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A phenomenological π^-p scattering length from pionic hydrogen

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We derive a closed, model independent, expression for the electromagnetic correction factor to a phenomenological hadronic scattering length a^h extracted from a hydrogenic atom. It is obtained in a non-relativistic approach and in the limit of a short ranged hadronic interaction to terms of order $\alpha^2 \log \alpha$ using an extended charge distribution. A hadronic πN scattering length $a^h_{\pi^- p} = 0.0870\,(5) m_\pi^{-1}$ is deduced leading to a πNN coupling constant from the GMO relation $g_c^2/(4\pi) = 14.04\,(17)$.

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1. Introduction

The strong interaction energy shifts ϵ_{1s} and total decay width Γ_{1s} in pionic hydrogen have been measured to a remarkable precision [1]

$$\epsilon_{1s} = [-7.108 \pm 0.013(\text{stat}) \pm 0.034(\text{syst})] \text{ eV}, \quad (1)$$
$$\Gamma_{1s} = [0.868 \pm 0.040(\text{stat}) \pm 0.038(\text{syst})] \text{ eV}. \quad (2)$$

It is well known [2,3] that the (complex) strong interaction shift in the 1s state of hadronic atoms is closely linked to the (complex) elastic threshold scattering amplitude a defined in the absence of the Coulomb field. We refer in the following to this quantity as the (complex) scattering length. This is conventionally expressed in the ratio of the shift to the Bohr energy $E_B = -m\alpha^2/2$:

$$\frac{\epsilon_{1s}}{E_B} = \frac{\epsilon^0_{1s}}{E_B}(1 + \delta_{1s}) = 4m\ \alpha\ a\ (1 + \delta_{1s}), \quad (3)$$

where δ_{1s} conveniently measures the deviation of the shift from the lowest order estimate

$$\epsilon^0_{1s} = -\frac{4\pi}{2m}\ \phi^2_B(0)\ a. \quad (4)$$

Here $\phi_B(r)$ is the non-relativistic 1s Bohr wave function of a point charge and m the reduced mass, which in the present case is that of the π^-p system. It is important to understand the correction δ_{1s} transparently and reliably to an accuracy matching the high experimental precision, since the hadronic πN scattering lengths are key testing quantities for chiral physics. In addition, they are needed phenomenologically to about 1 % for the precision determination of the πNN coupling constant using the GMO relation [4].

The standard conversion of experimental data to a scattering length uses the potential approach of Sigg et al. [5], which describes the πN interaction in terms of coupled equations using physical pion masses and an isospin invariant non-diagonal potential matched to scattering lengths calculated by setting the neutral pion mass equal to the charged one. This gives $\delta_{1s}(\text{Sigg}) = (-2.1 \pm 0.5)\%$. The procedure is model dependent and it is not consistent with the πN low-energy expansion [4]. Their results must therefore be used with caution.

The classical way to obtain Eq. (3) is based on analytical approaches using Coulomb wave functions (see Refs. [2,3,6,7,8] and references therein). To our knowledge these papers do not explore the effect of the extended charge distribution on the strong interaction shift. This paper discusses this question.

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The authors of Refs. [9,10] have calculated the ground state energy of the $\pi^- p$ system in the framework of QCD+QED, using effective field theory (EFT) techniques. The shift of the ground state energy is related to the scattering lengths in pure QCD, evaluated in the isospin symmetric limit $m_u = m_d$ The corresponding correction δ_ϵ in the energy shift is evaluated in the framework of Chiral Perturbation Theory (ChPT). At leading order [9], one has δ_ϵ =$(-4.3 \pm 2.8)\%$, while the next-to-leading order result [10] is δ_ϵ= $(-7.2 \pm 2.9)\%$. The uncertainty in δ_ϵ is due to the poor knowledge of one of the low-energy constants occurring in the effective theory. The correction δ_ϵ is also considerably larger than what was found in Ref. [5]. This EFT approach makes inherently no distinction between the atomic corrections due to Coulomb effects discussed here and other contributions. We come back in section 4 to a comparison with the present work.

Our aim here concerns only the connection of the strong atomic energy shift to the scattering length a^h defined as the one which would be observed if the Coulomb field of the extended charge could be removed and considered as due to an external source. This scattering length is directly related to the one appearing in forward dispersion relations for πN scattering. In this spirit, no correction is made for the internal e. m. contributions to the masses. The physical scattering lengths for the $\pi^\pm$n scattering correspond closely to the present definition, neglecting the very small e. m. correction from the charged pion interaction with the neutron charge distribution. These scattering lengths with physical masses are natural 'observables' for the study of isospin breaking. The (complex) π^+n scattering length coincides with the π^-p one in the limit of exact charge symmetry: it has the corresponding open charge exchange channel $\pi^+ \mathrm{n} \to \pi^0 \mathrm{p}$ and the corresponding open radiative decay channel $\pi^+ \mathrm{n} \to \gamma \mathrm{p}$. This definition is different from that of the QCD scattering length used in the EFT approach. We use here the r-representation, which is more transparent for the present problem than the equivalent momentum representation. Since the π^-p atom is highly non-relativistic, most of the discussion will be made using non-relativistic concepts. The result will be expressed in terms of the empirical on-shell parameters of the πN low-energy expansion. For the electromagnetic corrections, this approach gives intuitively interpretable expressions, exact up to terms in $\alpha^2 \log \alpha$, provided the Coulomb potential of the extended charge varies little over the range of the strong interaction.

In section 2 we solve the problem exactly to all orders in α in the limit of a short ranged strong interaction with the charge located to a spherical shell. The correction for an arbitrary charge distribution is then derived perturbatively to the same order in α as in the EFT expansion [10]. Corrections for the finite interaction range are explored. We explicitly include the correction for the vacuum polarization. In section 3 we discuss the magnitudes of the corrections and their physical structure. In section 4, we compare our results to those of previous approaches.

We denote by E_{fs}, E and $\epsilon_{1s} = E - E_{fs}$ the 1s finite size e.m. binding energy, the total 1s binding energy with strong interaction and finite size, and the strong interaction shift, respectively. The non-relativistic wave numbers are κ_B, κ_{fs} and κ for E_B, E_{fs} and E, respectively. Since this paper concerns atomic corrections we also use the Bohr radius $r_B = \kappa_B^{-1} = (m\alpha)^{-1}$.

2. A Model for the π^-p Atom

The aim of this section is to explore the consequences of the extended charge distribution in a pedagogically transparent and soluble model of the π^-p system. This will serve as a prototype for the later more general discussion and it will reveal the nature of the contributions to the correction term in Eq. (3). In the absence of the Coulomb potential the threshold expansion for the angular momentum $l = 0$, typical of a weak scattering length, is related to the phase shift $\delta_{l=0}$ and to the momentum q by the relation

$$\frac{\tan \delta^h_{l=0}}{q} = a^h + b^h q^2 + ... \qquad (5)$$

Here a^h is the hadronic scattering length and b^h is the range parameter.

The model is constructed as follows. We first consider the case of a single channel. This avoids

the complications of several open channels with different masses for which the equivalent of the single channel scattering length is ambiguous. The generalization will be discussed later. The charge is assumed to be concentrated to a spherical shell of radius R, outside the range of the hadronic interaction. The system is taken to be non-relativistic. For the moment we neglect the effect of the vacuum polarization potential. As defined, this problem can be solved exactly in terms of the on-shell hadronic s-wave scattering amplitude, although we will only evaluate it for contributions to the correction term up to order $\alpha^2 \log \alpha$. In this case the definition of the scattering length is clear. The model does not have problems with the intermingling of the Coulomb and hadronic interaction, contrary to other descriptions. Such problems are particularly acute in any description with a pointlike charge distribution, since the Coulomb interaction is then divergent at $r = 0$.

Inside a typical shell radius R of the order of 1 fm, the Coulomb potential is constant with $V_C(R) = -\alpha/R \simeq 1.4$ MeV. This motivates the non-relativistic approximation in this region to order α. The inside wave number q_c is constant:

$$q_c^2 = \frac{2m\alpha}{R} - \kappa^2 \simeq \frac{2m\alpha}{R}. \tag{6}$$

The $1s$ binding energy $E \simeq -3.2$ keV is negligible compared to the Coulomb field and the strong interaction inside the charge distribution region, although its exact value governs the scale of the atom. The external 1s wave function for $r \geq R$ is a Whittaker function $\mathcal{W}_{\lambda;1/2}(z)$ (see e. g., [11], Eq. (9.237)) with $\lambda = \kappa_B/\kappa$ and $z = 2\kappa r$. The inside wave function for $r \leq R$ is a standing wave outside the strong interaction region. Neglecting terms of order$(1-\lambda)^2 \simeq (\alpha m a^h)^2 \simeq 10^{-6}$, one has for $r \leq R$,

$$u_{in}(r) = N\left[\frac{\sin\,(q_c r)}{q_c} + \frac{\tan\delta^h_{l=0}}{q_c}\cos\,(q_c r)\right] \tag{7}$$

and for $r \geq R$,

$$u_{out}(r) = (4\pi)^{1/2})^{-1}\kappa \exp(-z/2)$$

$$\times\left\{z\big[(1+(1-\lambda)(1-\gamma-\log\ z)\big] + \frac{1-\lambda}{\lambda}\right\}. \tag{8}$$

Here $\gamma = 0.577...$ is the Euler constant. Note that the term $\tan(\delta^h_{l=0})$ is determined by the 'physical' hadronic phase shift in the absence of the Coulomb field taken at the energy $-V_C(R)$. The wave function corresponding to Eqs. (7) and (8) is normalized to order α^2.

The energy shift ϵ_{1s} produced by the strong interaction is obtained by matching the logarithmic derivative of the wave function at the radius R. In accordance with standard practice, it is defined as the difference between the total binding energy E and the electromagnetic binding energy with a finite size charge distribution [5]. This corresponds to the removal of the 'scattering length' corresponding to the extended charge distribution:

$$a_{fs} = -\frac{\alpha m R^2}{3};\ \epsilon^{fs}_{1s} = \frac{2\pi\alpha R^2}{3}\phi_B^2(0). \tag{9}$$

Expanding the exact analytical expression to terms of order $\alpha^2 \log\alpha$ by a straightforward algebraic calculation gives the following correction factors, where the hadronic scattering length a^h takes the place of a in Eq. (3):

$$\begin{aligned}\delta_{1s} &= -2\frac{R}{r_B} + 2\frac{a^h}{r_B}\left[2-\gamma-\log(2\alpha m R)\right] \\ &+ \frac{2m\alpha}{R}\frac{b^h}{a^h}.\end{aligned} \tag{10}$$

This expression serves as a guide for the later generalizations.

The assumption of a zero range hadronic interaction is unnecessary in our simple model. An interaction of any shape will give the same result provided its range is smaller than R. This follows from the matching condition for the wave functions (7) and (8), which is only required at R, such that any interaction with the same hadronic scattering near-threshold amplitude, $a^h + q_c^2 b^h$, gives the same result.

The terms in Eq. (10) have a clear physical interpretation. The extended charge wave function at $r = 0$ in the absence of strong interactions is $\phi_{in}(0) = \phi_B(0)(1 - R/r_B + ..)$ to the present

order in α. It is a better starting approximation than the wave function of the Bohr atom in Eq. (4), which then should be multiplied by a factor $(1 - R/r_B + ..)^2 \simeq 1 - 2R/r_B + ...$ This accounts for the first term in Eq. (10). The second term proportional to a^h is a renormalization due to the external wave function which is changed at R by the hadronic scattering itself by a factor $1 + 2a^h m\alpha[2 - \gamma - \log(2\alpha mR)]$. The outside wave function is determined by the energy shift. The matching of the inside and outside wave functions of Eqs. (7) and (8) gives near the origin

$$\begin{aligned} u_{in}(r) &= \{1 + 2a^h m\alpha[2 - \gamma - \log(2\alpha mR)]\} \\ &\times \quad (r + a^h)\ \phi_{in}(0). \end{aligned} \tag{11}$$

This result agrees with that obtained by matching the logarithmic derivative at R. This factor has little sensitivity to the exact value of the radius R. The leading $\alpha \log \alpha$ part of the term in a^h is well known from previous approaches and has also been found in the EFT approach where it corresponds to a "loop" term [9,10].

The last term in Eq. (10) follows from gauge invariance with the replacement $E \to E - V_C(0)$ in the scattering amplitude [4]. Alternatively, and more intuitively, it follows using the correct energy at the point of interaction. This is not the binding energy, but the finite depth of the Coulomb potential of the extended charge. (For the corresponding effect in higher Z pionic atoms, see Refs. [12]-[14]).

Exactly the same reasoning as for the 1s state can be applied to hadronic energy shift ϵ_{ns} in states of any n. The correction factor δ_{ns} is defined in complete analogy to Eq. (3):

$$\frac{\epsilon_{ns}}{E_{ns}} = \frac{\epsilon^0_{ns}}{E_{ns}}(1 + \delta_{ns}) = 4\frac{a^h}{nr_B}(1 + \delta_{ns}), \tag{12}$$

where $E_{ns} = -m\alpha^2/(2n^2)$ and the convenient comparison shift ϵ^0_{ns} is the simplest perturbative expression for the energy shift

$$\epsilon^0_{ns} = -\frac{4\pi}{2m}\ \phi^2_{B;ns}(0)\ a^h = \frac{\epsilon^0_{1s}}{n^3}. \tag{13}$$

One has to order $\alpha^2 \log \alpha$ in the correction

$$\delta_{1s} - \delta_{ns} = 2\frac{a^h}{r_B}\left(1 - \frac{2}{n} + \sum_1^n \frac{1}{k} - \log\ n\right). \tag{14}$$

In this expression all the dependence on the parameter R has disappeared, which reflects that all the short-ranged physics is identical but for a renormalization factor. In the limit $n \to \infty$

$$\delta_{1s} - \delta_{\infty s} = 2\frac{a^h}{r_B}(1 + \gamma), \tag{15}$$

where the correction term is given by

$$\begin{aligned} \delta_{\infty s} &= -2\frac{R}{r_B} + 2\frac{a^h}{r_B}\left[1 - 2\gamma - \log\ \left(\frac{2R}{r_B}\right)\right] \\ &+ \quad \frac{2m\alpha}{R}\frac{b^h}{a^h}. \end{aligned} \tag{16}$$

This semi-classical limit for $\kappa_n \equiv (\kappa_B/n) \to\ 0$ corresponds to the Coulomb scattering length [15] $a^c = a^h(1 + \delta_{\infty s})$ in our model. For the present case of a π^-p atom the numerical difference in the correction terms for different values of n in Eqs. (14) and (15) is less than 10^{-3} and of little practical importance.

Arbitrary Charge Distribution

The result (10) is the prototype for more general charge distributions. The difference between the Coulomb potential V_{Cfs} for a charge distribution from the observed π^- and proton form factors, $\rho(r)$, and V_{CR} corresponding to that for the spherical shell of radius R, $\rho_R(r)$, gives a perturbative potential, which includes the f.s. charge density associated with the anomalous magnetic moment:

$$\delta V_C(r) = V_{Cfs}(r) - V_{CR}(r)$$

$$= -\alpha \int_r^\infty \left(\frac{1}{r'} - \frac{1}{r}\right) \delta\rho(r') 4\pi r'^2 dr', \tag{17}$$

where $\delta\rho(r') = \rho(r) - \rho_R(r')$. Applying this perturbation to our soluble model gives a net correction independent of R to the present order in α. The explicit calculation leads to the following four changes in our model results. First, the e. m. finite size energy shift (9) is changed with the substitution of the model R^2 by $\langle r^2 \rangle_{em} = \langle r^2_p \rangle_{em} + \langle r^2_\pi \rangle_{em} = 1.15(2)$ fm^2 as in Ref. [5]. Likewise, the value of the overall Coulomb potential at the origin changes from the model value

α/R to $\alpha\langle 1/r\rangle_{em}$ and the wave function squared at the origin changes it value from $(1-2\alpha R+..)$ to $(1-2\alpha\langle r\rangle_{em}+..)$. Finally, the term $\log R$ is replaced by $\langle\log\ r\rangle_{em}$. The changes are independent of the hadronic interaction.

In the case of a single channel and in the hadronic zero range limit, the corrections are:

$$\delta_{1s}=-2\frac{\langle r\rangle_{em}}{r_B}+2\frac{a^h}{r_B}\left(2-\gamma-\left\langle\log\frac{2r}{r_B}\right\rangle_{em}\right)$$

$$+2m\alpha\left\langle\frac{1}{r}\right\rangle_{em}\frac{b^h}{a^h}\equiv\delta^{\langle r\rangle}+\delta^c+\delta^g. \qquad (18)$$

We now introduce the correction δ^{vp} for the vacuum polarization [16]. The first order vacuum polarization is described by a potential proportional to α^2 with a range much larger than that of the hadronic and charge distribution ones and it is insensitive to the strong interaction dynamics. The joint extended Coulomb potential and vacuum polarization one is a perfectly justifiable alternative to the point Coulomb potential as the starting point for the wave function in Eq. (3). The square of the unperturbed wave function at the origin changes by $\delta^{vp}=2\delta\phi_{vac}(0)/\phi_B(0)=0.48\%$ due to vacuum polarization [16] and by -0.85 % from the extended charge (see Table 1). This result is model-independent and it agrees with the prior numerical value implicit in Ref. [5].

In the derivation of the correction factors, we nowhere used that these quantities should be real. We can therefore take the energy shift to be complex, $\epsilon_{1s}-\mathrm{i}\Gamma_{1s}/2$, with a hadronic complex scattering length $a_r^h+\mathrm{i}a_i^h$. The complex energy shift is related to the corresponding correction factors by $a(1+\delta)\ \rightarrow\ a_r(1+\delta_r)+\mathrm{i}a_i(1+\delta_i)$, as in Eq. (3). The imaginary part corresponds to absorptive phenomena. In the notation of Eq. (18):

$$\delta_{1s,i}=-2\frac{\langle r\rangle_{em}}{r_B}+4\frac{a_r^h}{r_B}\left(2-\gamma-\left\langle\log\frac{2r}{r_B}\right\rangle_{em}\right)$$

$$+\ 2m\alpha\left\langle\frac{1}{r}\right\rangle_{em}\frac{b_i^h}{a_i^h}\equiv\delta^{\langle r\rangle}+2\delta^c+\delta_i^g. \qquad (19)$$

Here the imaginary amplitudes a_i^h and b_i^h refer to any absorptive channel such as the π^-p charge exchange scattering. Note the additional factor 2 in the middle term as compared to that for the real case in Eq. (18). Since $a_i << |a_r|$, the change in $\delta_{1s,r}$ due to absorption is negligible.

We conclude that most of the corrections to the width are due to the change of the wave function at origin: it is important to use wave functions corresponding to the finite size and vacuum polarization potentials. In addition, the non-linear renormalization term must also be included, but only the real part of the scattering length is relevant. To these should be added the amplitude change due to the gauge term in analogy to the case for the energy shift.

Coupled channels

The π^-p atom is a coupled system of the continuum $\pi^o n$ and γn channels in addition to the π^-p one. These three channels are denoted by indices $i\ (j)\ =\ c,o,f$, respectively. The low-energy expansion in multiple channel systems is defined in terms of energy dependent (symmetric) K-matrices which enter the standing wave solutions. The formalism is described briefly below and it is illustrated for the 2-channel situation. The single channel becomes a special case. The standing waves at distances larger than the charge radius $r>R$ are defined as [17]

$$u_j^i=u_r^i\delta_{i,j}+K_{i,j}^c u_s^j, \qquad (20)$$

where $K_{i,j}^c$ are the "Coulomb-corrected" K-matrix elements. The wave functions u_r^c and u_s^c are defined in terms of the standard regular and singular Coulomb functions F and G, respectively (see e.g. Refs. [6,17]). In the limit $\alpha\rightarrow 0$ and q fixed, these solutions correspond to $\sin(qr)/q$ and $\cos(qr)/q$, respectively. Furthermore

$$u_r^c=\frac{F(r)}{C(\eta)q}\underset{r\rightarrow\infty}{\rightarrow}\frac{\sin(\varphi)}{C(\eta)q}; \qquad (21)$$

$$u_s^c=G(r)C(\eta)-2\eta h(\eta)\frac{F(r)}{C(\eta)}$$

$$\underset{r\rightarrow\infty}{\rightarrow}C(\eta)\cos(\varphi)-\frac{2\eta h(\eta)\sin(\varphi)}{C(\eta)}, \qquad (22)$$

where $\eta=zz'\alpha m/q$ and $q\eta=-\kappa_B$. In these equations the digamma function $\psi(x)$ defines $h(\eta)=[\psi(i\eta)+\psi(-i\eta)]/2-\log(\eta^2)/2$ and

$C^2(\eta) = 2\pi\eta/[\exp(2\pi\eta) - 1]$ is the standard penetration factor. At large distances the phase $\varphi = qr - \eta\log(2qr) + \sigma$, where σ is the Coulomb phase shift.

From the K-matrix one obtains the scattering amplitude $\mathcal{T}$ by regrouping the standing waves into the regular and outgoing waves. With $C_c = C(\eta)$ and $C_o = C_f = 1$, one has

$$\mathcal{T}^c_{i,j} = C_i \left[-(K^c)^{-1} + f\right]^{-1}_{i,j} C_j. \quad (23)$$

In this equation we use a diagonal matrix with $f_{c,c} = 2\eta q_c h(\eta) + iq_c C^2(\eta)$ and $f_{0,0} = iq_0$.

In the single channel case, the textbook relation of K^c to the scattering amplitude $\mathcal{T}$ is [15]

$$\begin{aligned}\mathcal{T}^c &= C^2(\eta)\left[-\frac{1}{K^c} + 2\eta q h(\eta) + iqC^2(\eta)\right]^{-1} \\ &\equiv -[q\cot(\delta) - iq]^{-1}. \quad (24)\end{aligned}$$

The atomic level shift is obtained from the "well known" formula.

$$C^2(\eta)q\cot(\delta) + 2\eta q h(\eta) = \frac{1}{K^c} \quad (25)$$

and the bound state condition $\cot(\delta) = i$. As found by Trueman [3]

$$\epsilon_{1s} = -\frac{4\pi}{2m}\phi_B^2(0)a^c\left[1 + \frac{a^c}{r_B}(2 + 2\gamma)\right], \quad (26)$$

where a^c is given by the Coulomb K-matrix at threshold. Its relation to the hadronic scattering length of the present model is

$$a^c = a^h\left[1 - \frac{2R}{r_B} + a^h\Delta_G\right] + \frac{2m\alpha}{R}b^h; \quad (27)$$

$$\Delta_G = \frac{2}{r_B}\left[1 - 2\gamma - \log\left(\frac{2R}{r_B}\right)\right]. \quad (28)$$

XXX This relation leads to the correction given by Eq. (16).

In the 2-channel case with $(i, j) = (c, o)$, the leading order in the level shift follows by the replacement of a^c by the threshold amplitude in the charged channel $A^c_{c,c}$, which is obtained from Eq. (23). Including terms to order q_o^2

$$A^c_{c,c} = K^c_{c,c} + iq_o(K^c_{c,o})^2 - q_o^2 K^c_{o,o}(K^c_{c,o})^2. \quad (29)$$

At the charged threshold the phase space left in the open neutral channel is described by the momentum q_o. Eq. (26) should now be used with a complex $A^c_{c,c}$:

$$\epsilon_{1s} - i\Gamma_{1s}/2 = -\frac{4\pi}{2m}\phi_B^2(0)A^c_{c,c}\left[1 + \frac{A^c_{c,c}}{r_B}(2 + 2\gamma)\right]. \quad (30)$$

In the zero range limit, the hadronic interaction in the charged channel occurs at a momentum $q_c^2 = 2m\alpha/R$, while that in the neutral channel still occurs at the momentum q_o, since the atomic binding energy is negligible. The K-matrix elements are energy dependent with a low-energy expansion is analogous to that of Eq. (5).

$$K_{c,c} = a^h_{c,c} + b^h_{c,c}\,\frac{2m\alpha}{R}, \quad (31)$$

$$K_{c,o} = a^h_{c,o} + \frac{1}{2}(q_c^2 + q_o^2)\; b^h_{c,o}, \quad (32)$$

$$K_{o,o} = a^h_{o,o} + q_0^2 b^h_{o,o}. \quad (33)$$

We assume isospin invariance for the range parameters $b^h_{c,c}$ and $b^h_{c,o}$ since they only appear in correction terms. In the internal region $r < R$ the standing waves are

$$u^i_j \propto \sin(q_i r)\delta_{i,j} + K_{i,j}\cos(q_j r). \quad (34)$$

The continuity of the wave function matrix $\hat{u}$ and its logarithmic derivative $\hat{u}^{-1}d\hat{u}/dr$ at the radius of the charged shell R gives:

$$K^c_{c,c} = K_{c,c}\left[1 - \frac{2R}{r_B} + K_{c,c}\;\Delta_G\right]\;; \quad (35)$$

$$K^c_{c,o} = K_{c,o}\left[1 - \frac{R}{r_B} + K_{c,c}\;\Delta_G\right]\;; \quad (36)$$

$$K^c_{o,o} = K_{o,o} + K_{o,c}\;\Delta_G K_{c,o}. \quad (37)$$

These corrections are implicit in the single channel equations (10) and (16).

The extension to the (γ, n) channel is obtained with the substitution $K^c_{i,j} \to K^c_{i,j} + (iq_f K^c_{i,f}K^c_{f,j})/(1 - iq_f K^c_{f,f})$ to every matrix element of the two channel system. The higher order terms in the neutral and photon channels $K^c_{o,o}$ and $K^c_{f,f}$ are negligible such that

$$A^c_{c,c} \approx K^c_{c,c} + iq_o(K^c_{c,o})^2 + iq_f(K^c_{c,f})^2. \quad (38)$$

The corrections (35) to (38) can be introduced into the Trueman formula (30) in order to express it in the form of Eq. (3).

The result for an arbitrary charge distribution follows in complete analogy to the single channel case discussed in Eq. (17) and following. with the substitutions $R \to \langle r \rangle_{em}$ etc.. The corrective terms are given by Eq. (18) with the only change that now $a^h = a^h_{c,c}$ and $b^h = b^h_{c,c}$. The effects from neutral channels are negligible.

The total level width Γ_{1s} has two components of comparable magnitude corresponding to the decay via the charge exchange and radiative channel, respectively:

$$\Gamma_{1s} = \Gamma^{\pi^0 n}_{1s} + \Gamma^{\gamma n}_{1s}. \tag{39}$$

They can be physically separated using the Panofsky ratio $P = 1.546(9)$ [18]. Of special interest in the present context is the hadronic charge exchange channel. Here the ratio b^h_i/a^h_i of Eq. (19) is b^h_{co}/a^h_{co}, since the charge exchange width depends quadratically on the amplitude (32) and

$$\Gamma^{\pi^0 n}_{1s} = \frac{\Gamma_{1s}}{1+P^{-1}} = \frac{4\pi}{m}\phi^2_B(0) q_o \left[a^h_{c,o}\,(1+\delta_\Gamma)\right]^2 . \tag{40}$$

Here δ_Γ is the counterpart of $\delta_{1s,i}/2$ of Eq. (19):

$$\delta_\Gamma \equiv \frac{1}{2}\delta^{<r>} + \delta^c + \frac{1}{2}(q_c^2 + q_o^2)\frac{b^h_{c,o}}{a^h_{c,o}} + \frac{1}{2}\delta^{vp}. \tag{41}$$

We have corrected the charge exchange amplitude $a^h_{c,o}$ in Eq. (40) not only for the effective interaction energy in the charged channel (gauge term), but also for the non-atomic π^0 energy in the open charge exchange channel (32). This is justified, since this correction can be 'tuned' externally, for example by binding the proton into a potential. It is thus of a different nature than the non-trivial corrections for the mass splittings.

3. Numerical results

We now apply these results to the π^-p atom. We assume for the moment that the correction for the finite range of hadronic interaction only enters via the range parameter b. Isospin invariance is assumed for hadronic scattering parameters appearing in correction terms. The e.m. expectation values appearing in Eq. (18) follow from the folded (π^-, p) charge distributions corresponding to the observed form factors [19]: $\langle r \rangle_{em} = 0.95(1)$ fm; $\langle 1/r \rangle_{em} = 1.48(1)$ fm^{-1}; $\langle \log(mr) \rangle_{em} = -0.687(9)$ with $V_C(0) = \langle \alpha/r \rangle_{em} = 2.13(2)$ MeV. We use the empirical values for the range terms $b_{\pi^- p} = b_{\pi^+ n} = -0.031(9)\ m_\pi^{-3}$; $b_{\pi^- n} = b_{\pi^+ p} = -0.058(9)\ m_\pi^{-3}$ or $b^+ = -0.0044(7)\ m_\pi^{-3}$; $b^- = -0.0013(6)\ m_\pi^{-3}$ from πN scattering data [20].

The correction terms are given in Table 1. For the $\pi^- p$ energy shift they are obtained from Eq. (18) by a two step iteration and do not require the knowledge of $a_{\pi^- p}$. The width corrections, calculated from Eqs. (40) and (41), require that one knows the sign of $a^h_{c,o}$. We also give the corrections for the π^+p Coulomb scattering length a^c [15], which is similar to the π^-p case, but for appropriate sign changes in parameters. The $a_{\pi^+ \mathrm{p}}$ correction terms follow from our determination of $a^h_{c,c}$ and $a^h_{c,o}$ assuming them to be isospin invariant. The physical $\pi^\pm$n scattering lengths have Coulomb corrections of less than 0.1% and can safely be identified with the hadronic ones at the present level of precision.

There is little uncertainty in any of the corrections within our assumptions. It comes mostly from the experimental value of the range term $b_{\pi^- p} = b^+ + b^-$. Here the b^- part contributes 50% to the error of the energy shift and nearly all to that of the width. From a purely phenomenological standpoint its theoretical origin is irrelevant. However, to leading order it is simply generated by the energy dependence of the Weinberg-Tomozawa amplitude on the one hand and by the nucleon Born term of opposite sign on the other one (Eqs. (44-46) in Ref. [21]), consistent with the experimental value. In the case of δ_Γ, the non-atomic correction in Eq. (40) for the neutral pion energy is responsible for 60% of the 'gauge term'.

The low-energy expansion for the K-matrix depends symmetrically on the initial and final momenta as $(q_i^2 + q_j^2)/2$. For the terms proportional to b^- this is explicitly the case in the non-relativistic limit when the initial and final pion are separately on the mass shell [21]. The situation is similar for the isoscalar effective range

Table 1
Coulomb corrections in percent as described in the text. δ^{vp} is included in δ_{1s} and $\delta_{\pi^+p\to\pi^+\mathrm{p}}$.

	Extended charge	Renormalization	Gauge term	Vacuum polarization	Total
δ_{1s}	−0.853(8)	0.701(4)	−0.95(29)	0.48	−0.62(29)
δ_Γ	−0.427(4)	0.701(4)	0.50(23)	0.24	1.02(23)
$\delta_{\pi^+p\to\pi^+\mathrm{p}}$	0.853(8)	0.72(5)	−1.71(29)	0.48	0.35(29)

term b^+. The dominant contribution is in this case proportional to the scalar form factor $\sigma(t)$ (see, e.g, Eq. (10.1) and following in Ref. [22]). The corresponding nucleon Born terms have the same structure.

We have therefore a good quantitative picture of the precision to which the hadronic scattering length can be extracted from the strong energy shift in the limit of a short ranged energy dependent hadronic interaction. To fully exploit the present experimental information content, the theoretical corrections must controlled at least to 0.5%. This has been achieved even using the errors of the phenomenological parameters. The true theoretical precision in our approach is far higher. The model with a spherical shell charge distribution suggests that the results are robust with small modifications as long as the interaction range is inside a characteristic charge scale.

The correction terms of Table 1 applied to the experimental value of the pionic atom energy shift and width of Eq. (1) give the following hadronic scattering lengths

$$a^h_{\pi^-p} \quad = \quad a^h_{c,c} \;=\; 0.0870(5)\; m_\pi^{-1}\;; \qquad (42)$$

$$a^h_{\pi^-p\to\pi^0n} \;=\; a^h_{c,o} \;=\; -0.125(4)\; m_\pi^{-1}\;. \qquad (43)$$

Here the masses are the physical ones and the π^0n and γn decay channels are open. The result (42) is 1.5% smaller and outside the quoted uncertainty of the value 0.0883(8) deduced in Ref. [1] which is based on the Sigg analysis [5]. We have made no correction in Eq. (42) for the e. m. terms outside our Coulomb potential approach, in particular for the effect of the γn and the $\gamma\Delta$ intermediate states in the direct and crossed channels. This will be discussed in the next section. Our value (43) for $a^h_{\pi^-p\to\pi^0n}$ differs by 2.4% from that deduced in Ref. [1].

The scattering length $a^h_{\pi^-p}$ can be analyzed jointly with the π^-D scattering length to give an isovector scattering length $(a_{\pi^-p} - a_{\pi^-n})/2$. We follow the procedure of Ref. [4] with two minor additions. First, the triple scattering term in the multiple scattering was only partly included. The full triple scattering term according to Refs. [23, 24] represents a contribution of $+0.0027\, m_\pi^{-1}$ [24] to the theoretical π^-D scattering length in Ref. [4] in the limit of point interactions. Following the procedure in Ref. [4] we reduce the overall factor $< 1/r^2 >_D = 0.314\,(25)\,\mathrm{fm}^{-2}$ in the triple scattering term by a form factor to give $< f(r)^2/r^2 >_D = 0.238\,(24)\,\mathrm{fm}^{-2}$. The theoretical π^-D scattering length in Ref. [4] should therefore be increased by $+0.0019\,(2)\; m_\pi^{-1}$. Second, the Fermi motion ('boost') correction now includes not only the dominant contribution from p-wave πN scattering, but also a smaller one from the energy dependence of s-wave isoscalar amplitude at threshold as first found in a chiral approach [23]. In fact, a more accurate description of this correction term is obtained using the momentum expansion of the forward scattering amplitude near threshold [25]. The p-wave coefficient c_0 in Ref. [4], Eq. (14), should then be replaced by $b^+ + c_0$ with $b^+ = (b_{\pi^-p} + b_{\pi^-n})/2$. The Fermi motion correction to the π^-D scattering length changes then from $0.0061\,(7)\; m_\pi^{-1}$ to $0.0047\,(6)\; m_\pi^{-1}$. The net contribution of these two changes to the determination of the isovector scattering length is only about 0.2% of its numerical value. This is small in comparison to the systematic theoretical uncertainty quoted in Table IV of Ref. [4]. The individual contributions of both changes are also within this uncertainty.

Using a_{π^-p} from Eq. (42) in conjunction with the procedure of Ref. [4] Eq.(B8), including the corrections just described leads to the improved isovector combination value $(a_{\pi^-n} - a_{\pi^-p})/\sqrt{2} =$

$-0.125\,(1)\,m_\pi^{-1}$. It can therefore be directly compared to the corresponding value $-0.125\,(4)\,m_\pi^{-1}$ deduced in Eq. (43) from the charge exchange width $\Gamma_{1s}^{\pi^0 n}$. Note that when the deuteron data are used to extract the isovector scattering length, the further e. m. contributions cancel in the limit of charge symmetry [4]. Such is the situation for the leading dispersive contributions from processes such as the $\gamma N(\gamma\Delta)$ channels and their crossed terms. Thus, there is no indication of isospin violation in the isovector amplitude at the present level of precision.

4. Comparison to previous approaches

In previous analytical approaches using wave functions little attention was given to the effect of the electromagnetic finite size effects and to the issue of the correct energy of the interaction. In a recent discussion, the Coulomb interaction is cut off entirely at the range of the strong interaction [8]. Several authors starting from Trueman [3,8] consider the influence of a threshold expansion for the hadronic amplitude. However, they incorrectly identify the momentum q_c with the Bohr momentum, which leads to a very small correction of order α^2. The correction to the scattering length proportional to $(a^c)^2$ is obtained to leading order $\alpha \log \alpha$, but these previous treatments give incorrectly such terms of order α. The numerical consequences of this difference is small.

As discussed in the introduction, the numerical approach using coupled channels [5] correctly includes the effect of the finite size and vacuum polarization in the wave function correction as well as the renormalization term. It is, however, inconsistent with the low-energy expansion. In addition, it makes model dependent corrections for isospin violation and radiative decay effects (see also comments in Refs. [9,10]). The numerical result for a single channel does not have these problems. It agrees with our explicit result using the same input parameters.

In Refs. [9,10], the energy shift ϵ_{1s} is related to the scattering length in pure QCD. The calculations are performed perturbatively to the same order in α as here in the framework of an effective quantum field theory. The correction to Eq. (3) with this scattering length is obtained using an expansion in in powers of α and of the quark masses $(\bar{m}_d, \bar{m}_u)$. The leading and next-to-leading order terms can be given in closed form in terms of πN scattering lengths [9]. The higher order contributions have been evaluated in the framework of ChPT in Refs. [9,10]. Since our approach is phenomenological and uses the physical masses, the e. m. EFT corrections for mass splittings are implicitly included in our scattering length. One cannot therefore match the expansions in detail to each other. In spite of that it is possible to make some observations. First, the next-to-leading order non-analytic term $\alpha \log \alpha$ proportional to $(a^h)^2$ (our renormalization term) results quantitatively also in the EFT approach as in all previous potential approaches. This is natural, since it describes long-range physics. Inside our model, the charge distribution acts as a regulator and leads quantitatively to additional terms of order α.

As discussed at the end of the previous section, our Coulomb scattering length $A_{c,c}^c$ in Eq. (29) contains additional non-potential e. m. contributions, which are an integral part of our phenomenological scattering length. Such contributions have been calculated in a chiral quark model with quark wave functions [26]; the authors obtain expressions for the EFT empirical constants $f_{1,2,3}$ in an approximation corresponding to γN and $\gamma\Delta$ appearing in intermediate states. The lack of knowledge of f_1 is the largest uncertainty quoted in the EFT approach ($\pm 2.9\%$) [9,10]. The authors of Ref. [26] find that the constant f_1 in their picture is to 95% associated with the axial baryon form factor. The corresponding direct and crossed contribution with γn as the intermediate state has recently been calculated using soft pion techniques in the heavy baryon limit, which gives a 3.4(7)% attractive contribution [27]. Such attraction is also found in the chiral quark model of Ref. [26]. This follows generically from the predominance of intermediate states of energy higher than the threshold one. In Ref. [27] the leading term has the symmetry structure corresponding to that associated with the ChPT parameter f_1. The next order term proportional to $m_\pi \log m_\pi$ has the same coefficient as in the corresponding

term $\delta\mathcal{T}_3^{em}$ of Ref. [10], which indicates an equivalent physical expansion.

The main difference of our approach to the e. m. corrections of the EFT approach [9,10] is the wave function and energy shift corrections which are linear, respectively inverse, in the e. m. charge radius. In spite of their ready intuitive interpretation, we have been unable to identify explicit counterparts in the EFT approach in which only even powers $\langle r^{2n}\rangle_{em}$ appear, contrary to our result with a leading linear power. This difference occurs already at the non-relativistic level [28]. It is due to a different handling of the e. m. charge form factor, which is seen clearly in a configuration space representation. We modify the short range part of the Coulomb potential by the form factor and iterate its effect on the strong interaction. To present order in the expansion the authors of Refs. [9,10] consider the effect of the form factor to be equivalent to an additional short range effective Lagrangian term proportional to $\langle r_{em}^2\rangle$; the electromagnetic Lagrangian is otherwise taken to be independent of the form factor. This generates an e. m. contribution a_{fs} to the scattering length independent of the strong scattering, which is identical to our result to this order. However, the term in question is generated only by the fact that the charge distribution has a range. Contrary to the corresponding situation for a hadronic scattering length one cannot keep a_{fs} (see Eq. (9)) unchanged in a strict zero range limit, since it would vanish. In addition, the Bohr wave function varies linearly with r near the origin, contrary to the quadratic dependence for regular potentials. Both effects make it delicate to iterate a finite range e. m. interaction using an equivalent zero range Lagrangian. We conjecture that this difference is only technical. The EFT approach should generate our terms in higher orders, since form factors are obtained using such descriptions.

The π^-p scattering length is the dominant contribution in the direct determination of the πNN coupling constant $g_c^2/(4\pi)$ via the GMO sum-rule as given in Eq. (4) of Ref. [4]. Our result (42) for the scattering length, together with the reanalysis of the $\pi^- D$ scattering length as described above, allows an improved evaluation with a new value for $g_c^2/(4\pi) = 14.04\,(17)$, as compared to $14.11\,(19)$ found in Ref. [4]. As discussed above, non-potential terms from the γN ($\gamma\Delta$) channel contribute both to $a^h_{\pi^- p}$ and $a^h_{\pi^- d}$, but cancel to leading order in m_π in the combination required for the determination [4]. Additional contributions are within the systematic uncertainties.

5. Conclusions

The aim of the present paper was to extract the phenomenological threshold scattering amplitude to high precision and model independently from the corresponding strong interaction energy shift and width in a hadronic atom such as the π^-p atom. Our scattering length is differently defined with respect to the QCD one used in Refs. [9,10] as discussed above. We reach our goal within a non-relativistic picture by the following key observation. The Coulomb potential of the extended charge distribution is perfectly regular at short distances and it is useful to consider it to be an externally applied binding potential in addition to the hadronic interaction. This allows us to solve the problem exactly for a model with respect to which perturbations can be applied. In this situation the hadron masses are the physical ones and the extracted hadron amplitude does not assume isospin invariance. The regularizing effect of the extended charge is an essential feature for the understanding of the correction terms to the Deser-Trueman relation (3). It has previously been included only in a numerical study using potentials [5].

We show then that an accurate relation can only be achieved if three physical effects are properly included. First, the relevant wave function at the origin is not the Bohr one, but should correspond to an extended charge distribution including vacuum polarization. The extended charge distribution is at present important beyond the purely e. m. energy shift it produces in the atom. Second, the correct long-range behavior of the wave function induces a characteristic change of the wave function near the origin. The result of this feed-back is a quadratic correction to the scattering length proportional to $\alpha m\ (a^h)^2 \log\alpha$ to leading order. The leading term is known from

many investigations. Here we obtain a more accurate result including the terms of order α with little sensitivity to the model used. Thirdly, the low-energy expansion of the scattering amplitude leads to a characteristic 'gauge' correction which expresses that the scattering occurs at an energy typical of that of the extended charge Coulomb potential close to the origin [4,12]. This effect is as important as the other corrections. Therefore, approaches which do not respect the empirical low-energy expansion cannot extract accurate values for scattering lengths from atomic data.

The present investigation assumes that the charge distribution produces a Coulomb potential that varies little over the range of the strong interaction. The results of our model in Eq. (10) and following suggest that the results are nearly unchanged as long as the hadronic range is within that of the scale set by the charge distribution. Although we have not yet investigated these aspects, the results appear to be robust with compensations between contributions [25]. The sign and magnitude of these corrections are unlikely to change. Our considerations also apply to other hadronic systems, in particular to the $\pi^+\pi^-$ atom as will be discussed in a more detailed version [25].

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