha_i(\hat{x}) = \frac{1-\hat{x}_i}{2}\cdot\Delta\alpha_i^- + \frac{1+\hat{x}_i}{2}\cdot\Delta\alpha_i^+ , \tag{1a}$$

$$\Delta\alpha_m(\hat{x}) = \begin{cases} \hat{x}_m\cdot\Delta\alpha_m^+ & \text{for } \hat{x}_m \geq 0 \\ -\hat{x}_m\cdot\Delta\alpha_m^- & \text{for } \hat{x}_m < 0 \end{cases} \quad \text{and} \tag{1b}$$

$$\Delta\alpha_o(\hat{x}) = \begin{cases} (1-\hat{x}_o)\cdot\Delta\alpha_o^0 + \hat{x}_o\cdot\Delta\alpha_o^+ & \text{for } \hat{x}_o \geq 0 \\ (1+\hat{x}_o)\cdot\Delta\alpha_o^0 - \hat{x}_o\cdot\Delta\alpha_o^- & \text{for } \hat{x}_o < 0 \end{cases} . \tag{1c}$$

This gives

$$\alpha_i^{real} = \alpha_i^{meas} + \frac{1-\hat{x}_i}{2}\cdot\Delta\alpha_i^- + \frac{1+\hat{x}_i}{2}\cdot\Delta\alpha_i^+ ,$$

$$\alpha_m^{real} = \alpha_m^{meas} + \begin{cases} \hat{x}_m\cdot\Delta\alpha_m^+ & \text{for } \hat{x}_m \geq 0 \\ -\hat{x}_m\cdot\Delta\alpha_m^- & \text{for } \hat{x}_m < 0 \end{cases} \quad \text{and}$$

$$\alpha_o^{real} = \alpha_o^{meas} + \begin{cases} (1-\hat{x}_o)\cdot\Delta\alpha_o^0 + \hat{x}_o\cdot\Delta\alpha_o^+ & \text{for } \hat{x}_o \geq 0 \\ (1+\hat{x}_o)\cdot\Delta\alpha_o^0 - \hat{x}_o\cdot\Delta\alpha_o^- & \text{for } \hat{x}_o < 0 \end{cases} .$$

Although the above equations are exact for a wire, they are only exact for a track segment if the rotation angle is the same all over the track segment. This is true if there is *no torque* in the chamber *and no rotation* of the complete chamber around the y and z axes *and no chamber sag* in y and z (or the track should be vertical in the x-y plane, but that is not an interesting case). Any of these cases will make the rotation angle vary along the track segment, since the track segment extends in x. The exact correction on the angle can only be obtained by re-doing the complete trackfit using displaced wire positions. This is however quite slow. We therefore use two approximations.

1. We correct the track angles before feeding them into the fit. We take two ‘super-points’ on the track, each halfway (in y) either multilayer. These points we move (using the positional parameters) and rotate (with the angular parameters) using the interpolated (exact!) formulas. From the two new points we calculate the new angle, which is fed into the fit as the ‘measurement’
2. In the χ^2 itself we use the above simple interpolated formulas.

We now replace the upper index ‘meas’ with the track index number k.

4.1 Fitting BIL - BML

We first deal with the $\chi^2{}_{i,m}$ part of the χ^2, using track segments in the BIL and BML chamber. The error on the measurement is given by $(\sigma^k_{i,m})^2 = (\sigma^k_i)^2 + (\sigma^k_m)^2$, which is dominated by multiple scattering.

$$\chi^2_{i,m} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,m})^2} \begin{cases} \left[\frac{1-\hat{x}^k_i}{2} \cdot \Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2} \cdot \Delta\alpha^+_i - \hat{x}^k_m \cdot \Delta\alpha^+_m + (\alpha^k_i - \alpha^k_m)\right]^2 & \text{for } \hat{x}^k_m \geq 0 \\ \left[\frac{1-\hat{x}^k_i}{2} \cdot \Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2} \cdot \Delta\alpha^+_i + \hat{x}^k_m \cdot \Delta\alpha^-_m + (\alpha^k_i - \alpha^k_m)\right]^2 & \text{for } \hat{x}^k_m < 0 \end{cases}$$

The derivatives of $\chi^2{}_{i,m}$ are for $\hat{x}^k_m \geq 0$:

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^-_i} = 2\sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,m})^2}\left[\frac{1-\hat{x}^k_i}{2} \cdot \Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2} \cdot \Delta\alpha^+_i - \hat{x}^k_m \cdot \Delta\alpha^+_m + (\alpha^k_i - \alpha^k_m)\right] \cdot \frac{1-\hat{x}^k_i}{2},$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^+_i} = 2\sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,m})^2}\left[\frac{1-\hat{x}^k_i}{2} \cdot \Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2} \cdot \Delta\alpha^+_i - \hat{x}^k_m \cdot \Delta\alpha^+_m + (\alpha^k_i - \alpha^k_m)\right] \cdot \frac{1+\hat{x}^k_i}{2},$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^-_m} = 0,$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^+_m} = 2\sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,m})^2}\left[\frac{1-\hat{x}^k_i}{2} \cdot \Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2} \cdot \Delta\alpha^+_i - \hat{x}^k_m \cdot \Delta\alpha^+_m + (\alpha^k_i - \alpha^k_m)\right] \cdot -\hat{x}^k_m,$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^-_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^0_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta\alpha^+_o} = 0.$$

To minimise this χ^2, we set all the derivatives to zero. This can be written in matrix notation as $A \cdot \overrightarrow{\Delta\alpha} + \vec{b} = \vec{0}$ with A a symmetric 7x7 matrix, $\vec{b}$ a column vector of size 7, $\vec{b} = \vec{b}^+ + \vec{b}^-$ and $A = A^+ + A^-$. The upper index ‘+’ stands for $\hat{x}^k_m \geq 0$, and ‘-’ stands for $\hat{x}^k_m < 0$. The matrices and vectors are given in appendix A, and the angles vector is given by

$$\overrightarrow{\Delta\alpha} = [\Delta\alpha^-_i, \Delta\alpha^+_i, \Delta\alpha^-_m, \Delta\alpha^+_m, \Delta\alpha^-_o, \Delta\alpha^0_o, \Delta\alpha^+_o]^T.$$

4.2 Fitting BIL-BOL

Next we deal with the $\chi^2{}_{i,o}$ part, using track segments in the BIL and BOL chamber. The error on the measurement is given by $(\sigma^k_{i,o})^2 = (\sigma^k_i)^2 + (\sigma^k_o)^2$, which is dominated by multiple scattering. We follow the same procedure as for BIL-BML.

$$\chi^2_{i,o} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,o})^2} \begin{cases} \left[\frac{1-\hat{x}^k_i}{2}\cdot\Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2}\cdot\Delta\alpha^+_i - (1-\hat{x}^k_o)\cdot\Delta\alpha^0_o - \hat{x}^k_o\cdot\Delta\alpha^+_o + (\alpha^k_i - \alpha^k_o)\right]^2 & \text{for } \hat{x}^k_o \ge 0 \\ \left[\frac{1-\hat{x}^k_i}{2}\cdot\Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2}\cdot\Delta\alpha^+_i - (1+\hat{x}^k_o)\cdot\Delta\alpha^0_o + \hat{x}^k_o\cdot\Delta\alpha^-_o + (\alpha^k_i - \alpha^k_o)\right]^2 & \text{for } \hat{x}^k_o < 0 \end{cases}$$

The derivatives of $\chi^2{}_{i,o}$ are for $\hat{x}^k_o \ge 0$:

$$\frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^-_i} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,o})^2}\left[\frac{1-\hat{x}^k_i}{2}\Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2}\Delta\alpha^+_i - (1-\hat{x}^k_o)\Delta\alpha^0_o - \hat{x}^k_o\Delta\alpha^+_o + (\alpha^k_i - \alpha^k_o)\right]\cdot\frac{1-\hat{x}^k_i}{2}$$

$$\frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^+_i} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,o})^2}\left[\frac{1-\hat{x}^k_i}{2}\Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2}\Delta\alpha^+_i - (1-\hat{x}^k_o)\Delta\alpha^0_o - \hat{x}^k_o\Delta\alpha^+_o + (\alpha^k_i - \alpha^k_o)\right]\cdot\frac{1+\hat{x}^k_i}{2}$$

$$\frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^-_m} = \frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^+_m} = \frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^-_o} = 0$$

$$\frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^0_o} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,o})^2}\left[\frac{1-\hat{x}^k_i}{2}\Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2}\Delta\alpha^+_i - (1-\hat{x}^k_o)\Delta\alpha^0_o - \hat{x}^k_o\Delta\alpha^+_o + (\alpha^k_i - \alpha^k_o)\right]\cdot -(1-\hat{x}^k_o)$$

$$\frac{\partial\chi^2_{i,o}}{\partial\Delta\alpha^+_o} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,o})^2}\left[\frac{1-\hat{x}^k_i}{2}\Delta\alpha^-_i + \frac{1+\hat{x}^k_i}{2}\Delta\alpha^+_i - (1-\hat{x}^k_o)\Delta\alpha^0_o - \hat{x}^k_o\Delta\alpha^+_o + (\alpha^k_i - \alpha^k_o)\right]\cdot -\hat{x}^k_o .$$

Setting these derivatives to zero can again be expressed in a matrix equation. See appendix A.

4.3 Fitting BOL-BML

Finally we deal with the $\chi^2{}_{\mathrm{o,m}}$ part, using track segments in the BOL and BML chamber. The error on the measurement is given by $(\sigma^k_{\mathrm{o,m}})^2 = (\sigma^k_{\mathrm{o}})^2 + (\sigma^k_{\mathrm{m}})^2$, again dominated by multiple scattering. We follow the same procedure as for the other two combinations.

$$\chi^2_{\mathrm{o,m}} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{\mathrm{o,m}})^2} \begin{cases} [(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta\alpha^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta\alpha^+_{\mathrm{o}} - \hat{x}^k_{\mathrm{m}}\cdot\Delta\alpha^+_{\mathrm{m}} + (\alpha^k_{\mathrm{o}} - \alpha^k_{\mathrm{m}})]^2 & \text{for } \hat{x}^k_{\mathrm{o}} \ge 0, \hat{x}^k_{\mathrm{m}} \ge 0 \\ [(1+\hat{x}^k_{\mathrm{o}})\cdot\Delta\alpha^0_{\mathrm{o}} - \hat{x}^k_{\mathrm{o}}\cdot\Delta\alpha^-_{\mathrm{o}} - \hat{x}^k_{\mathrm{m}}\cdot\Delta\alpha^+_{\mathrm{m}} + (\alpha^k_{\mathrm{o}} - \alpha^k_{\mathrm{m}})]^2 & \text{for } \hat{x}^k_{\mathrm{o}} < 0, \hat{x}^k_{\mathrm{m}} \ge 0 \\ [(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta\alpha^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta\alpha^+_{\mathrm{o}} + \hat{x}^k_{\mathrm{m}}\cdot\Delta\alpha^-_{\mathrm{m}} + (\alpha^k_{\mathrm{o}} - \alpha^k_{\mathrm{m}})]^2 & \text{for } \hat{x}^k_{\mathrm{o}} \ge 0, \hat{x}^k_{\mathrm{m}} < 0 \\ [(1+\hat{x}^k_{\mathrm{o}})\cdot\Delta\alpha^0_{\mathrm{o}} - \hat{x}^k_{\mathrm{o}}\cdot\Delta\alpha^-_{\mathrm{o}} + \hat{x}^k_{\mathrm{m}}\cdot\Delta\alpha^-_{\mathrm{m}} + (\alpha^k_{\mathrm{o}} - \alpha^k_{\mathrm{m}})]^2 & \text{for } \hat{x}^k_{\mathrm{o}} < 0, \hat{x}^k_{\mathrm{m}} < 0 \end{cases}$$

To minimise this part of the χ^2, we again put the derivatives to zero. This time we have 4 contributions to the matrices, which are indicated by the upper indices '++', '-+', '+-' and '--', depending on the signs of $\hat{x}^k_{\mathrm{o}}$ and $\hat{x}^k_{\mathrm{m}}$. The matrices and vectors are given in appendix A.

The 7 parameters $\Delta\alpha$ can be found in one go, including all correlations by adding all the matrices, and then doing the inversion and multiplication:

$$\overrightarrow{\Delta\alpha} = -A^{-1}\cdot\vec{b}$$

with

$$\vec{b} = \vec{b}^+_{\mathrm{i,m}} + \vec{b}^-_{\mathrm{i,m}} + \vec{b}^+_{\mathrm{i,o}} + \vec{b}^-_{\mathrm{i,o}} + \vec{b}^{++}_{\mathrm{o,m}} + \vec{b}^{-+}_{\mathrm{o,m}} + \vec{b}^{+-}_{\mathrm{o,m}} + \vec{b}^{--}_{\mathrm{o,m}} \text{ and}$$

$$A = A^+_{\mathrm{i,m}} + A^-_{\mathrm{i,m}} + A^+_{\mathrm{i,o}} + A^-_{\mathrm{i,o}} + A^{++}_{\mathrm{o,m}} + A^{-+}_{\mathrm{o,m}} + A^{+-}_{\mathrm{o,m}} + A^{--}_{\mathrm{o,m}}.$$

The covariance matrix is $C_{\overrightarrow{\Delta\alpha}} = A^{-1}$ and the errors on the $\Delta\alpha$ are given by $\sigma_{\overrightarrow{\Delta\alpha}} = \sqrt{C_{jj}}$.

5 Fitting the positions

For the positions we follow a similar procedure as for the angles. The local (in x) correction of a *wire* position in y *and* z is again a simple linear interpolation in between the two neighbouring cross-plates, so the real wire positions are given by the following equations, after y^{meas} and z^{meas} have been rotated around the chamber centre by $\Delta\alpha$ (equations 1a-1c):

$$z_i^{real} = z_i^{meas} + \frac{1-\hat{x}_i}{2} \cdot \Delta z_i^- + \frac{1+\hat{x}_i}{2} \cdot \Delta z_i^+ ,$$

$$y_i^{real} = y_i^{meas} + \frac{1-\hat{x}_i}{2} \cdot \Delta y_i^- + \frac{1+\hat{x}_i}{2} \cdot \Delta y_i^+ ,$$

$$z_m^{real} = z_m^{meas} + \begin{cases} \hat{x}_m \cdot \Delta z_m^+ & \text{for } \hat{x}_m \geq 0 \\ -\hat{x}_m \cdot \Delta z_m^- & \text{for } \hat{x}_m < 0 \end{cases} ,$$

$$y_m^{real} = y_m^{meas} + \begin{cases} \hat{x}_m \cdot \Delta y_m^+ & \text{for } \hat{x}_m \geq 0 \\ -\hat{x}_m \cdot \Delta y_m^- & \text{for } \hat{x}_m < 0 \end{cases} ,$$

$$z_o^{real} = z_o^{meas} + \begin{cases} (1-\hat{x}_o) \cdot \Delta z_o^0 + \hat{x}_o \cdot \Delta z_o^+ & \text{for } \hat{x}_o \geq 0 \\ (1+\hat{x}_o) \cdot \Delta z_o^0 - \hat{x}_o \cdot \Delta z_o^- & \text{for } \hat{x}_o < 0 \end{cases} ,$$

$$y_o^{real} = y_o^{meas} + \begin{cases} (1-\hat{x}_o) \cdot \Delta y_o^0 + \hat{x}_o \cdot \Delta y_o^+ & \text{for } \hat{x}_o \geq 0 \\ (1+\hat{x}_o) \cdot \Delta y_o^0 - \hat{x}_o \cdot \Delta y_o^- & \text{for } \hat{x}_o < 0 \end{cases} .$$

The displacement of the track segment is equal to the wire displacement given by the equations above if and only if the total displacement is the same all over the track. Total meaning here the displacement because of Δy, Δz, *and rotation* $\Delta\alpha$. The approach is therefore the same as for the angles. We make two approximations:

1. We correct the track angles before feeding them into the fit. We take two ‘super-points’ on the track, each halfway (in y) either multilayer. These points we move (using the positional parameters) and rotate (with the angular parameters) using the interpolated (exact!) formulas. From the two new points we calculate the new track positions in y and z, which is fed into the fit as the ‘measurement’
2. In the χ^2 itself we use the above simple interpolated formulas.

In the χ^2 we have to put somehow the distance between the track segments in the chambers.

In principle the angles of the track segments should be the same for one track. This is however not true because of multiple scattering. But we know that they should be the same, so we force the angles of the two track segments to be the average of all three (corrected) angles: $\bar{\alpha}$. Then the χ^2 is simply the square of the distance between two parallel lines!

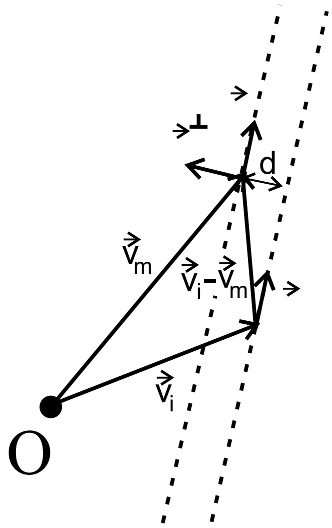

FIGURE 3. Calculation of the distance d between two parallel lines with support vectors v_i and v_m and direction vector α.

The distance between two parallel lines can be calculated directly using vector algebra and is given by (see figure 3):

$$d = \left|(\vec{v}_i - \vec{v}_m) \bullet \vec{\alpha}^{\perp}\right| ,$$

where $\vec{v}_i$ and $\vec{v}_m$ are the supports vector w.r.t. an arbitrary zero ('O') and $\vec{\alpha}^{\perp}$ the unit vector perpendicular to the direction vector $\vec{\alpha}$.

5.1 Fitting BIL-BML

We first take the track segments in BIL and BML with direction vector

$$\vec{\alpha}_{\mathrm{i,m}} = \begin{bmatrix} \cos(\bar{\alpha}) \\ \sin(\bar{\alpha}) \end{bmatrix}$$

and support vectors

$$\vec{v}_{\mathrm{i}} = \begin{bmatrix} z_{\mathrm{i}} \\ y_{\mathrm{i}} \end{bmatrix},$$

$$\vec{v}_{\mathrm{m}}^{+} = \begin{bmatrix} z_{\mathrm{m}} + (1-\hat{x}_{\mathrm{m}})\Delta z_{\mathrm{m}}^{0} + \hat{x}_{\mathrm{m}}\Delta z_{\mathrm{m}}^{+} \\ y_{\mathrm{m}} + (1-\hat{x}_{\mathrm{m}})\Delta y_{\mathrm{m}}^{0} + \hat{x}_{\mathrm{m}}\Delta y_{\mathrm{m}}^{+} \end{bmatrix} \quad (\hat{x}_{\mathrm{m}} \geq 0),$$

$$\vec{v}_{\mathrm{m}}^{-} = \begin{bmatrix} z_{\mathrm{m}} + (1+\hat{x}_{\mathrm{m}})\Delta z_{\mathrm{m}}^{0} - \hat{x}_{\mathrm{m}}\Delta z_{\mathrm{m}}^{-} \\ y_{\mathrm{m}} + (1+\hat{x}_{\mathrm{m}})\Delta y_{\mathrm{m}}^{0} - \hat{x}_{\mathrm{m}}\Delta y_{\mathrm{m}}^{-} \end{bmatrix} \quad (\hat{x}_{\mathrm{m}} < 0).$$

where (z_{i},y_{i}) and (z_{m},y_{m}) are arbitrary points on the extrapolated track segment. The distance between the two lines is now simply $\left|(\vec{v}_{\mathrm{i}} - \vec{v}_{\mathrm{m}}) \bullet \vec{\alpha}^{\perp}\right|$.

The χ^2 is given by the square of this, with the upper indices ‘meas’ are replaced by the upper index k (so k is *not* a power!):

$$\chi^2_{\mathrm{i,m}} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{\mathrm{i,m}})^2} \begin{cases} \left(\begin{bmatrix} -(1-\hat{x}_{\mathrm{m}})\Delta z_{\mathrm{m}}^{0} - \hat{x}^k_{\mathrm{m}}\Delta z_{\mathrm{m}}^{+} + (z^k_{\mathrm{i}} - z^k_{\mathrm{m}}) \\ -(1(-\hat{x}_{\mathrm{m}}))\Delta y_{\mathrm{m}}^{0} - \hat{x}^k_{\mathrm{m}}\Delta y_{\mathrm{m}}^{+} + (y^k_{\mathrm{i}} - y^k_{\mathrm{m}}) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix} \right)^2 & \text{for } \hat{x}^k_{\mathrm{m}} \geq 0 \\ \left(\begin{bmatrix} -(1+\hat{x}_{\mathrm{m}})\Delta z_{\mathrm{m}}^{0} + \hat{x}^k_{\mathrm{m}}\Delta z_{\mathrm{m}}^{-} + (z^k_{\mathrm{i}} - z^k_{\mathrm{m}}) \\ -(1+\hat{x}_{\mathrm{m}})\Delta y_{\mathrm{m}}^{0} + \hat{x}^k_{\mathrm{m}}\Delta y_{\mathrm{m}}^{-} + (y^k_{\mathrm{i}} - y^k_{\mathrm{m}}) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix} \right)^2 & \text{for } \hat{x}^k_{\mathrm{m}} < 0 \end{cases},$$

with the error

$$(\sigma^k_{\mathrm{i,m}})^2 = ((z^k_{\mathrm{i}} - z^k_{\mathrm{m}})^2 + (y^k_{\mathrm{i}} - y^k_{\mathrm{m}})^2)(\sigma^k_{\bar{\alpha}})^2 + (\sigma^k_{\mathrm{pos,i}})^2 + (\sigma^k_{\mathrm{pos,m}})^2, \qquad (2)$$

where σ_{α} is dominated by multiple scattering and gives the major contribution to the error; σ_{pos} is the positional error perpendicular to the track.

The derivatives are for $\hat{x}_m^k \geq 0$:

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta z^-_m} = \frac{\partial\chi^2_{i,m}}{\partial\Delta y^-_m} = 0$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta z^0_m} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,m})^2}\begin{bmatrix}-(1-\hat{x}_m)\Delta z^0_m - \hat{x}^k_m\Delta z^+_m + (z^k_i - z^k_m)\\ -(1-\hat{x}_m)\Delta y^0_m - \hat{x}^k_m\Delta y^+_m + (y^k_i - y^k_m)\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\ -\cos(\bar{\alpha}^k)\end{bmatrix}\cdot -(1-\hat{x}_m)\sin(\bar{\alpha}^k)$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta y^0_m} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,m})^2}\begin{bmatrix}-(1-\hat{x}_m)\Delta z^0_m - \hat{x}^k_m\Delta z^+_m + (z^k_i - z^k_m)\\ -(1-\hat{x}_m)\Delta y^0_m - \hat{x}^k_m\Delta y^+_m + (y^k_i - y^k_m)\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\ -\cos(\bar{\alpha}^k)\end{bmatrix}\cdot (1-\hat{x}_m)\cos(\bar{\alpha}^k)$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta z^+_m} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,m})^2}\begin{bmatrix}-(1-\hat{x}_m)\Delta z^0_m - \hat{x}^k_m\Delta z^+_m + (z^k_i - z^k_m)\\ -(1-\hat{x}_m)\Delta y^0_m - \hat{x}^k_m\Delta y^+_m + (y^k_i - y^k_m)\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\ -\cos(\bar{\alpha}^k)\end{bmatrix}\cdot -\hat{x}^k_m\sin(\bar{\alpha}^k)$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta y^+_m} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{i,m})^2}\begin{bmatrix}-(1-\hat{x}_m)\Delta z^0_m - \hat{x}^k_m\Delta z^+_m + (z^k_i - z^k_m)\\ -(1-\hat{x}_m)\Delta y^0_m - \hat{x}^k_m\Delta y^+_m + (y^k_i - y^k_m)\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\ -\cos(\bar{\alpha}^k)\end{bmatrix}\cdot \hat{x}^k_m\cos(\bar{\alpha}^k)$$

$$\frac{\partial\chi^2_{i,m}}{\partial\Delta z^-_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta y^-_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta z^0_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta y^0_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta z^+_o} = \frac{\partial\chi^2_{i,m}}{\partial\Delta y^+_o} = 0\,.$$

In analogy to the angles fit, the positions can be found by solving the matrix equation $A\cdot\overrightarrow{\Delta zy} + \vec{b} = \vec{0}$ with A a symmetric 12x12 matrix, $\vec{b}$ a column vector of size 12, where the A and $\vec{b}$ are the sum of matrices and vectors depending on the chambers included in the fit.
The vector with the parameters to be fitted $\overrightarrow{\Delta zy}$ is given by

$$\overrightarrow{\Delta zy} = [\Delta z^-_m, \Delta y^-_m, \Delta z^0_m, \Delta y^0_m, \Delta z^+_m, \Delta y^+_m, \Delta z^-_o, \Delta y^-_o, \Delta z^0_o, \Delta y^0_o, \Delta z^+_o, \Delta y^+_o]^T\,. \quad (3)$$

The matrices A and vectors $\vec{b}$ are given in appendix B. (4)

5.2 Fitting BIL-BOL

Next we take the track segments in BIL and BOL with direction vector

$$\vec{\alpha}_{i,m} = \begin{bmatrix} \cos(\bar{\alpha}) \\ \sin(\bar{\alpha}) \end{bmatrix}$$

and support vectors

$$\vec{v}_i = \begin{bmatrix} z_i \\ y_i \end{bmatrix},$$

$$\vec{v}_o^{\,+} = \begin{bmatrix} z_o + (1-\hat{x}_o)\Delta z_o^0 + \hat{x}_o \Delta z_o^+ \\ y_o + (1-\hat{x}_o)\Delta y_o^0 + \hat{x}_o \Delta y_o^+ \end{bmatrix} \quad (\hat{x}_o \geq 0),$$

$$\vec{v}_o^{\,-} = \begin{bmatrix} z_o + (1+\hat{x}_o)\Delta z_o^0 - \hat{x}_o \Delta z_o^- \\ y_o + (1+\hat{x}_o)\Delta y_o^0 - \hat{x}_o \Delta y_o^- \end{bmatrix} \quad (\hat{x}_o < 0).$$

where (z_i,y_i) and (z_o,y_o) are arbitrary points on the extrapolated track segment. The distance between the two lines is now simply $\left|(\vec{v}_i - \vec{v}_o) \bullet \vec{\alpha}^{\perp}\right|$.

The χ^2 is again the square of this:

$$\chi^2_{i,o} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,o})^2} \begin{cases} \left(\begin{bmatrix} -(1-\hat{x}^k_o)\Delta z^0_o - \hat{x}^k_o \Delta z^+_o + (z^k_i - z^k_o) \\ -(1-\hat{x}^k_o)\Delta y^0_o - \hat{x}^k_o \Delta y^+_o + (y^k_i - y^k_o) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix} \right)^2 & \text{for } \hat{x}^k_o \geq 0 \\ \left(\begin{bmatrix} -(1+\hat{x}^k_o)\Delta z^0_o + \hat{x}^k_o \Delta z^-_o + (z^k_i - z^k_o) \\ -(1+\hat{x}^k_o)\Delta y^0_o + \hat{x}^k_o \Delta y^-_o + (y^k_i - y^k_o) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix} \right)^2 & \text{for } \hat{x}^k_o < 0 \end{cases}$$

with the error

$$(\sigma^k_{i,o})^2 = ((z^k_i - z^k_o)^2 + (y^k_i - y^k_o)^2)(\sigma^k_{\bar{\alpha}})^2 + (\sigma^k_{pos,i})^2 + (\sigma^k_{pos,o})^2 \quad , \qquad (5)$$

where σ_{α} is dominated by multiple scattering and gives the major contribution to the error; σ_{pos} is the positional error perpendicular to the track.

The matrices and vectors are given in appendix B.

5.3 Fitting BOL-BML

Finally we take the track segments in BOL and BML (the most complicated one) with direction vector

$$\vec{\alpha}_{o,m} = \begin{bmatrix} \cos(\bar{\alpha}) \\ \sin(\bar{\alpha}) \end{bmatrix}$$

and support vectors

$$\vec{v}_o^+ = \begin{bmatrix} z_o + (1-\hat{x}_o)\cdot\Delta z_o + \hat{x}_o\cdot\Delta z_o^+ \\ y_o + (1-\hat{x}_o)\cdot\Delta y_o + \hat{x}_o\cdot\Delta y_o^+ \end{bmatrix} \quad (\hat{x}_o \geq 0),$$

$$\vec{v}_o^- = \begin{bmatrix} z_o + (1+\hat{x}_o)\cdot\Delta z_o - \hat{x}_o\cdot\Delta z_o^+ \\ y_o + (1+\hat{x}_o)\cdot\Delta y_o - \hat{x}_o\cdot\Delta y_o^+ \end{bmatrix} \quad (\hat{x}_o < 0),$$

$$\vec{v}_m^+ = \begin{bmatrix} z_m + (1-\hat{x}_m)\Delta z_m^0 + \hat{x}_m\Delta z_m^+ \\ y_m + (1-\hat{x}_m)\Delta y_m^0 + \hat{x}_m\Delta y_m^+ \end{bmatrix} \quad (\hat{x}_m \geq 0),$$

$$\vec{v}_m^- = \begin{bmatrix} z_m + (1+\hat{x}_m)\Delta z_m^0 - \hat{x}_m\Delta z_m^- \\ y_m + (1+\hat{x}_m)\Delta y_m^0 - \hat{x}_m\Delta y_m^- \end{bmatrix} \quad (\hat{x}_m < 0).$$

where (z_o, y_o) and (z_m, y_m) are arbitrary points on the extrapolated track segment. The distance between the two lines is now simply $\left|(\vec{v}_o - \vec{v}_m) \bullet \vec{\alpha}^{\perp}\right|$.

The χ^2 is given by the square of this:

$$\chi^2_{o,m} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{o,m})^2} \begin{cases} \left(\begin{bmatrix} (1-\hat{x}_o^k)\cdot\Delta z_o^0 + \hat{x}_o^k\cdot\Delta z_o^+ - (1-\hat{x}_m^k)\cdot\Delta z_m^0 - \hat{x}_m^k\Delta z_m^+ + (z_o^k - z_m^k) \\ (1-\hat{x}_o^k)\cdot\Delta y_o^0 + \hat{x}_o^k\cdot\Delta y_o^+ - (1-\hat{x}_m^k)\cdot\Delta y_m^0 - \hat{x}_m^k\Delta y_m^+ + (y_o^k - y_m^k) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix}\right)^2 & \text{for } \hat{x}_o^k \geq 0,\ \hat{x}_m^k \geq 0 \\ \left(\begin{bmatrix} (1+\hat{x}_o^k)\cdot\Delta z_o^0 - \hat{x}_o^k\cdot\Delta z_o^- - (1-\hat{x}_m^k)\cdot\Delta z_m^0 - \hat{x}_m^k\Delta z_m^+ + (z_o^k - z_m^k) \\ (1+\hat{x}_o^k)\cdot\Delta y_o^0 - \hat{x}_o^k\cdot\Delta y_o^- - (1-\hat{x}_m^k)\cdot\Delta y_m^0 - \hat{x}_m^k\Delta y_m^+ + (y_o^k - y_m^k) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix}\right)^2 & \text{for } \hat{x}_o^k < 0,\ \hat{x}_m^k \geq 0 \\ \left(\begin{bmatrix} (1-\hat{x}_o^k)\cdot\Delta z_o^0 + \hat{x}_o^k\cdot\Delta z_o^+ - (1+\hat{x}_m^k)\cdot\Delta z_m^0 + \hat{x}_m^k\Delta z_m^- + (z_o^k - z_m^k) \\ (1-\hat{x}_o^k)\cdot\Delta y_o^0 + \hat{x}_o^k\cdot\Delta y_o^+ - (1+\hat{x}_m^k)\cdot\Delta y_m^0 + \hat{x}_m^k\Delta y_m^- + (y_o^k - y_m^k) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix}\right)^2 & \text{for } \hat{x}_o^k \geq 0,\ \hat{x}_m^k < 0 \\ \left(\begin{bmatrix} (1+\hat{x}_o^k)\cdot\Delta z_o^0 - \hat{x}_o^k\cdot\Delta z_o^- - (1+\hat{x}_m^k)\cdot\Delta z_m^0 + \hat{x}_m^k\Delta z_m^- + (z_o^k - z_m^k) \\ (1+\hat{x}_o^k)\cdot\Delta y_o^0 - \hat{x}_o^k\cdot\Delta y_o^- - (1+\hat{x}_m^k)\cdot\Delta y_m^0 + \hat{x}_m^k\Delta y_m^- + (y_o^k - y_m^k) \end{bmatrix} \bullet \begin{bmatrix} \sin(\bar{\alpha}^k) \\ -\cos(\bar{\alpha}^k) \end{bmatrix}\right)^2 & \text{for } \hat{x}_o^k < 0,\ \hat{x}_m^k < 0 \end{cases}$$

with the error

$$(\sigma^k_{\mathrm{o,m}})^2 = ((z^k_{\mathrm{o}} - z^k_{\mathrm{m}})^2 + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})^2)(\sigma^k_{\bar{\alpha}})^2 + (\sigma^k_{\mathrm{pos,o}})^2 + (\sigma^k_{\mathrm{pos,m}})^2,$$

where σ_α is dominated by multiple scattering and gives the major contribution to the error; σ_{pos} is the positional error perpendicular to the track.

The derivatives for $\hat{x}^k_{\mathrm{o}} \ge 0$ and $\hat{x}^k_{\mathrm{m}} \ge 0$ are:

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^-_{\mathrm{i}}} = \frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^-_{\mathrm{i}}} = \frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^+_{\mathrm{i}}} = \frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^+_{\mathrm{i}}} = 0,$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^-_{\mathrm{m}}} = \frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^-_{\mathrm{m}}} = 0,$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^0_{\mathrm{m}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot -(1-\hat{x}^k_{\mathrm{m}})\sin(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^0_{\mathrm{m}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot (1-\hat{x}^k_{\mathrm{m}})\cos(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^+_{\mathrm{m}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot -\hat{x}^k_{\mathrm{m}}\sin(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^+_{\mathrm{m}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot \hat{x}^k_{\mathrm{m}}\cos(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^-_{\mathrm{o}}} = \frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^-_{\mathrm{o}}} = 0,$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^0_{\mathrm{o}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot (1-\hat{x}^k_{\mathrm{o}})\sin(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^0_{\mathrm{o}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot -(1-\hat{x}^k_{\mathrm{o}})\cos(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta z^+_{\mathrm{o}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot \hat{x}^k_{\mathrm{o}}\sin(\bar{\alpha}^k),$$

$$\frac{\partial\chi^2_{\mathrm{o,m}}}{\partial\Delta y^+_{\mathrm{o}}} = 2\sum_{k=1}^{N}\frac{1}{(\sigma^k_{\mathrm{o,m}})^2}\begin{bmatrix}(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta z^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta z^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta z^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta z^+_{\mathrm{m}} + (z^k_{\mathrm{o}} - z^k_{\mathrm{m}})\\(1-\hat{x}^k_{\mathrm{o}})\cdot\Delta y^0_{\mathrm{o}} + \hat{x}^k_{\mathrm{o}}\cdot\Delta y^+_{\mathrm{o}} - (1-\hat{x}^k_{\mathrm{m}})\cdot\Delta y^0_{\mathrm{m}} - \hat{x}^k_{\mathrm{m}}\Delta y^+_{\mathrm{m}} + (y^k_{\mathrm{o}} - y^k_{\mathrm{m}})\end{bmatrix}\bullet\begin{bmatrix}\sin(\bar{\alpha}^k)\\-\cos(\bar{\alpha}^k)\end{bmatrix}\cdot -\hat{x}^k_{\mathrm{o}}\cos(\bar{\alpha}^k).$$

The matrices and vectors for all four cases are given in appendix B.

6 Results

6.1 Monte Carlo Results

The algorithm as describes is implemented in C++, and is tested on a Monte Carlo of the DATCHA set-up which is produced using ARVE/GISMO [3]. The tracks were reconstructed using the DATCHA muon track reconstruction [2] in the mutdat framework [4]. In the track reconstruction we start from the ideal geometry and displace the positions of the hits using the cross-plate parameters before doing the reconstruction. This way we generate a ‘distorted’ DST file with track segments. The auto-calibrated r-t relation of the Monte Carlo is used. The first 60.000 tracks of MC run 3 are used to find back the input cross-plate parameters using the muon geometry fit. All cross-plates (that are not fixed in the fit) are moved in y and z and rotated around x (order mm and mrad). The y and z parameters per cross-plate are 99% correlated, as can be seen in table 1. This is because the tracks all have an angle of about -117 degrees with respect to the cross-plates. It is therefore not useful to compare the input/output parameters in y and z. Instead, we compare displacements in the sagitta direction (“sag”) and in the direction ‘parallel’ to the tracks (“par”). These values are obtained by rotating the coordinate system around x with the ideal average angle of the tracks, which is determined by the centres of the BIL and BOL chambers (in the ideal geometry), i.e. -117 degrees. Table 2 shows the input and output values in this rotated coordinate system. The angles are of course not influenced by this transformation. The correlations between those rotated parameters are shown in table 3 and table 4. Indeed the correlations are small in this coordinate system.

The fit reproduces the input parameters within a few sigma (see table 2) , which is good considering the fact that the auto-calibrated r-t relation was used, which is known to have some (small) systematics [2]. This gives confidence in the fit, and now we can start fitting real data, which is the subject of the next section.

TABLE 1. Correlation matrix of position parameters (Monte Carlo)

Δz_m^-	1.000											
Δy_m^-	-0.985	1.000										
Δz_m^0	-0.427	0.420	1.000									
Δy_m^0	0.421	-0.427	-0.985	1.000								
Δz_m^+	0.189	-0.186	-0.427	0.421	1.000							
Δy_m^+	-0.186	0.189	0.421	-0.428	-0.985	1.000						
Δz_o^-	0.470	-0.462	-0.126	0.124	0.044	-0.043	1.000					
Δy_o^-	-0.464	0.470	0.124	-0.126	-0.043	0.044	-0.985	1.000				
Δz_o^0	-0.154	0.152	0.460	-0.453	-0.156	0.154	-0.351	0.346	1.000			
Δy_o^0	0.152	-0.155	-0.452	0.460	0.154	-0.157	0.345	-0.351	-0.985	1.000		
Δz_o^+	0.042	-0.041	-0.123	0.121	0.467	-0.459	0.130	-0.128	-0.354	0.347	1.000	
Δy_o^+	-0.042	0.042	0.121	-0.123	-0.461	0.467	-0.129	0.130	0.349	-0.354	-0.985	1.000

TABLE 2. Monte Carlo comparison rotated input and fitted parameters

Parameter	Input Value	Fit Value	Error	Difference	Diff/Error
$\Delta\alpha_i^-$ **(mrad)**	1.000	1.003	0.011	0.003	0.30
$\Delta\alpha_i^+$ **(mrad)**	-1.000	-1.008	0.011	-0.008	-0.72
$\Delta\alpha_m^-$ **(mrad)**	-2.000	-1.999	0.019	0.001	0.04
$\Delta\alpha_m^+$ **(mrad)**	2.000	2.017	0.019	0.017	0.90
$\Delta\alpha_o^-$ **(mrad)**	3.000	2.983	0.016	-0.017	-1.10
$\Delta\alpha_o^0$ **(mrad)**	1.000	1.009	0.008	0.009	1.09
$\Delta\alpha_o^+$ **(mrad)**	-1.000	-0.998	0.016	0.002	0.12
Δsag_m^- **(mm)**	-2.236	-2.224	0.019	0.012	0.64
Δpar_m^- **(mm)**	-0.016	0.003	0.264	0.020	0.08
Δsag_m^0 **(mm)**	-0.891	-0.899	0.010	-0.008	-0.84
Δpar_m^0 **(mm)**	-0.454	-0.679	0.141	-0.225	-1.60
Δsag_m^+ **(mm)**	-2.220	-2.238	0.019	-0.019	-1.01
Δpar_m^+ **(mm)**	-2.253	-2.288	0.267	-0.036	-0.13
Δsag_o^- **(mm)**	1.345	1.407	0.022	0.062	2.81
Δpar_o^- **(mm)**	-0.437	-0.708	0.315	-0.270	-0.86
Δsag_o^0 **(mm)**	1.766	1.719	0.014	-0.046	-3.31
Δpar_o^0 **(mm)**	3.144	2.856	0.198	-0.288	-1.45
Δsag_o^+ **(mm)**	1.328	1.263	0.022	-0.065	-2.96
Δpar_o^+ **(mm)**	1.799	1.125	0.318	-0.674	-2.12

sag = sagitta direction, par = parallel to track direction

TABLE 3. Correlation matrix of angle parameters (Monte Carlo)

$\Delta\alpha_i^-$	1.000						
$\Delta\alpha_i^+$	-0.278	1.000					
$\Delta\alpha_m^-$	0.700	0.097	1.000				
$\Delta\alpha_m^+$	0.098	0.700	0.303	1.000			
$\Delta\alpha_o^-$	0.577	0.059	0.745	0.224	1.000		
$\Delta\alpha_o^0$	0.215	0.215	0.273	0.273	-0.037	1.000	
$\Delta\alpha_o^+$	0.060	0.575	0.224	0.744	0.239	-0.040	1.000

TABLE 4. Correlation matrix of rotated position parameters (Monte Carlo)

Δsag_m^-	1.000											
Δpar_m^-	-0.168	1.000										
Δsag_m^0	-0.427	0.071	1.000									
Δpar_m^0	0.070	-0.427	-0.154	1.000								
Δsag_m^+	0.189	-0.031	-0.428	0.067	1.000							
Δpar_m^+	-0.031	0.189	0.066	-0.428	-0.155	1.000						
Δsag_o^-	0.472	-0.076	-0.126	0.020	0.045	-0.007	1.000					
Δpar_o^-	-0.076	0.470	0.020	-0.126	-0.007	0.044	-0.154	1.000				
Δsag_o^0	-0.158	0.028	0.462	-0.074	-0.159	0.027	-0.353	0.058	1.000			
Δpar_o^0	0.028	-0.155	-0.074	0.460	0.027	-0.156	0.059	-0.351	-0.166	1.000		
Δsag_o^+	0.043	-0.006	-0.125	0.017	0.469	-0.069	0.132	-0.020	-0.355	0.055	1.000	
Δpar_o^+	-0.006	0.042	0.017	-0.123	-0.068	0.467	-0.020	0.130	0.054	-0.354	-0.136	1.000

6.2 Geometry fit results of real data

Data run numbers 2011 - 2020 (see appendix A of [2]) have been fit. Each run has about 50k tracks (except 2017: 26k). The parameter values (again in the rotated coordinate system) are given in table 5. The correlation matrices are given in table 6 and table 7. The errors on the angles are of the order of 10 μrad, and on the positions a few tens of μm in the sagitta direction and a few hundred μm 'parallel' to the tracks. The χ^2/d.o.f. is 0.93 for the angle fits and 1.3 for the position fits.

The main check on the consistency of the fit comes from the sagitta values. These should be zero all over the chamber. The cross-plate parameters found in the fit are fed back into the trackfit, where the parameters are used to correct each hit position before the trackfit is done. This is a sure and robust test of the fitresults. As an example this is done for run 2015. Figure 4 shows the sagitta distribution before and after the fit. The average sagitta is about -8 mm before any corrections are made. After the geometry fit the mean of a gauss fit on the sagitta distribution is $1.5 \pm 5\ \mu\mathrm{m}$, which is a reduction of four orders of magnitude. Remember that we do not fit the sagitta directly. The r.m.s. of 1.7 mm is mainly caused by multiple scattering. The average sagitta vs the normalised x (along the wire) and z (perpendicular to the wire) of the impact point in the BML chamber is shown in figure 5. The sagitta is consistent with zero all over the chamber surface. Figure 6 shows the distributions of the angle differences between pairs of chambers. These are the quantities that go into the χ^2, and should be consistent with zero, which they are.

To get higher statistics the corrected sagittas and angles of all 10 runs are combined into one histogram. To gain time, these histograms are not filled inside the full-blown trackfit, but in the geometry fit using the approximated corrections as used in the fit ('superpoints', see sections 4 and 5). The histograms made in this way agree well with the ones obtained from the complete trackfit. Figures 7 through 9 show, for the combined 10 runs, the sagitta distribution, the average sagitta vs x and z of BML, and the angle differences distributions. The exponential function $y = const \cdot e^{-\left|\frac{x - mean}{width}\right|}$ fitted to the correction sagitta distribution has a mean of $-0.9 \pm 1.4\ \mu\mathrm{m}$, which is consistent with zero. The average sagitta is zero within the statistical error of 5-10 μm all over BML. For completeness, figures 10 through 12 show the average of the angle differences as a function of the normalised x and z of the respective chambers. The distributions are flat as a function of x. There seem to be some systemetics left as a function of z.

TABLE 5. Parameter values of chamber displacement data runs

Run	Error	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Param.	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**	**mrad**
$\Delta\alpha_i^-$	0.012	1.217	1.168	1.176	1.211	1.233	1.254	1.239	1.219	1.213	1.205
$\Delta\alpha_i^+$	0.012	2.519	2.515	2.520	2.560	2.553	2.542	2.546	2.577	2.565	2.578
$\Delta\alpha_m^-$	0.02	0.048	0.045	0.053	0.043	0.046	0.054	0.037	0.007	0.052	0.122
$\Delta\alpha_m^+$	0.02	0.097	0.059	0.073	0.050	0.091	0.105	0.049	0.092	0.094	0.099
$\Delta\alpha_o^-$	0.019	1.151	1.113	1.162	1.194	1.185	1.186	1.203	1.189	1.186	1.176
$\Delta\alpha_o^0$	0.008	1.397	1.391	1.377	1.428	1.451	1.437	1.447	1.436	1.419	1.425
$\Delta\alpha_o^+$	0.019	1.370	1.328	1.333	1.364	1.361	1.381	1.353	1.384	1.363	1.376
Param.	**mm**	**mm**	**mm**	**mm**	**mm**	**mm**	**mm**	**mm**	**mm**	**mm**	**mm**
Δsag_m^-	0.02	-9.36	-9.30	-9.38	-9.40	-9.44	-7.79	-8.66	-11.15	-10.22	-8.54
Δpar_m^-	0.4	-4.8	-3.4	-4.2	-2.7	-3.2	-3.1	-4.9	-6.0	-5.1	-3.9
Δsag_m^0	0.011	-10.898	-10.749	-11.033	-11.278	-10.808	-8.996	-9.936	-12.515	-11.571	-9.984
Δpar_m^0	0.19	-6.23	-6.28	-5.35	-5.72	-6.47	-5.54	-5.86	-6.71	-7.03	-6.55
Δsag_m^+	0.02	-13.84	-13.47	-14.08	-14.42	-13.50	-11.71	-12.53	-15.22	-14.26	-12.63
Δpar_m^+	0.4	-6.1	-6.1	-5.4	-3.9	-6.1	-5.4	-4.9	-7.3	-6.7	-7.1
Δsag_o^-	0.03	-2.33	-2.12	-2.29	-2.55	-2.68	-2.64	-2.60	-2.48	-2.60	-2.48
Δpar_o^-	0.5	0.3	1.1	0.0	3.7	3.0	0.7	0.7	0.3	-0.1	1.6
Δsag_o^0	0.015	-6.673	-6.561	-6.573	-6.854	-6.933	-6.933	-6.918	-6.933	-6.856	-6.971
Δpar_o^0	0.3	0.1	-0.1	0.4	-0.1	-0.1	0.1	-0.3	0.8	0.0	-1.0
Δsag_o^+	0.03	-12.42	-12.13	-12.19	-12.43	-12.72	-12.66	-12.37	-12.63	-12.59	-12.51
Δpar_o^+	0.5	-0.8	2.6	0.9	2.9	2.9	2.4	4.3	1.4	2.6	-0.5

Note: sag = sagitta direction, par = parallel to track direction. The χ^2/d.o.f. is 0.93 for the angle fits and 1.3 for the position fits.

TABLE 6. Correlation matrix of angle parameters (Real Data)

$\Delta\alpha_i^-$	1.000						
$\Delta\alpha_i^+$	-0.326	1.000					
$\Delta\alpha_m^-$	0.687	0.071	1.000				
$\Delta\alpha_m^+$	0.078	0.688	0.295	1.000			
$\Delta\alpha_o^-$	0.547	0.023	0.713	0.193	1.000		
$\Delta\alpha_o^0$	0.194	0.237	0.266	0.305	-0.086	1.000	
$\Delta\alpha_o^+$	0.041	0.537	0.205	0.707	0.229	-0.063	1.000

TABLE 7. Correlation matrix of rotated position parameters (Real Data)

Δsag_m^-	1.000											
Δpar_m^-	-0.235	1.000										
Δsag_m^0	-0.444	0.107	1.000									
Δpar_m^0	0.108	-0.447	-0.256	1.000								
Δsag_m^+	0.213	-0.049	-0.464	0.115	1.000							
Δpar_m^+	-0.049	0.214	0.114	-0.463	-0.235	1.000						
Δsag_o^-	0.460	-0.106	-0.129	0.029	0.044	-0.009	1.000					
Δpar_o^-	-0.107	0.459	0.029	-0.128	-0.009	0.044	-0.216	1.000				
Δsag_o^0	-0.167	0.042	0.457	-0.118	-0.157	0.039	-0.381	0.087	1.000			
Δpar_o^0	0.042	-0.168	-0.117	0.457	0.039	-0.158	0.086	-0.382	-0.258	1.000		
Δsag_o^+	0.058	-0.012	-0.149	0.033	0.457	-0.101	0.155	-0.030	-0.387	0.088	1.000	
Δpar_o^+	-0.012	0.058	0.032	-0.147	-0.102	0.458	-0.030	0.155	0.088	-0.386	-0.212	1.000

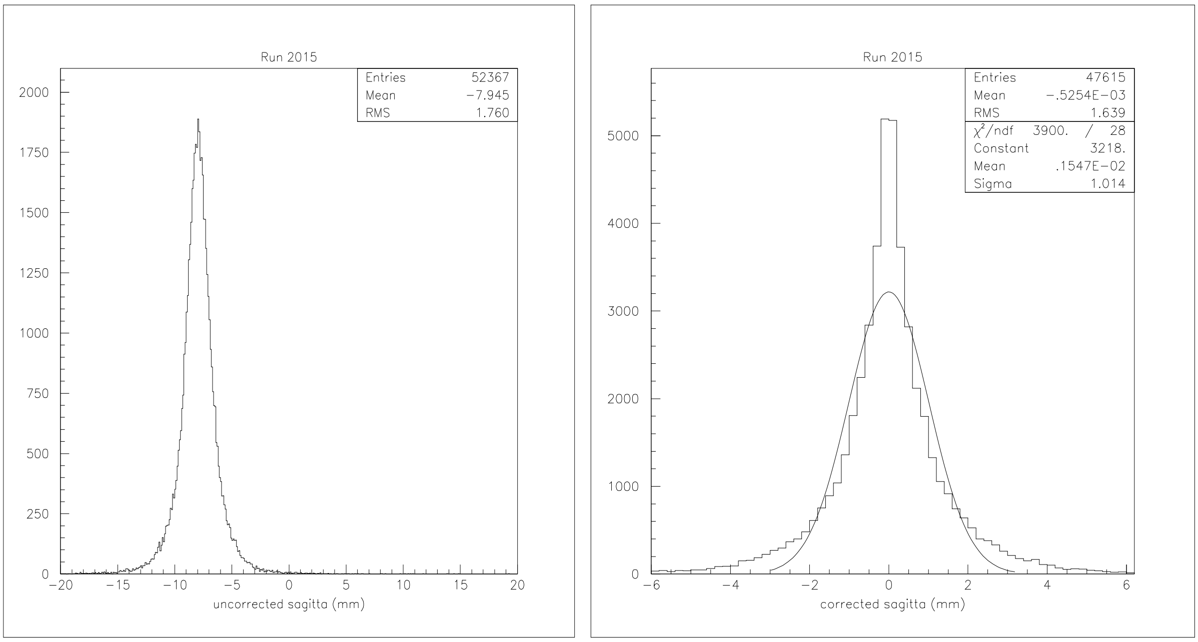

FIGURE 4. Sagitta distribution of one run before (left) and after (right) fit.

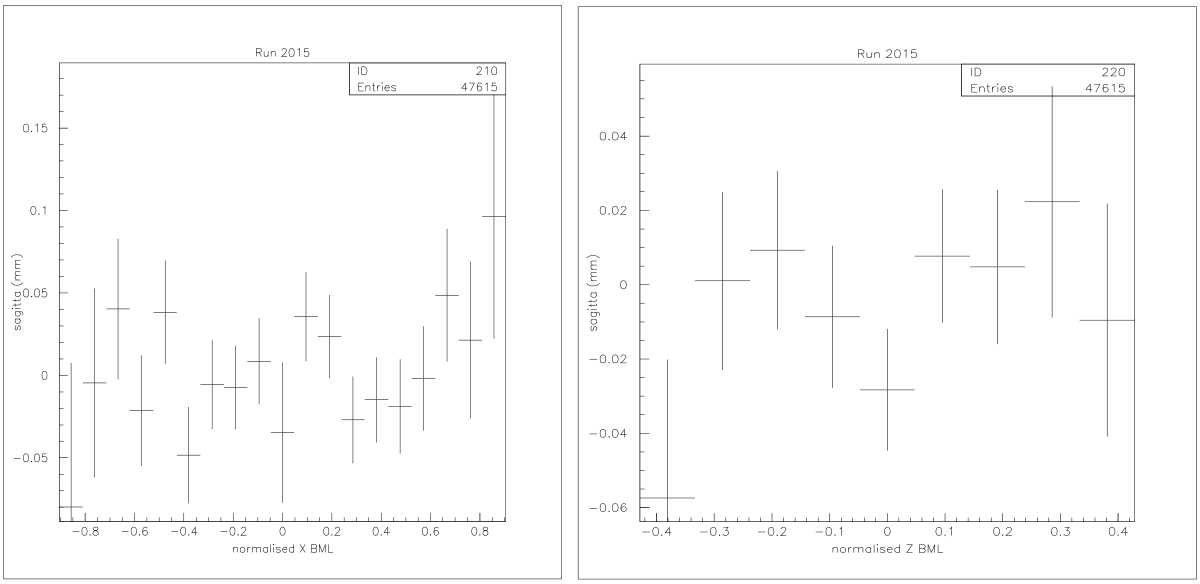

FIGURE 5. Average sagitta of one run vs normalised x and z of BML.

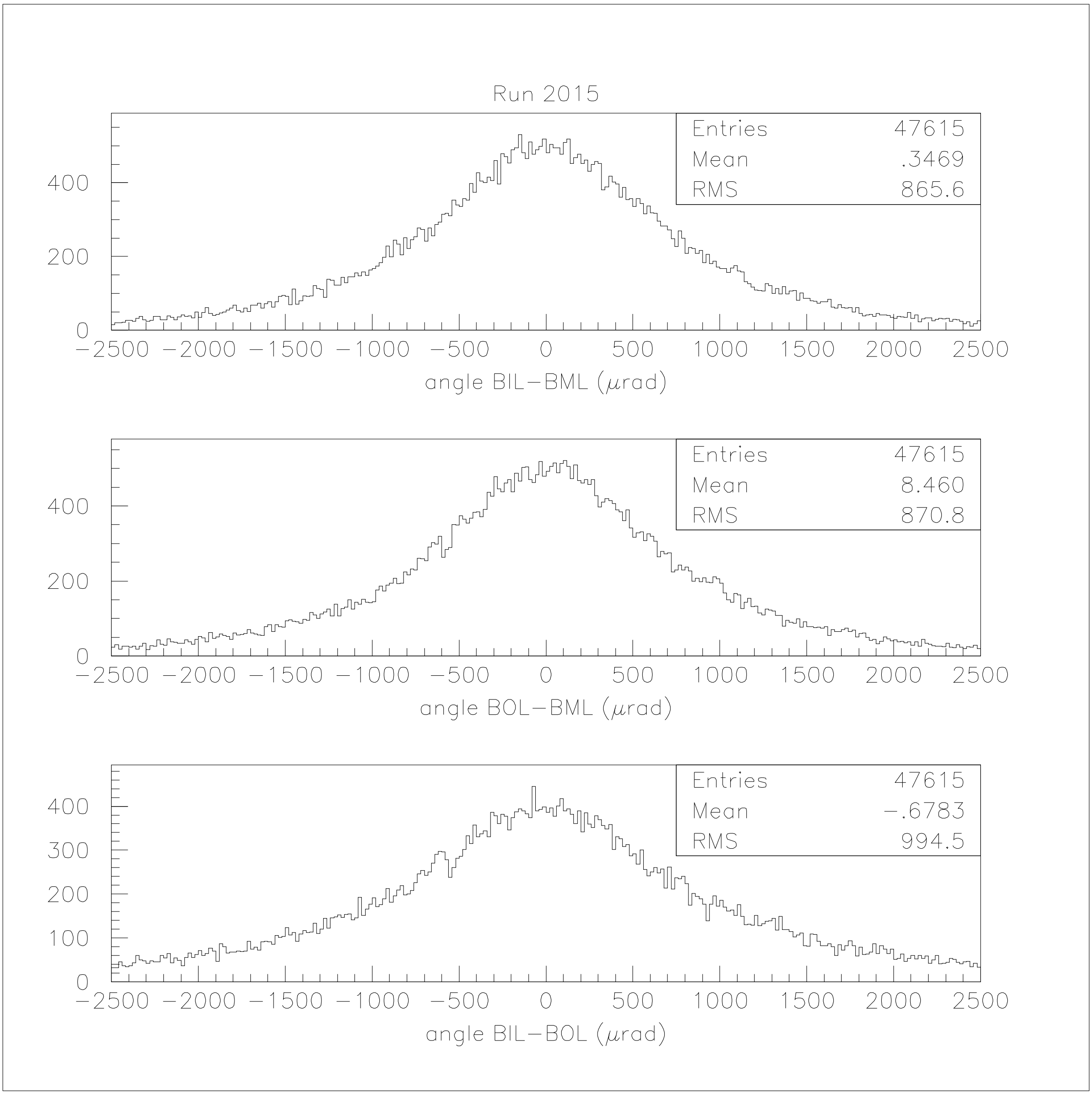

FIGURE 6. Angle differences of one run between track segments in pairs of chambers.

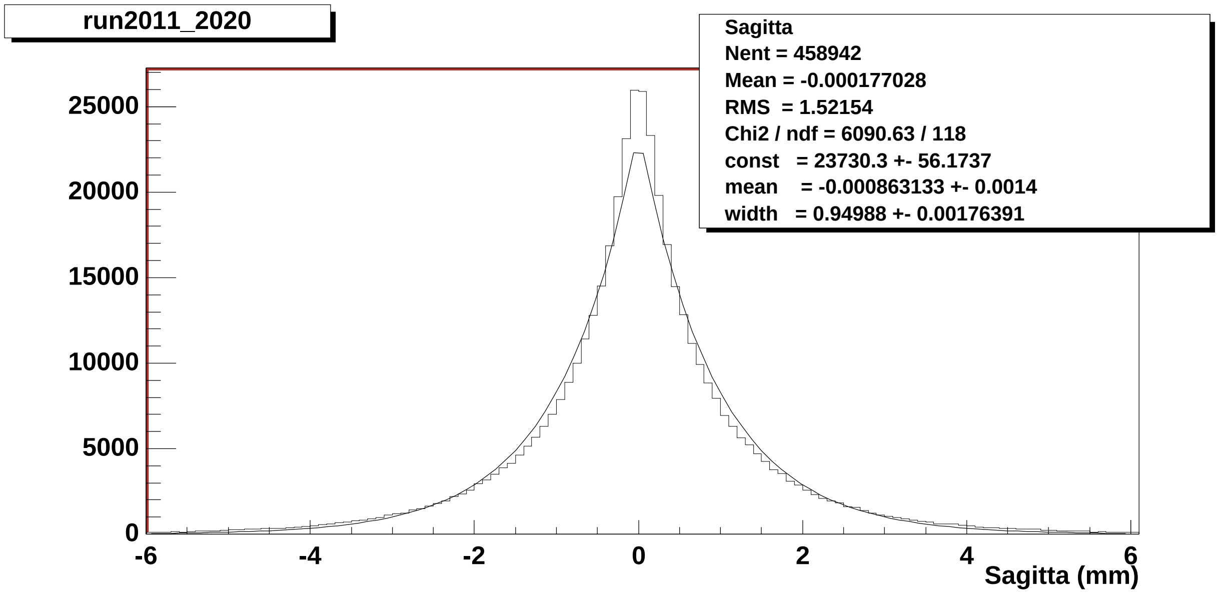

FIGURE 7. Sagitta distribution of 10 runs after geometry fits.

Fitted function: $y = const \cdot e^{-\left|\frac{x - mean}{width}\right|}$

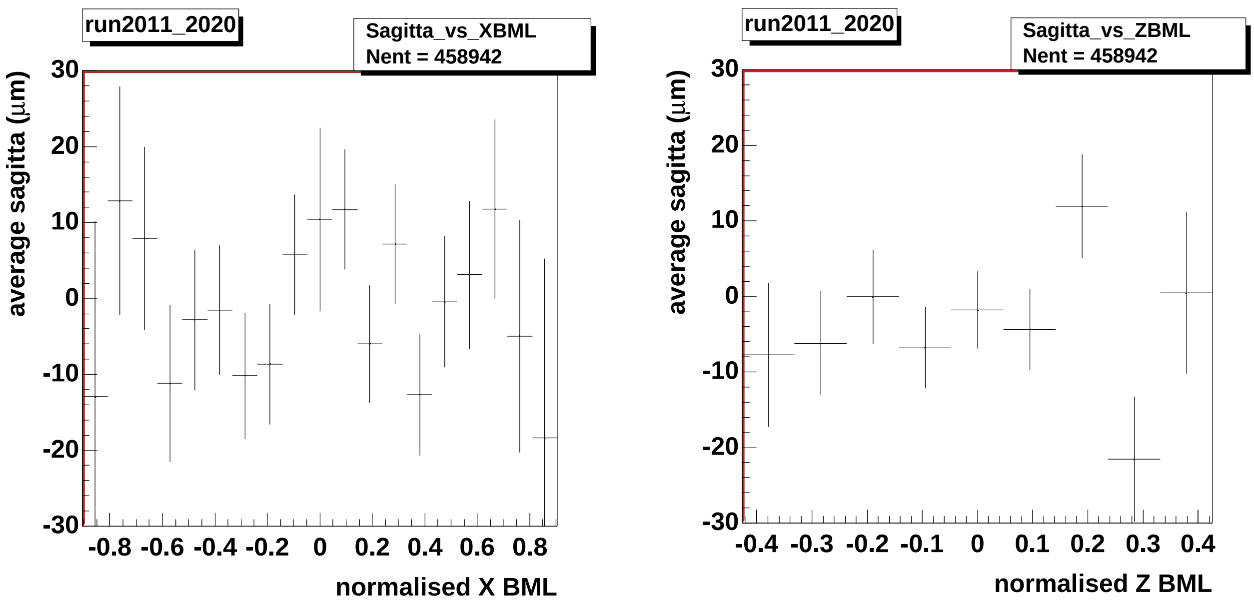

FIGURE 8. Average sagitta of 10 runs as a function of normalised X and Z of BML.

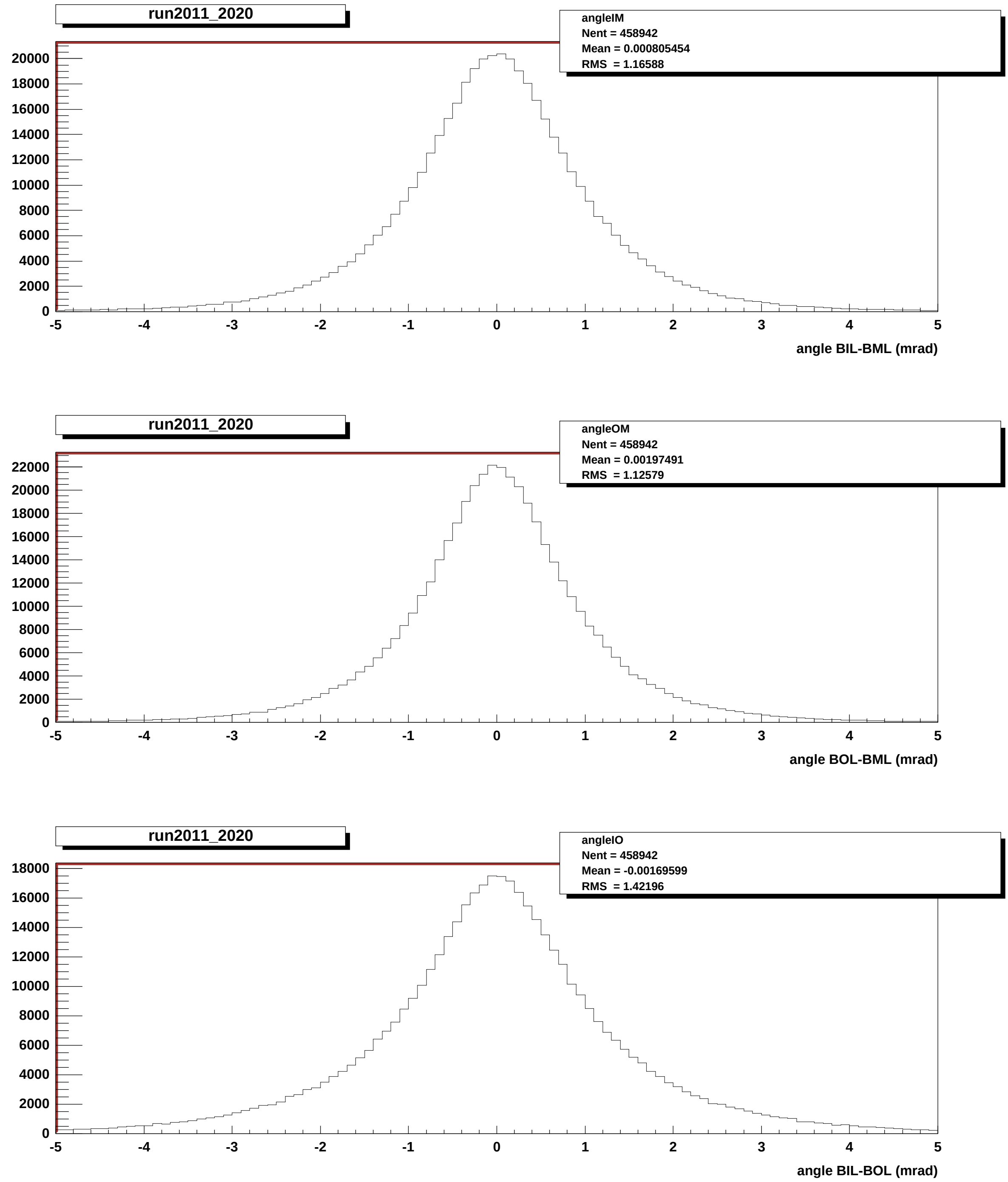

FIGURE 9. Angle differences distributions of 10 runs after geometry fit.

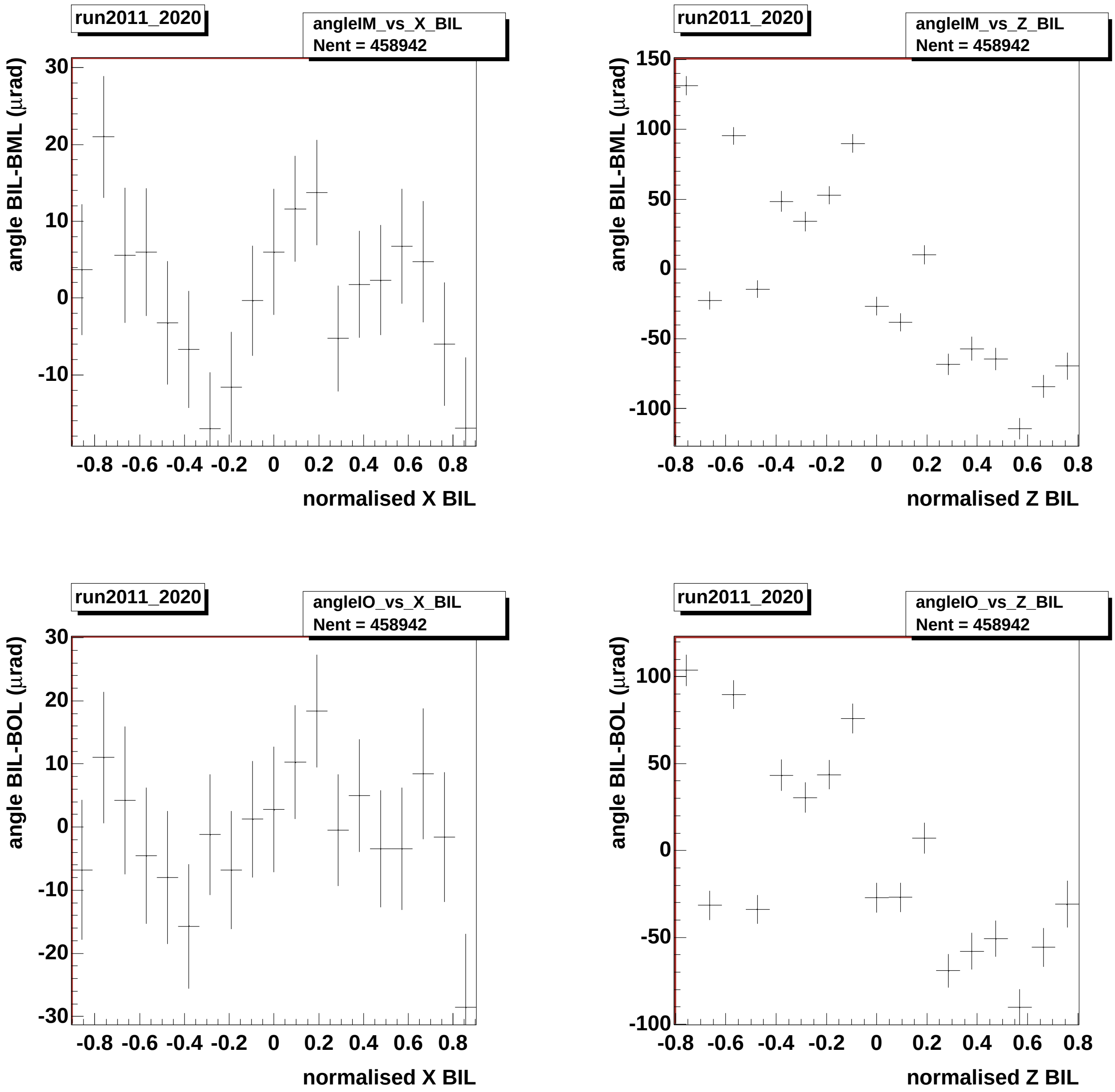

FIGURE 10. Average angle differences of 10 runs between tracks in BIL and the other two chambers vs the normalised x and z of BIL.

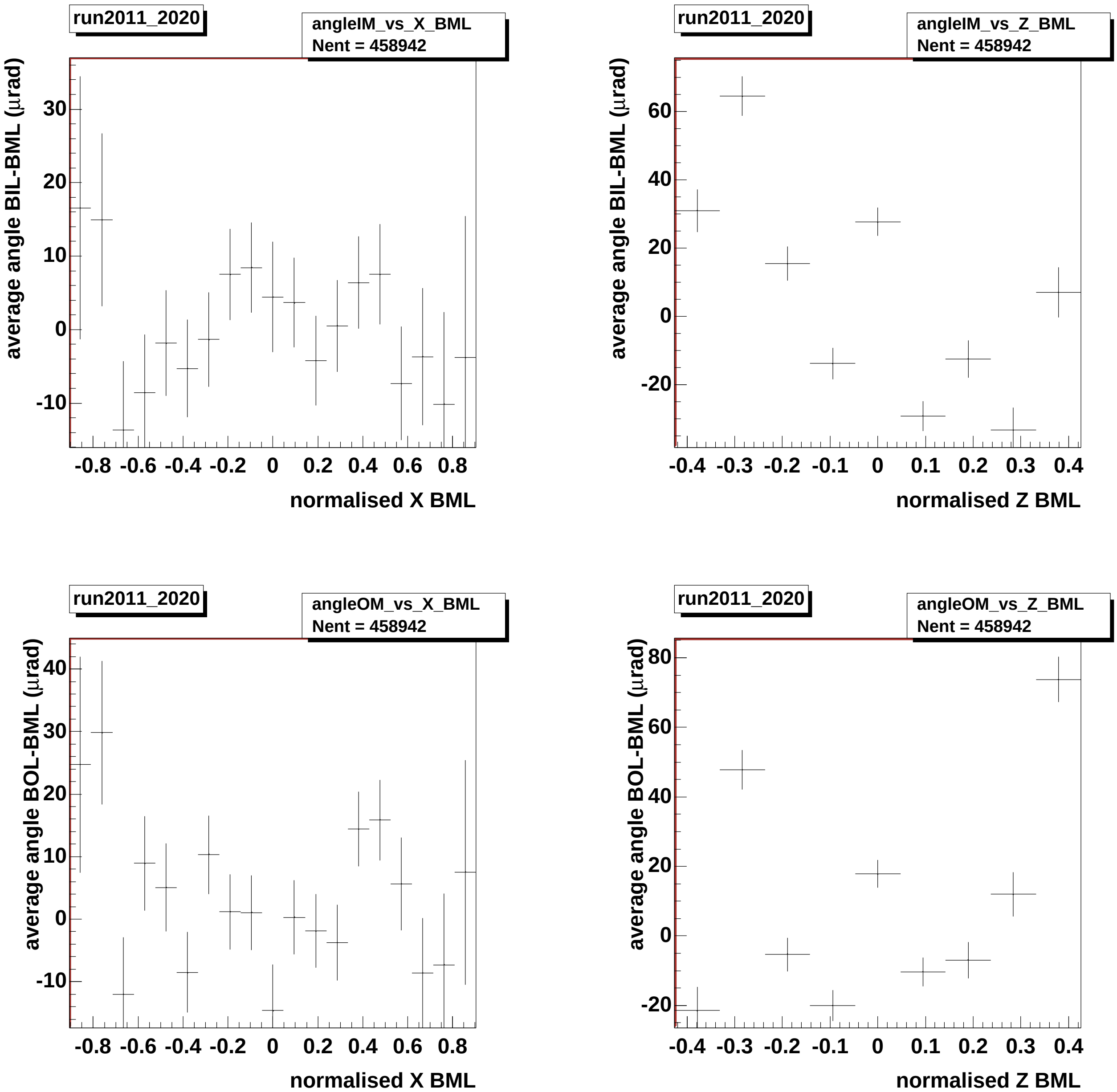

FIGURE 11. Average angle differences of 10 runs between tracks in BML and the other two chambers vs the normalised x and z of BML.

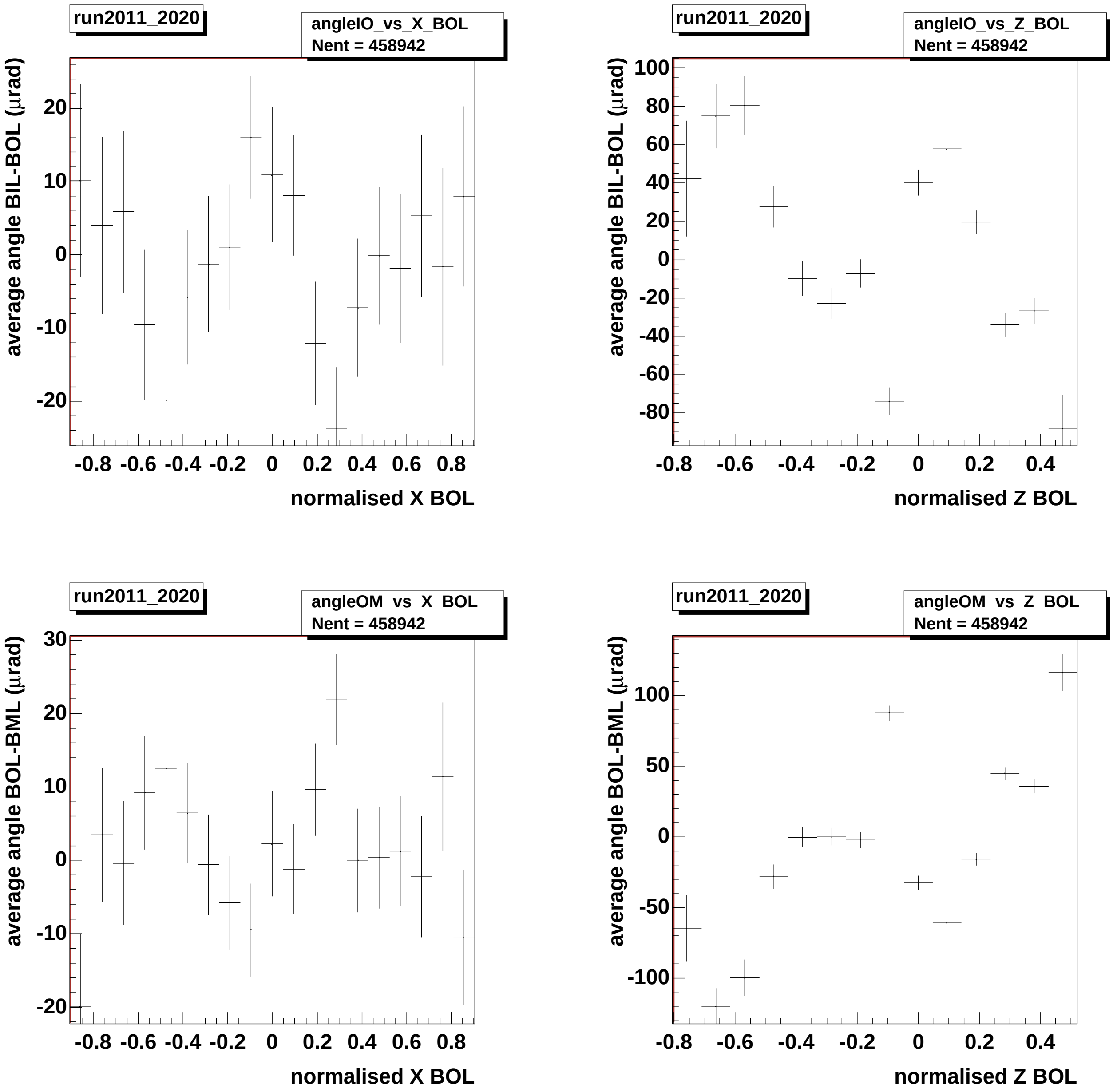

FIGURE 12. Average angle differences of 10 runs between tracks in BOL and the other two chambers vs the normalised x and z of BOL.

6.3 Calibration of the 16 RASNIK systems

Now the geometry has been found using the muon tracks, the RASNIK systems can be calibrated. This is done by calculating the rasnik component positions and rotations based on the ideal geometry using the parameters found in the muon geometry fits. The RASNIK readings can now be predicted and the difference between these predicted readings and the measured readings give directly the RASNIK calibration values. Tables 9 through 11 show the values of x,y and the rotation around z (RASNIK coordinate system) for the 16 RASNIK systems (3x4 in-plane and 4 projective). It also lists the calibration value in the sagitta direction, since this is the one we really determine in the fit (the x and y calibration values are 99% correlated!).

As a reminder of the RASNIK coordinate system:

in-plane: RASNIK	<=>	Global
x		-z
y		y
rotZ		rotX

projective: RASNIK	<=>	Global
x		~sagitta
y		~wire

The errors on the calibration values are calculated from the errrors on the crossplate parameters, taking into account the correlations between those parameters. The calibration is done on all 10 runs. The r.m.s. of the calibration values are almost all twice as large as the errors. It is not clear why. The calibration in the relevant (=sagitta) direction has an error of 40-60 µm per system per run of 50k reconstructed tracks, which is not sufficient for ATLAS (dominant error is due to multiple scattering of 1.7 mm r.m.s. which will be much smaller in ATLAS). We would like to obtain an error at least a factor of 3 smaller, so 10x more statistics are needed. The uncertainty on the average calibration values of the 10 runs is 25-45 µm. In the present case we can combine runs to increase statistics, but it is more elegant to calibrate with high statistics per chamber position.

TABLE 8. RASNIK calibration values of BIL in-plane systems.

	error	average	r.m.s.	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
bil0_x		5.6	0.0	-7	-2	-3	-1	4	2	1	1	6	0
bil0_y		11.5	0.0	0	-1	1	1	-1	0	1	0	-1	1
bil0_rotZ	0.019	4.171	0.030	-62	9	0	15	1	-32	-27	30	24	41
bil1_x		6.7	0.0	-7	-2	-3	-1	4	2	1	1	5	0
bil1_y		13.8	0.0	2	-2	0	0	0	2	3	-2	-2	-1
bil1_rotZ	0.019	-12.121	0.032	-55	-4	-10	2	13	-36	-24	25	51	39
bil2_x		6.7	0.0	-7	-3	-3	-1	4	2	1	1	5	0
bil2_y		13.8	0.0	-3	0	1	2	-2	-4	-1	1	0	4
bil2_rotZ	0.019	-0.143	0.029	-53	12	11	16	-7	-41	-27	29	19	37
bil3_x		5.8	0.0	-8	-3	-3	-1	4	2	1	1	5	1
bil3_y		13.3	0.0	0	0	1	1	-1	-1	1	0	-1	1
bil3_rotZ	0.019	-12.483	0.037	-78	27	24	26	-14	-43	-24	23	13	47

Columns: error from muon fit (mm,mrad), average of the 10 runs (mm,mrad), r.m.s. of the 10 runs (mm,mrad), difference between each run and the average (µm,µrad)

TABLE 9. RASNIK calibration values of BML in-plane systems.

	error	average	rms	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
bml0_x	0.4	6.7	0.6	-382	388	-651	1204	622	-26	-173	-1013	-8	37
bml0_y	0.8	14.0	1.1	-581	605	-1145	2184	1180	354	-396	-2000	-18	-183
bml0_rotZ	0.03	9.35	0.02	-2	-19	-15	-29	12	14	-12	47	0	2
bml1_x	0.4	10.2	0.6	-386	389	-650	1204	623	-25	-172	-1011	-6	36
bml1_y	0.8	14.1	1.1	-600	628	-1136	2221	1170	328	-357	-1999	-32	-223
bml1_rotZ	0.03	0.25	0.03	7	-11	-8	-27	17	22	-19	61	11	-55
bml2_x	0.4	9.0	0.6	-388	387	-651	1204	622	-26	-172	-1012	-7	42
bml2_y	0.8	12.0	1.1	-579	610	-1146	2189	1185	351	-385	-1965	-20	-240
bml2_rotZ	0.03	8.28	0.02	14	-10	-10	-23	14	14	-14	55	2	-40
bml3_x	0.4	7.7	0.6	-388	388	-650	1204	622	-27	-172	-1012	-6	42
bml3_y	0.8	13.7	1.1	-598	634	-1139	2226	1175	325	-346	-1964	-31	-281
bml3_rotZ	0.03	-4.99	0.03	25	-9	-13	-33	11	23	-20	61	7	-53
bml0_sag	*0.05*	*-0.41*	*0.08*	*-76*	*72*	*-60*	*82*	*19*	*-184*	*26*	*5*	*1*	*116*
bml1_sag	*0.05*	*2.72*	*0.08*	*-72*	*61*	*-64*	*65*	*24*	*-171*	*9*	*6*	*9*	*133*
bml2_sag	*0.05*	*2.59*	*0.09*	*-83*	*68*	*-60*	*80*	*17*	*-182*	*22*	*-10*	*3*	*146*
bml3_sag	*0.05*	*0.64*	*0.09*	*-74*	*58*	*-63*	*63*	*21*	*-171*	*4*	*-11*	*9*	*165*

Columns: error from muon fit (mm,mrad), average of the 10 runs (mm,mrad), r.m.s. of the 10 runs (mm,mrad), difference between each run and the average (µm,µrad). _sag=sagitta direction

TABLE 10. RASNIK calibration values of BOL in-plane systems.

system/run	error	average	rms	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
bol0_x	0.5	5.3	1.1	-1753	609	-1309	1767	980	-207	1310	-1262	-428	294
bol0_y	1.0	10.0	2.1	-3176	777	-2449	3411	2366	-112	2188	-2598	-436	29
bol0_rotZ	0.02	-12.86	0.03	48	33	-29	-16	-5	11	-39	15	-13	-2
bol1_x	0.5	7.9	1.1	-1752	608	-1309	1767	982	-206	1309	-1263	-425	290
bol1_y	1.0	12.9	2.2	-3236	800	-2494	3415	2436	-117	2253	-2613	-450	6
bol1_rotZ	0.02	-9.33	0.03	40	45	-20	-13	-8	5	-43	20	4	-30
bol2_x	0.5	5.4	1.1	-1750	610	-1308	1767	981	-208	1307	-1263	-425	289
bol2_y	1.0	11.9	2.1	-3173	824	-2494	3394	2395	-107	2183	-2598	-456	33
bol2_rotZ	0.02	-13.29	0.08	-89	48	133	127	-10	-40	8	-48	-4	-122
bol3_x	0.5	8.8	1.1	-1752	609	-1309	1766	980	-208	1310	-1263	-424	293
bol3_y	1.0	13.8	2.2	-3234	848	-2539	3397	2466	-112	2248	-2613	-469	9
bol3_rotZ	0.02	-12.96	0.03	54	17	-26	-32	-31	12	-36	3	-25	65
bol0_sag	*0.06*	*0.22*	*0.16*	*-121*	*190*	*-55*	*27*	*-200*	*-133*	*174*	*54*	*-183*	*249*
bol1_sag	*0.06*	*1.18*	*0.15*	*-93*	*179*	*-35*	*25*	*-231*	*-131*	*144*	*61*	*-175*	*255*
bol2_sag	*0.06*	*-0.58*	*0.15*	*-120*	*169*	*-34*	*35*	*-212*	*-137*	*174*	*53*	*-172*	*243*
bol3_sag	*0.06*	*1.56*	*0.15*	*-94*	*158*	*-14*	*32*	*-246*	*-135*	*147*	*60*	*-165*	*257*

Columns: error from muon fit (mm,mrad), average of the 10 runs (mm,mrad), r.m.s. of the 10 runs (mm,mrad), difference between each run and the average (µm,µrad). _sag=sagitta direct

TABLE 11. RASNIK calibration values of projective systems.

system/run	error	average	rms	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Q0_x	0.05	6.31	0.07	5	-120	-61	73	45	-119	32	12	66	67
Q0_y	0.14	11.03	0.25	249	-399	-166	-123	-124	-217	468	273	-92	130
Q0_rotZ		-4.239	0.117	-75	127	-101	-92	-49	232	54	117	-132	-81
P0_x	0.06	14.37	0.09	-50	26	-15	121	80	-62	-171	-103	96	76
P0_y	0.14	4.91	0.12	-30	-144	-103	-105	-22	-32	250	20	-28	194
P0_rotZ		15.319	0.329	170	194	211	227	-294	172	320	-575	-589	163
Q1_x	0.05	15.70	0.04	75	-63	10	18	-58	41	-19	0	-35	30
Q1_y	0.14	12.29	0.18	-419	272	95	-9	112	90	-117	-133	117	-7
Q1_rotZ		-12.526	0.260	-91	133	76	-606	-177	231	-39	251	-114	336
P1_x	0.06	14.98	0.04	-56	11	26	-16	-56	31	17	-8	-13	64
P1_y	0.14	9.91	0.07	-170	21	-1	-76	29	11	126	24	27	9
P1_rotZ		-14.809	0.155	-19	57	-29	5	258	-27	188	-240	-267	74
Q0_sag	*0.04*	*-7.62*	*0.09*	*-36*	*169*	*81*	*-58*	*-29*	*145*	*-90*	*-45*	*-54*	*-82*
P0_sag	*0.04*	*-14.78*	*0.09*	*52*	*-12*	*24*	*-111*	*-78*	*65*	*146*	*100*	*-92*	*-95*
Q1_sag	*0.04*	*14.07*	*0.06*	*126*	*-96*	*-2*	*19*	*-71*	*30*	*-5*	*16*	*-49*	*31*
P1_sag	*0.04*	*13.97*	*0.03*	*-40*	*9*	*26*	*-9*	*-58*	*30*	*5*	*-11*	*-16*	*63*

Columns: error from muon fit (mm,mrad), average of the 10 runs (mm,mrad), r.m.s. of the 10 runs (mm,mrad), difference between each run and the average (µm,µrad). _sag=sagitta direction

7 Conclusion

The DATCHA geometry has been reconstructed using straight muon tracks in an iterative fit. We have taken the cross-plates as the basic units to be fitted with three first order parameters per cross-plate: two position parameters (Δz, Δy) and one angle parameter ($\Delta\alpha$=rotation around the x-axis). Taking the BML middle cross-plate as a reference for the angles, and the two outer cross-plates of BIL as a reference for the positions, we are left with 19 parameters (2 BIL, 5 BML, 9 BOL), i.e. 7 angles and 12 positions.

We split the fit in two by first fitting the 7 angles, then correcting the tracks for the angles and positions, fitting the 12 positions, and then correcting the tracks for the positions and the angles. The fits themselves are linearised around zero, whereas the corrections in between the fits are not linearised, but are an approximation based on ‘superpoints’. In an iterative procedure the fit converges to within 2% of the errors in typically 9 iterations, which takes less than 5 minutes for 50k tracks (on a HP 712/80).

The fit is tested on Monte Carlo with a known deformation. The fit finds the input values back within a few sigma.

Applied to real data, the consistency of the fitted parameters is tested by feeding them back into the trackfit, which applies a hit-by-hit correction before doing the trackfit. This is done for one run as an example. The average corrected sagitta is consistent with zero all over the surface of BML. The 10 runs are combined into one histogram using the approximated corrections as used in the fit (‘superpoints’). The average corrected sagitta of the 10 runs is consistent with zero within the statistical error of 5-10 μm all over BML. For the angles there seem to be still some systematics left as a function of z of the chambers.

Using 50k reconstructed straight tracks, the cross-plate positions (y and z) are determined with a precision of 10-25 μm in the sagitta direction and 0.2-0.5 mm ‘parallel’ to the tracks. The rotation of the cross-plates around x is determined with 10-20 μrad precision.

From the geometry reconstructed with the muons, the 16 RASNIK systems are calibrated. Using 50k tracks the calibration in the sagitta direction has and error of 40-60 μm. The r.m.s. of the calibration values over the 10 runs is a factor 2 larger than the errors. This is probably due to the correlations between the calibration values. A factor of 10 more statistics is needed to get the errors down to an acceptable level.

We conclude by saying that a simple and elegant method has been presented to fit the DATCHA geometry using straight muon tracks. The parameters are directly related to chamber construction. The performance is satisfactory.

8 References

[1] M.Vreeswijk *et al.*, *Global alignment of MDT chambers in DATCHA-CERN*, ATLAS Internal Note ATL-MUON-98-246

[2] F.Linde *et al.*, μ-*Track Reconstruction in DATCHA*, ATLAS Internal Note ATL-MUON-98-220

[3] see several documents on WWW page http://wwwinfo.cern.ch/~patrickh/Presentations/Presentations.html

[4] N.Hessey, M.Woudstra, *Mutdat User Guide for DATCHA Event Analysis*, /afs/cern.ch/atlas/project/datcha/mutdat/manual.ps

Appendix A Matrices and vectors used in angles fit

For track segment pairs **BIL - BML** we find for $\hat{x}_m^k \geq 0$ (upper index '+'):

$$\tilde{b}^+_{i,m} = \sum_{k=1}^{N} \frac{\alpha_i^k - \alpha_m^k}{(\sigma_{i,m}^k)^2} \left[\frac{1-\hat{x}_i^k}{2}, \frac{1+\hat{x}_i^k}{2}, 0, -\hat{x}_m^k, 0, 0, 0 \right]^T$$

$$A^+_{i,m} = \sum_{k=1}^{N} \frac{1}{(\sigma_{i,m}^k)^2} \begin{bmatrix} \frac{(1-\hat{x}_i^k)^2}{4} & \frac{1-(\hat{x}_i^k)^2}{4} & 0 & -\frac{1-\hat{x}_i^k}{2}\hat{x}_m^k & 0 & 0 & 0 \\ . & \frac{(1+\hat{x}_i^k)^2}{4} & 0 & -\frac{1+\hat{x}_i^k}{2}\hat{x}_m^k & 0 & 0 & 0 \\ . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & (\hat{x}_m^k)^2 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & 0 \end{bmatrix} .$$

For $\hat{x}_m^k < 0$ (upper index '-') we find

$$\tilde{b}^-_{i,m} = \sum_{k=1}^{N} \frac{\alpha_i^k - \alpha_m^k}{(\sigma_{i,m}^k)^2} \left[\frac{1-\hat{x}_i^k}{2}, \frac{1+\hat{x}_i^k}{2}, \hat{x}_m^k, 0, 0, 0, 0 \right]^T$$

$$A^-_{i,m} = \sum_{k=1}^{N} \frac{1}{(\sigma_{i,m}^k)^2} \begin{bmatrix} \frac{(1-\hat{x}_i^k)^2}{4} & \frac{1-(\hat{x}_i^k)^2}{4} & \frac{1-\hat{x}_i^k}{2}\hat{x}_m^k & 0 & 0 & 0 & 0 \\ . & \frac{(1+\hat{x}_i^k)^2}{4} & \frac{1+\hat{x}_i^k}{2}\hat{x}_m^k & 0 & 0 & 0 & 0 \\ . & . & (\hat{x}_m^k)^2 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & 0 \end{bmatrix}$$

For track segment pairs **BIL - BOL** we find for $\hat{x}_o^k \geq 0$

$$\vec{b}_{i,o}^{+} = \sum_{k=1}^{N} \frac{\alpha_i^k - \alpha_o^k}{(\sigma_{i,o}^k)^2} \left[\frac{1-\hat{x}_i^k}{2}, \frac{1+\hat{x}_i^k}{2}, 0, 0, 0, -(1-\hat{x}_o^k), -\hat{x}_o^k \right]^T$$

$$A_{i,o}^{+} = \sum_{k=1}^{N} \frac{1}{(\sigma_{i,o}^k)^2} \begin{bmatrix} \frac{(1-\hat{x}_i^k)^2}{4} & \frac{1-(\hat{x}_i^k)^2}{4} & 0 & 0 & 0 & -\frac{(1-\hat{x}_i^k)}{2}(1-\hat{x}_o^k) & -\frac{(1-\hat{x}_i^k)}{2}\hat{x}_o^k \\ . & \frac{(1+\hat{x}_i^k)^2}{4} & 0 & 0 & 0 & -\frac{(1+\hat{x}_i^k)}{2}(1-\hat{x}_o^k) & -\frac{(1+\hat{x}_i^k)}{2}\hat{x}_o^k \\ . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & (1-\hat{x}_o^k)^2 & (1-\hat{x}_o^k)\hat{x}_o^k \\ . & . & . & . & . & . & (\hat{x}_o^k)^2 \end{bmatrix}.$$

For $\hat{x}_o^k < 0$:

$$\vec{b}_{i,o}^{-} = \sum_{k=1}^{N} \frac{\alpha_i^k - \alpha_o^k}{(\sigma_{i,o}^k)^2} \left[\frac{1-\hat{x}_i^k}{2}, \frac{1+\hat{x}_i^k}{2}, 0, 0, \hat{x}_o^k, -(1+\hat{x}_o^k), 0 \right]^T \text{ and}$$

$$A_{i,o}^{-} = \sum_{k=1}^{N} \frac{1}{(\sigma_{i,o}^k)^2} \begin{bmatrix} \frac{(1-\hat{x}_i^k)^2}{4} & \frac{1-(\hat{x}_i^k)^2}{4} & 0 & 0 & \frac{(1-\hat{x}_i^k)}{2}\hat{x}_o^k & -\frac{(1-\hat{x}_i^k)}{2}(1+\hat{x}_o^k) & 0 \\ . & \frac{(1+\hat{x}_i^k)^2}{4} & 0 & 0 & \frac{(1+\hat{x}_i^k)}{2}\hat{x}_o^k & -\frac{(1+\hat{x}_i^k)}{2}(1+\hat{x}_o^k) & 0 \\ . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & (\hat{x}_o^k)^2 & -(1+\hat{x}_o^k)\hat{x}_o^k & 0 \\ . & . & . & . & . & (1+\hat{x}_o^k)^2 & 0 \\ . & . & . & . & . & . & 0 \end{bmatrix}$$

For track segment pairs **BOL - BML** we find for $\hat{x}_o^k \geq 0$ and $\hat{x}_m^k \geq 0$ (upper index '++'):

$$\vec{b}_{o,m}^{++} = \sum_{k=1}^{N} \frac{\alpha_o^k - \alpha_m^k}{(\sigma_{o,m}^k)^2} [0, 0, 0, -\hat{x}_m^k, 0, (1-\hat{x}_o^k), \hat{x}_o^k]^T$$

$$A_{o,m}^{++} = \sum_{k=1}^{N} \frac{1}{(\sigma_{o,m}^k)^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & (\hat{x}_m^k)^2 & 0 & -\hat{x}_m^k(1-\hat{x}_o^k) & -\hat{x}_m^k\hat{x}_o^k \\ . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & (1-\hat{x}_o^k)^2 & \hat{x}_o^k(1-\hat{x}_o^k) \\ . & . & . & . & . & . & (\hat{x}_o^k)^2 \end{bmatrix}$$

And for $\hat{x}_o^k < 0$ and $\hat{x}_m^k \geq 0$ (upper index '-+'):

$$\vec{b}_{o,m}^{-+} = \sum_{k=1}^{N} \frac{\alpha_o^k - \alpha_m^k}{(\sigma_{o,m}^k)^2} [0, 0, 0, -\hat{x}_m^k, -\hat{x}_o^k, (1+\hat{x}_o^k), 0]^T$$

$$A_{o,m}^{-+} = \sum_{k=1}^{N} \frac{1}{(\sigma_{o,m}^k)^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & (\hat{x}_m^k)^2 & \hat{x}_m^k\hat{x}_o^k & -\hat{x}_m^k(1+\hat{x}_o^k) & 0 \\ . & . & . & . & (\hat{x}_o^k)^2 & -\hat{x}_o^k(1+\hat{x}_o^k) & 0 \\ . & . & . & . & . & (1+\hat{x}_o^k)^2 & 0 \\ . & . & . & . & . & . & 0 \end{bmatrix}$$

And for $\hat{x}_{\mathrm{o}}^{k} \geq 0$ and $\hat{x}_{\mathrm{m}}^{k} < 0$ (with upper index ‘+-’):

$$\vec{b}_{\mathrm{o,m}}^{+-} = \sum_{k=1}^{N} \frac{\alpha_{\mathrm{o}}^{k} - \alpha_{\mathrm{m}}^{k}}{(\sigma_{\mathrm{o,m}}^{k})^{2}} [0, 0, \hat{x}_{\mathrm{m}}^{k}, 0, 0, (1 - \hat{x}_{\mathrm{o}}^{k}), \hat{x}_{\mathrm{o}}^{k}]^{T}$$

$$A_{\mathrm{o,m}}^{+-} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & (\hat{x}_{\mathrm{m}}^{k})^{2} & 0 & 0 & \hat{x}_{\mathrm{m}}^{k}(1 - \hat{x}_{\mathrm{o}}^{k}) & \hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k} \\ . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & (1 - \hat{x}_{\mathrm{o}}^{k})^{2} & \hat{x}_{\mathrm{o}}^{k}(1 - \hat{x}_{\mathrm{o}}^{k}) \\ . & . & . & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2} \end{bmatrix}$$

And finally for $\hat{x}_{\mathrm{o}}^{k} < 0$ and $\hat{x}_{\mathrm{m}}^{k} < 0$ (with upper index ‘--’):

$$\vec{b}_{\mathrm{o,m}}^{--} = \sum_{k=1}^{N} \frac{\alpha_{\mathrm{o}}^{k} - \alpha_{\mathrm{m}}^{k}}{(\sigma_{\mathrm{o,m}}^{k})^{2}} [0, 0, \hat{x}_{\mathrm{m}}^{k}, 0, -\hat{x}_{\mathrm{o}}^{k}, (1 + \hat{x}_{\mathrm{o}}^{k}), 0]^{T}$$

$$A_{\mathrm{o,m}}^{--} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & (\hat{x}_{\mathrm{m}}^{k})^{2} & 0 & -\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k} & \hat{x}_{\mathrm{m}}^{k}(1 + \hat{x}_{\mathrm{o}}^{k}) & 0 \\ . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2} & -\hat{x}_{\mathrm{o}}^{k}(1 + \hat{x}_{\mathrm{o}}^{k}) & 0 \\ . & . & . & . & . & (1 + \hat{x}_{\mathrm{o}}^{k})^{2} & 0 \\ . & . & . & . & . & . & 0 \end{bmatrix}$$

Appendix B Matrices and vectors used in positions fit

For track segment pairs **BIL - BML** we find for $\hat{x}_{\mathrm{m}}^{k} \geq 0$ ('+'):

$$\hat{b}_{\mathrm{i,m}}^{+} = \sum_{k=1}^{N} \frac{(z_{\mathrm{i}}^{k} - z_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) - (y_{\mathrm{i}}^{k} - y_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k})}{(\sigma_{\mathrm{i,m}}^{k})^{2}} \begin{bmatrix} 0 \\ 0 \\ -(1-\hat{x}_{\mathrm{m}}^{k}) \cdot \sin(\bar{\alpha}^{k}) \\ (1-\hat{x}_{\mathrm{m}}^{k}) \cdot \cos(\bar{\alpha}^{k}) \\ -\hat{x}_{\mathrm{m}}^{k}\sin(\bar{\alpha}^{k}) \\ \hat{x}_{\mathrm{m}}^{k}\cos(\bar{\alpha}^{k}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_{\mathrm{i,m}}^{+} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{i,m}}^{k})^{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & -(1-\hat{x}_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})\sin^{2}(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & (1-\hat{x}_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & (\hat{x}_{\mathrm{m}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & (\hat{x}_{\mathrm{m}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & 0 \end{bmatrix}.$$

For $\hat{x}_{\mathrm{m}}^{k} < 0$ ('-'):

$$\vec{b}_{\mathrm{i,m}}^{-} = \sum_{k=1}^{N} \frac{(z_{\mathrm{i}}^{k} - z_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) - (y_{\mathrm{i}}^{k} - y_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k})}{(\sigma_{\mathrm{i,m}}^{k})^{2}} \begin{bmatrix} \hat{x}_{\mathrm{m}}^{k}\sin(\bar{\alpha}^{k}) \\ -\hat{x}_{\mathrm{m}}^{k}\cos(\bar{\alpha}^{k}) \\ -(1+\hat{x}_{\mathrm{m}}^{k})\cdot\sin(\bar{\alpha}^{k}) \\ (1+\hat{x}_{\mathrm{m}}^{k})\cdot\cos(\bar{\alpha}^{k}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_{\mathrm{i,m}}^{-} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{i,m}}^{k})^{2}} \begin{bmatrix} (\hat{x}_{\mathrm{m}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & (\hat{x}_{\mathrm{m}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & -(1+\hat{x}_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) & -\frac{(1+\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & (1+\hat{x}_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & 0 \end{bmatrix}$$

For track segment pairs **BIL - BOL** we find for $\hat{x}_o^k \geq 0$ (‘+’):

$$\vec{b}^{+}_{i,o} = \sum_{k=1}^{N} \frac{(z_i^k - z_o^k)\sin(\bar{\alpha}^k) - (y_i^k - y_o^k)\cos(\bar{\alpha}^k)}{(\sigma^k_{i,o})^2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -(1-\hat{x}_o^k)\sin(\bar{\alpha}^k) \\ (1-\hat{x}_o^k)\cos(\bar{\alpha}^k) \\ -\hat{x}_o^k\sin(\bar{\alpha}^k) \\ \hat{x}_o^k\cos(\bar{\alpha}^k) \end{bmatrix}$$

$$A^{+}_{i,o} = \sum_{k=1}^{N} \frac{1}{(\sigma^k_{i,o})^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & (1-\hat{x}_o^k)^2\sin^2(\bar{\alpha}^k) & -\frac{(1-\hat{x}_o^k)^2}{2}\sin(2\bar{\alpha}^k) & \hat{x}_o^k(1-\hat{x}_o^k)\sin(\bar{\alpha}^k)^2 & -\frac{\hat{x}_o^k(1-\hat{x}_o^k)}{2}\sin(2\bar{\alpha}^k) \\ . & . & . & . & . & . & . & . & . & (1-\hat{x}_o^k)^2\cos^2(\bar{\alpha}^k) & -\frac{\hat{x}_o^k(1-\hat{x}_o^k)}{2}\sin(2\bar{\alpha}^k) & \hat{x}_o^k(1-\hat{x}_o^k)\cos(\bar{\alpha}^k)^2 \\ . & . & . & . & . & . & . & . & . & . & (\hat{x}_o^k)^2\sin^2(\bar{\alpha}^k) & -\frac{(\hat{x}_o^k)^2}{2}\sin(2\bar{\alpha}^k) \\ . & . & . & . & . & . & . & . & . & . & . & (\hat{x}_o^k)^2\cos^2(\bar{\alpha}^k) \end{bmatrix}$$

And for $\hat{x}_o^k < 0$ ('-'):

$$\vec{b}^{-}_{i,o} = \sum_{k=1}^{N} \frac{(z_i^k - z_o^k)\sin(\bar{\alpha}^k) - (y_i^k - y_o^k)\cos(\bar{\alpha}^k)}{(\sigma_{i,o}^k)^2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hat{x}_o^k \sin(\bar{\alpha}^k) \\ -\hat{x}_o^k \cos(\bar{\alpha}^k) \\ -(1+\hat{x}_o^k)\sin(\bar{\alpha}^k) \\ (1+\hat{x}_o^k)\cos(\bar{\alpha}^k) \\ 0 \\ 0 \end{bmatrix}$$

$$A^{-}_{i,o} = \sum_{k=1}^{N} \frac{1}{(\sigma_{i,o}^k)^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & (\hat{x}_o^k)^2 \sin^2(\bar{\alpha}^k) & -\frac{(\hat{x}_o^k)^2}{2}\sin(2\bar{\alpha}^k) & -\hat{x}_o^k(1+\hat{x}_o^k)\sin(\bar{\alpha}^k)^2 & \frac{\hat{x}_o^k(1+\hat{x}_o^k)}{2}\sin(2\bar{\alpha}^k) & 0 & 0 \\ . & . & . & . & . & . & . & (\hat{x}_o^k)^2\cos^2(\bar{\alpha}^k) & \frac{\hat{x}_o^k(1+\hat{x}_o^k)}{2}\sin(2\bar{\alpha}^k) & -\hat{x}_o^k(1+\hat{x}_o^k)\cos(\bar{\alpha}^k)^2 & 0 & 0 \\ . & . & . & . & . & . & . & . & (1+\hat{x}_o^k)^2\sin^2(\bar{\alpha}^k) & -\frac{(1+\hat{x}_o^k)^2}{2}\sin(2\bar{\alpha}^k) & 0 & 0 \\ . & . & . & . & . & . & . & . & . & (1+\hat{x}_o^k)^2\cos^2(\bar{\alpha}^k) & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & 0 \end{bmatrix}$$

Finally for track segment pairs **BIL - BML** first the vectors $\vec{b}$ for the four cases.

We find for $\hat{x}_{\mathrm{o}}^{k} \geq 0, \hat{x}_{\mathrm{m}}^{k} \geq 0$ ('++'):

$$\vec{b}_{\mathrm{o,m}}^{++} = \sum_{k=1}^{N} \frac{(z_{\mathrm{o}}^{k} - z_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) - (y_{\mathrm{o}}^{k} - y_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k})}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \begin{bmatrix} 0 \\ 0 \\ -(1-\hat{x}_{\mathrm{m}}^{k}) \cdot \sin(\bar{\alpha}^{k}) \\ (1-\hat{x}_{\mathrm{m}}^{k}) \cdot \cos(\bar{\alpha}^{k}) \\ -\hat{x}_{\mathrm{m}}^{k}\sin(\alpha^{k}) \\ \hat{x}_{\mathrm{m}}^{k}\cos(\bar{\alpha}^{k}) \\ 0 \\ 0 \\ (1-\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k}) \\ -(1-\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k}) \\ \hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k}) \\ -\hat{x}_{\mathrm{o}}^{k}\cos(\bar{\alpha}^{k}) \end{bmatrix}$$

For $\hat{x}_{\mathrm{o}}^{k} < 0, \hat{x}_{\mathrm{m}}^{k} \geq 0$ ('-+'):

$$\vec{b}_{\mathrm{o,m}}^{-+} = \sum_{k=1}^{N} \frac{(z_{\mathrm{o}}^{k} - z_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) - (y_{\mathrm{o}}^{k} - y_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k})}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \begin{bmatrix} 0 \\ 0 \\ -(1-\hat{x}_{\mathrm{m}}^{k}) \cdot \sin(\bar{\alpha}^{k}) \\ (1-\hat{x}_{\mathrm{m}}^{k}) \cdot \cos(\bar{\alpha}^{k}) \\ -\hat{x}_{\mathrm{m}}^{k}\sin(\bar{\alpha}^{k}) \\ \hat{x}_{\mathrm{m}}^{k}\cos(\bar{\alpha}^{k}) \\ -\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k}) \\ \hat{x}_{\mathrm{o}}^{k}\cos(\bar{\alpha}^{k}) \\ (1+\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k}) \\ -(1+\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k}) \\ 0 \\ 0 \end{bmatrix}$$

For $\hat{x}_o^k \geq 0, \hat{x}_m^k < 0$ ('+-'):

$$\tilde{b}_{o,m}^{+-} = \sum_{k=1}^{N} \frac{(z_o^k - z_m^k)\sin(\bar{\alpha}^k) - (y_o^k - y_m^k)\cos(\bar{\alpha}^k)}{(\sigma_{o,m}^k)^2} \begin{bmatrix} \hat{x}_m^k \sin(\bar{\alpha}^k) \\ -\hat{x}_m^k \cos(\bar{\alpha}^k) \\ -(1+\hat{x}_m^k)\cdot\sin(\bar{\alpha}^k) \\ (1+\hat{x}_m^k)\cdot\cos(\bar{\alpha}^k) \\ 0 \\ 0 \\ 0 \\ 0 \\ (1-\hat{x}_o^k)\sin(\bar{\alpha}^k) \\ -(1-\hat{x}_o^k)\cos(\bar{\alpha}^k) \\ \hat{x}_o^k \sin(\bar{\alpha}^k) \\ -\hat{x}_o^k \cos(\bar{\alpha}^k) \end{bmatrix}$$

And finally for $\hat{x}_o^k < 0, \hat{x}_m^k < 0$ ('--'):

$$\tilde{b}_{o,m}^{--} = \sum_{k=1}^{N} \frac{(z_o^k - z_m^k)\sin(\bar{\alpha}^k) - (y_o^k - y_m^k)\cos(\bar{\alpha}^k)}{(\sigma_{o,m}^k)^2} \begin{bmatrix} \hat{x}_m^k \sin(\bar{\alpha}^k) \\ -\hat{x}_m^k \cos(\bar{\alpha}^k) \\ -(1+\hat{x}_m^k)\cdot\sin(\bar{\alpha}^k) \\ (1+\hat{x}_m^k)\cdot\cos(\bar{\alpha}^k) \\ 0 \\ 0 \\ -\hat{x}_o^k \sin(\bar{\alpha}^k) \\ \hat{x}_o^k \cos(\bar{\alpha}^k) \\ (1+\hat{x}_o^k)\sin(\bar{\alpha}^k) \\ -(1+\hat{x}_o^k)\cos(\bar{\alpha}^k) \\ 0 \\ 0 \end{bmatrix}$$

Then the matices A for the four cases.

For $\hat{x}_{\mathrm{o}}^{k} \geq 0, \hat{x}_{\mathrm{m}}^{k} \geq 0$:

$$A_{\mathrm{o,m}}^{++} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
. & 0 & 0 & 0 & 0 & 0 \\
. & . & -(1-\hat{x}_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})\sin^{2}(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) \\
. & . & . & (1-\hat{x}_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})\cos^{2}(\bar{\alpha}^{k}) \\
. & . & . & . & (\hat{x}_{\mathrm{m}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) \\
. & . & . & . & . & (\hat{x}_{\mathrm{m}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & .
\end{array} \right.$$

$$\left. \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) \\
0 & 0 & \frac{(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})\cos^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) \\
0 & 0 & -\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) \\
0 & 0 & \frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
. & 0 & 0 & 0 & 0 & 0 \\
. & . & (1-\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) \\
. & . & . & (1-\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} \\
. & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) \\
. & . & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k})
\end{array} \right]$$

For $\hat{x}_{\mathrm{o}}^{k} < 0, \hat{x}_{\mathrm{m}}^{k} \geq 0:$

$$A_{\mathrm{o,m}}^{-+} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \left[\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & . & -(1-\hat{x}_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})\sin^{2}(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & (1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & (1-\hat{x}_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{m}}^{k})\cos^{2}(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & (1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & (\hat{x}_{\mathrm{m}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & . & (\hat{x}_{\mathrm{m}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} & 0 & 0 \\
. & . & . & . & . & . & -(\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} & 0 & 0 \\
. & . & . & . & . & . & . & . & (1+\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(1+\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & . & . & . & . & . & (1+\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & 0
\end{array}\right]$$

For $\hat{x}_{\mathrm{o}}^{k} \geq 0,\ \hat{x}_{\mathrm{m}}^{k} < 0$

$$A_{\mathrm{o,m}}^{+-} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \left[\begin{matrix}
(\hat{x}_{\mathrm{m}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & (\hat{x}_{\mathrm{m}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & -(1+\hat{x}_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) & -\frac{(1+\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & (1+\hat{x}_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & 0 & 0 \\
. & . & . & . & . & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & .
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) \\
0 & 0 & -\frac{\hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} \\
0 & 0 & -(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) \\
0 & 0 & \frac{(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})(1-\hat{x}_{\mathrm{o}}^{k})\cos^{2}(\bar{\alpha}^{k}) & \frac{(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -(1-\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
. & 0 & 0 & 0 & 0 & 0 \\
. & . & (1-\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(1-\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) \\
. & . & . & (1-\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & -\frac{\hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{o}}^{k}(1-\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} \\
. & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) \\
. & . & . & . & . & (\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k})
\end{matrix}\right]$$

For $\hat{x}_{\mathrm{o}}^{k} < 0, \hat{x}_{\mathrm{m}}^{k} < 0$:

$$
A_{\mathrm{o,m}}^{--} = \sum_{k=1}^{N} \frac{1}{(\sigma_{\mathrm{o,m}}^{k})^{2}} \left[\begin{array}{cccccc}
(\hat{x}_{\mathrm{m}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & (\hat{x}_{\mathrm{m}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{m}}^{k})\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & -(1+\hat{x}_{\mathrm{m}}^{k})\sin(\bar{\alpha}^{k}) & -\frac{(1+\hat{x}_{\mathrm{m}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & (1+\hat{x}_{\mathrm{m}}^{k})\cos(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & 0 & 0 \\
. & . & . & . & . & 0 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & .
\end{array} \right.
$$

$$
\left. \begin{array}{cccccc}
-\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
\frac{\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{m}}^{k}\hat{x}_{\mathrm{o}}^{k}\sin(\bar{\alpha}^{k})^{2} & -\frac{\hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & \hat{x}_{\mathrm{m}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} & 0 & 0 \\
(1+\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(1+\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & -(1+\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})\sin^{2}(\bar{\alpha}^{k}) & \frac{(1+\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
-\frac{(1+\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}}{2}\sin(2\bar{\alpha}^{k}) & (1+\hat{x}_{\mathrm{m}}^{k})\hat{x}_{\mathrm{o}}^{k}\sin^{2}(\bar{\alpha}^{k}) & \frac{(1+\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -(1+\hat{x}_{\mathrm{m}}^{k})(1+\hat{x}_{\mathrm{o}}^{k})\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-(\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\sin(\bar{\alpha}^{k})^{2} & \frac{\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & (\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & \frac{\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})}{2}\sin(2\bar{\alpha}^{k}) & -\hat{x}_{\mathrm{o}}^{k}(1+\hat{x}_{\mathrm{o}}^{k})\cos(\bar{\alpha}^{k})^{2} & 0 & 0 \\
. & . & (1+\hat{x}_{\mathrm{o}}^{k})^{2}\sin^{2}(\bar{\alpha}^{k}) & -\frac{(1+\hat{x}_{\mathrm{o}}^{k})^{2}}{2}\sin(2\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & (1+\hat{x}_{\mathrm{o}}^{k})^{2}\cos^{2}(\bar{\alpha}^{k}) & 0 & 0 \\
. & . & . & . & 0 & 0 \\
. & . & . & . & . & 0
\end{array} \right]
$$