HAMILTONIAN MONODROMY *VIA* PICARD-LEFSCHETZ THEORY

by

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Abstract. In this paper, we investigate the "Hamiltonian" monodromy of the fibration in Liouville tori of certain integrable systems *via* (real) algebraic geometry. Using Picard-Lefschetz theory in a relative Prym variety, we determine the Hamiltonian monodromy of the "geodesic flow on $\mathrm{SO}(4)$". Using a relative generalized Jacobian, we prove that the Hamiltonian monodromy of the spherical pendulum can also be obtained by Picard-Lefschetz formula.

Introduction

The aim of this paper is to relate the "Hamiltonian" monodromy of the fibration in Liouville tori of certain integrable systems with the monodromy of the fibration in (real) Abelian variety, *via* Picard-Lefschetz theory.

Consider an integrable system with n degrees of freedom. Assume the Hamiltonian is a proper map, so that, according to the Arnold-Liouville theorem (see [**2**]), the connected components of the regular level sets of the momentum mapping are n-dimensional compact tori, the Liouville tori. If the set of regular values of the momentum mapping is not simply connected, the fibration by the Liouville tori can have a non trivial monodromy.

Assume now that $n = 2$. When a singular point around which it is possible to turn is a "focus-focus" point (this is a nondegeneracy assumption), a general theorem of Nguyen Tien Zung [**17**] asserts that the Hamiltonian monodromy is non trivial.

This is, very classically, what happens for the spherical pendulum: the Hamiltonian monodromy phenomenon showed us precisely in this example investigated by Cushman to illustrate the possible non existence of global action-angle coordinates (see Duistermaat's foundational paper [**10**] and the book [**9**]). The case of the symmetric spinning top is completely similar.

Another example in which there is a component in the set of regular values which is not simply connected is the case of geodesic flows of invariant metrics on $\mathrm{SO}(4)$ (or "4-dimensional free rigid body") that I will investigate in this paper.

In these examples as in many others, the phase space is an algebraic submanifold of a space $\mathbf{R}^{2N}$ and the first integrals are polynomials. It is thus possible to complexify all the data. Very often (and in particular in the examples mentioned above) the differential system may be written as a Lax equation with a spectral parameter, the set of singular values of the momentum mapping is the discriminant locus of a family of affine plane curves and the fibration in Liouville tori appears (sometimes up to a covering map) as the real part of a fibration in Abelian varieties associated with this family.

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We will use this algebro-geometric framework to describe the "Hamiltonian" monodromy. Of course, there can be complex monodromy, turning around a branch of the discriminant hypersurface, but this is *not* what we are interested in: what we want is to turn around something real, *staying in the reals*. Thus we must turn around something having codimension 2, intersection of two branches of the (codimension-1!) discriminant: the singular spectral curves we will look at will have *two* double points.

Applying twice the Picard-Lefschetz formula and looking carefully at the real structures of the curves and their Jacobians, we will get the monodromy.

Here is a description of the results and organization of the paper.

- In § 1, I consider families of (real) algebraic curves and express, according to Picard-Lefschetz theory, the monodromy around a real point of the discriminant corresponding to two non real double points.
- In § 2, I concentrate on two specific families of elliptic curves. To get non trivial monodromy from the general considerations of § 1 in these examples, I will replace the relative Jacobian by another family of algebraic tori:
  - either the family of Pryms of a double branched cover of the elliptic curve
  - or the family of generalized Jacobians of a singular genus-2 curve (of which the original elliptic curve is the normalization).

  In both cases, I determine the monodromy of the fibration, for the family of complex tori and for its real part.
- In § 3, I come to integrable systems. The two families of curves considered abstractly in § 2 turn out to be the families of spectral curves for Lax equations describing respectively
  - the geodesic flow on $\mathrm{SO}(4)$
  - the spherical pendulum.

  Using the eigenvectors of the Lax matrices, I prove that the fibrations in algebraic tori of § 2 are models for the fibrations in Liouville tori and deduce the Hamiltonian monodromy from the algebro-geometric results. As far as I know, the explicit determination of the monodromy for the $\mathrm{SO}(4)$ case is new[(1)]. In the case of the spherical pendulum[(2)], I recover Cushman's classical results. As a byproduct, the Picard-Lefschetz formula shows the Hamiltonian monodromy as given by integrating a (non holomorphic) meromorphic differential form. This phenomenon was observed, using a direct computation, in [**8**].
- In § 4, I recall a few facts I use on real generalized Jacobians and in § 5, I have grouped a few computations used or mentioned in the paper.

1. General setting

Consider an n-dimensional family of affine (real) plane curves. Assume that two smooth local branches B_1 and B_2 of the discriminant intersect transversally in such a way that

$$B_1 \cap \mathbf{R}^n = B_2 \cap \mathbf{R}^n = (B_1 \cap B_2) \cap \mathbf{R}^n$$

is a codimension-2 submanifold of $\mathbf{R}^n$.

To be more specific, we will consider two families of elliptic curves.

Example 1.1. Consider the curves $E_{h,k}$ of equations

$$A(x)y^2 + B_{h,k}(x)y + C(x) = 0$$

(1) It would be a consequence of the results of [**17**] if the number of critical points and their non degeneracy were determined, but explicit computations are rather tedious...

(2) In this case, the connected components of the real part of the Jacobian are indeed compact 2-tori although the Jacobian itself is non compact. This is something we should expect, since one of the first integrals is a periodic Hamiltonian, inducing an S^1-action, which complexifies into a $\mathbf{C}^\star$-action.

where A, $B_{h,k}$ and C are real polynomials, the second depending on two real parameters h and k. Assume they are such that

$$\Delta_{h,k}(x) = B_{h,k}(x)^2 - 4A(x)C(x)$$

is a degree-4 polynomial. Let (h_0, k_0) be such that

$$\Delta_{h_0,k_0}(x) = (ax^2 + bx + c)^2$$

for some real numbers a, b and c such that $b^2 - 4ac < 0$. Then (h_0, k_0) is a point in $\mathbf{R}^2$ at which two non real branches of the discriminant of the family of curves intersect.

Example 1.2. Consider the curves $C_{h,k}$ of equations

$$y^2 + P_{h,k}(x) = 0$$

where $P_{h,k}$ is a monic degree-4 polynomial, real when the parameters h, k are real. Assume $(h_0, k_0) \in \mathbf{R}^2$ is such that

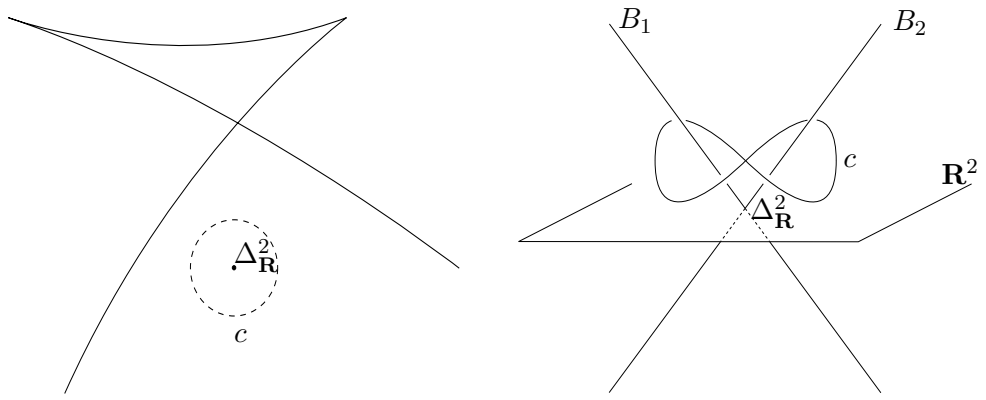
$$P_{h_0,k_0}(x) = (x^2 + ax + b)^2 \text{ for some real numbers } a, b \text{ satisfying } a^2 - 4b < 0.$$

Here again, (h_0, k_0) is a point in $\mathbf{R}^2$ at which two non real branches of the discriminant of the family of curves intersect.

Remark 1.3. Except for the way we have chosen to write the affine curves, the only difference between these two examples is that, for (h, k) close to (h_0, k_0), the curve $E_{h,k}$ has a non empty real part (with two connected components) while $C_{h,k}$ has no real point.

Back to the general situation, call $\Delta^2 = B_1 \cap B_2$ and $\Delta^2_{\mathbf{R}}$ its real part. Let U be a neighbourhood of Δ^2 in $\mathbf{C}^n$ and $U_{\mathbf{R}}$ its real part. Taking transversal linear subspaces $\mathbf{C}^2$ and $\mathbf{R}^2$, we see that the fundamental group of the complement of $\Delta^2_{\mathbf{R}}$ in $\mathbf{R}^n$ is an infinite cyclic group while $\pi_1(U - \Delta^2)$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$. More precisely, we have

Lemma 1.4. *The natural map $\pi_1(U_{\mathbf{R}} - \Delta^2_{\mathbf{R}}) \to \pi_1(U - \Delta^2) \cong \mathbf{Z} \oplus \mathbf{Z}$ maps a generator to $(1, -1)$.*



(a) The real part of the discriminant (b) The loop c

FIGURE 1. The local fundamental group

Proof. Having taken 2-dimensional transversals, we can assume that $n = 2$. Then the two branches can be locally described by an equation

$$P(x, y)\bar{P}(x, y) = 0$$

for some complex linear form $P : \mathbf{C}^2 \to \mathbf{C}$. An explicit isomorphism from the fundamental group $\pi_1(\mathbf{C}^2 - \Delta^2)$ to $\mathbf{Z} \times \mathbf{Z}$ is given by

$$\gamma \longmapsto \left(\frac{1}{2i\pi} \int_\gamma \frac{dP}{P}, \frac{1}{2i\pi} \int_\gamma \frac{d\bar{P}}{\bar{P}} \right).$$

Our assumption is that $P(x,y)\bar{P}(x,y) = 0$ at $(x,y) \in \mathbf{R}^2$ if and only if $(x,y) = (0,0)$, namely that $|P(x,y)|^2 = \varepsilon$ is the equation of an ellipse in $\mathbf{R}^2$. Let c be a generator of $\pi_1(\mathbf{R}^2 - \{(0,0)\})$ (see Figure 1) and evaluate

$$\frac{1}{2i\pi} \left(\int_c \frac{dP}{P} + \int_c \frac{d\bar{P}}{\bar{P}} \right) = \frac{1}{2i\pi} \int_c \frac{d\,|P|^2}{|P|^2} = 0$$

as one can choose a parametrization of $|P|^2 = \varepsilon$ for c. Also

$$\frac{1}{2i\pi} \left(\int_c \frac{dP}{P} - \int_c \frac{d\bar{P}}{\bar{P}} \right) = \frac{1}{2i\pi} \int_c \frac{\bar{P}dP - Pd\bar{P}}{|P|^2} = \pm 2$$

according to the orientation of c. Thus $\pm c$ is, indeed, mapped to $(1,-1) \in \mathbf{Z} \times \mathbf{Z}$. □

Remark 1.5. Notice that a complex transversal to a branch of the complex discriminant has a natural orientation, but this is not the case for an $\mathbf{R}^2$ transversal to $\Delta^2_{\mathbf{R}}$.

In the family of curves, those corresponding to parameters in Δ^2 have two double points. Choose a smooth curve nearby and call δ, $\bar{\delta}$ the vanishing cycles corresponding to the two double points. The Picard-Lefschetz formula [**15**] gives, for the monodromy along the real loop c

$$\gamma \longmapsto \gamma + \varepsilon((\gamma \cdot \delta)\delta - (\gamma \cdot \bar{\delta})\bar{\delta}) \qquad \varepsilon = \pm 1 \text{ according to the orientation of } c \tag{1}$$

(this applies to cycles on the complex curve).

2. Elliptic curves with two double points

In Examples 1.2 and 1.1 above, the curves have genus 1. On a punctured neighbourhood of (h_0, k_0), the two vanishing cycles corresponding to the two double points at (h_0, k_0) are in the same homology class, so that formula (1) gives nothing in the homology of the curve, which is too small. The problem is that (the Jacobian of) the curve has complex dimension 1.

We are going, for each of these examples, to construct a fibration of complex tori of dimension 2 for which the Picard-Lefschetz formula (1) will indeed give a non trivial monodromy. We use different approaches in these different examples.

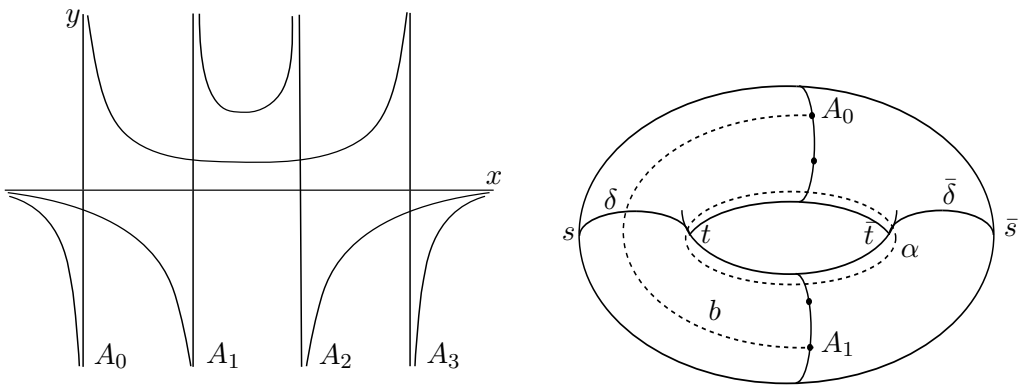
2.1. Double coverings of elliptic curves—Prym varieties. Let us start with elliptic curves[(3)] $E = E_{h,k}$ as in Example 1.1. For (h,k) in a neighbourhood U of (h_0,k_0), the real part of E has two connected components. We will choose four real points A_i (for $0 \leq i \leq 3$), two on each component, and look at the double cover of E branched at these points. To be more specific in the way these points depend on the parameters, I will assume that, in the equation of E

$$A(x)y^2 + B_{h,k}(x)y + C(x) = 0,$$

C is a constant and the polynomial A has four distinct real roots $a_0 < a_1 < a_2 < a_3$, so that $A_i = (a_i, \infty)$ is indeed a real point of E and the A_i's are suitably divided on the real part, A_0 and A_3 on a component, A_1 and A_2 on the other (see the real part of the affine curve $E_{h,k}$, for (h,k) in U, on the left part of Figure 2).

Look at the covering map $x : E \to \mathbf{P}^1$. The branch points are the four roots of $\Delta_{h,k}(x)$. As $\Delta_{h_0,k_0}(x) = (ax^2+bx+c)^2$ with $b^2 - 4ac < 0$, for the nearby value (h,k), the polynomial $\Delta_{h,k}$ has non real distinct roots. Call s, t the two roots in the upper half plane, $\bar{s}$ and $\bar{t}$ their conjugates. The right part of Figure 2 shows the complex curve $E_{h,k}$ with its real part, the vanishing cycles δ and $\bar{\delta}$ (which are, indeed, conjugate to each other).

(3) In this §, I will use the same symbol $E = E_{h,k}$ for the affine curve and for its non singular completion.



(a) The real part of E

(b) The complex curve

FIGURE 2. The curve E

The heavy points are the four points A_i where $y = \infty$. Notice that $\delta \cup \bar{\delta}$ separates the curve in two connected components, each of which contains two of the points A_i's.

Let us now look at the curve $D_{h,k}$ (non singular completion of the curve) of equation

$$z^4 A(x) + z^2 B_{h,k}(x) + C = 0.$$

This is a real curve, endowed with an involution $\tau(x, z) = (x, -z)$ which preserves the real structure and shows $D_{h,k}$ as a double cover of $E_{h,k}$ (by $y = z^2$) branched at the points A_i's. Hence, $D_{h,k}$ is a genus-3 curve, smooth exactly when $E_{h,k}$ is smooth (recall that the points A_i are assumed to be distinct).

The projection $\pi : D \to E$ induces an injective map

$$\pi^\star : H^0(\Omega^1_E) \longrightarrow H^0(\Omega^1_D)$$

the image of which is the vector space of holomorphic forms η on D such that $\tau^\star \eta = \eta$. Dualizing and factoring the period lattice, this defines a surjective morphism

$$\mathrm{Jac}(D) \longrightarrow \mathrm{Jac}(E)$$

the kernel of which is an Abelian dimension-2 subvariety of $\mathrm{Jac}(D)$, the Prym variety $\mathrm{Prym}(D|E)$. The relative Prym is a fibration of the complement of (h_0, k_0) in complex 2-dimensional tori. This is the fibration for which we are going to compute the monodromy.

What we need to compute is the monodromy of the lattice of anti-invariant (with respect to $\tau_\star$ elements of $H_1(D; \mathbf{Z})$. Let us use our description of D and E to define a basis of this group.

We take two copies of E and identify the corresponding points A_i in both. On the i-th copy of E, we have the curves α_i, δ_i and $\bar{\delta}_i$ corresponding to α, δ and $\bar{\delta}$. Call β a curve on D obtained in the following way:

- go from A_0 to A_1 on the path b of Figure 2 on the first copy of E, meeting δ_1 once transversally with intersection number 1
- then go back from A_1 to A_0 on the second copy of E, meeting $\bar{\delta}_2$ once transversally, with intersection number -1 (see Figure 3).

Then α_1, α_2, β, δ_1, $\bar{\delta}_1$ and $\bar{\delta}_2$ form a basis of $H_1(D; \mathbf{Z})$.

Notice that we have succeeded in the sense that $\delta_i \neq \bar{\delta}_i$ in $H_1(D; \mathbf{Z})$. In this group, we also have $\delta_1 + \delta_2 = \bar{\delta}_1 + \bar{\delta}_2$ so that $\delta_2 = \bar{\delta}_2 - (\delta_1 - \bar{\delta}_1)$. The orientations have been chosen so that $\alpha_i \cdot \delta_i = \alpha_i \cdot \bar{\delta}_i = 1$ and $\beta \cdot \bar{\delta}_1 = 1$, $\beta \cdot \bar{\delta}_2 = -1$.

The real structure. Notice firstly that the real part of D has two connected components (as the branch points are real and two on each component of the real part of E). The effect of the real involution S on the cycles

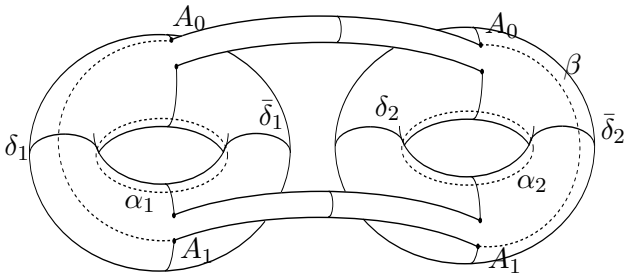


FIGURE 3. The curve D

defined above is transparent, as S comes from the real structure on E. We have

$$S(\alpha_i) = -\alpha_i, \qquad S(\delta_i) = \bar\delta_i.$$

Performing the complex conjugation on the path b, we also get

$$S(\beta) = \beta - \alpha_1 + \alpha_2.$$

The involution τ. Its effect is also transparent: the cycles with a label 1, 2 are sent to the same cycle with the other label, and $\tau(\beta) = -\beta$, so that the anti-invariant cycles are generated by

$$\alpha_1 - \alpha_2, \quad \beta, \quad \delta_1 - \bar\delta_1, \quad \bar\delta_1 - \bar\delta_2.$$

The real cycles that are anti-invariant are thus generated by

$$\alpha_1 - \alpha_2 - 2\beta \quad \text{and} \quad \delta_1 - \bar\delta_2.$$

Monodromy. Look now at the monodromy of the relative Prym. Formula (1) gives

$$\gamma \longmapsto \gamma + \varepsilon((\gamma\cdot\delta_1)\delta_1 + (\gamma\cdot\delta_2)\delta_2 - (\gamma\cdot\bar\delta_1)\bar\delta_1 - (\gamma\cdot\bar\delta_2)\bar\delta_2),$$

namely

$$\alpha_i \longmapsto \alpha_i + \varepsilon(\delta_i - \bar\delta_i), \qquad \beta \longmapsto \beta + \varepsilon(-\bar\delta_1 + \bar\delta_2)$$

and the others fixed.

If we restrict our attention to the real part, we find

$$\alpha_1 - \alpha_2 - 2\beta \longmapsto \alpha_1 - \alpha_2 - 2\beta + 2\varepsilon(\delta_1 - \bar\delta_2).$$

Eventually, we have proved:

Theorem 2.1. *The relative Prym on the complement of the discriminant in $U_{\mathbf{C}}$ is a fibration in complex 2-dimensional tori with monodromy matrix conjugated to*

$$\left(\begin{array}{ccc|c} & \mathrm{Id} & & 0 \\ \hline \varepsilon & -\varepsilon & 0 & \\ -\varepsilon & \varepsilon & -\varepsilon & \mathrm{Id} \\ 0 & 0 & \varepsilon & \end{array}\right).$$

The real part is a fibration in real tori of the complement of (h_0, k_0) in $U_{\mathbf{R}}$ with monodromy matrix conjugated to $\begin{pmatrix} 1 & 0 \\ 2\varepsilon & 1 \end{pmatrix}$. □

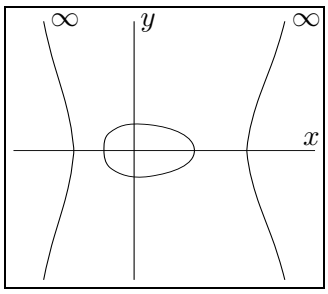


FIGURE 4. The affine curve C

2.2. Elliptic vs singular genus-2 curves—generalized Jacobians. Let us now investigate the example of the family $y^2 + P_{h,k}(x) = 0$ (Example 1.2). The polynomial has degree 4 so that the curve $C_{h,k}$ (or C) is the complement of two points in an elliptic curve, with only one point at infinity, which is a singular point (Figure 4, note that this picture is not real).

Call $\bar{C}_{h,k}$ (or $\bar{C}$) the complete singular curve obtained by adding this point at infinity to the affine curve and $\widetilde{C}_{h,k}$ (or $\widetilde{C}$) the normalized curve, that has two points $\infty_\pm$ at infinity. In other words, C can be completed in two different ways, and I will use both:

- Add a unique point at infinity, completing the affine curve in the completion $\mathbf{C}^2 \subset \mathbf{P}^2(\mathbf{C})$. In homogeneous coordinates $[x, y, Z]$, this is the point $\infty = [0, 1, 0]$. This is the completion $\bar{C}$.
- Normalize at infinity, that is, define a complete curve $\widetilde{C}$ by separating the two branches. Note that $\widetilde{C}$ is smooth at infinity (by definition) and has a degree-2 function to $\mathbf{P}^1(\mathbf{C})$ (extending x), with

$$x^{-1}(\infty) = \{\infty_+, \infty_-\}\,.$$

  The curve $\widetilde{C}$ is a double cover of $\mathbf{P}^1(\mathbf{C})$ branched at the four roots of the polynomial $P_{H,K}$ and thus, when these roots are distinct, this is a genus-1 curve.

  The simplest way to construct $\widetilde{C}$ is to complete the affine curve, not in $\mathbf{P}^2(\mathbf{C})$, but in the total space of the bundle $\mathcal{O}(2) \to \mathbf{P}^1(\mathbf{C})$. The affine coordinate x becomes an element of $\mathbf{P}^1(\mathbf{C})$, the fibration $(x, y) \mapsto x$ from $\mathbf{C}^2$ to $\mathbf{C}$ becomes the fibration $\mathcal{O}(2) \to \mathbf{P}^1(\mathbf{C})$ defined by gluing $\big(\mathbf{P}^1 - \{\infty\}\big) \times \mathbf{C}$ and $\big(\mathbf{P}^1 - \{0\}\big) \times \mathbf{C}$ by the map

$$(x, y) \longmapsto \left(\frac{1}{x}, \frac{y}{x^2}\right).$$

  The complete curve $\widetilde{C}$ is naturally embedded in this surface, the two points at infinity are the points $x = \infty$, $\dfrac{y}{x^2} = \pm i$.

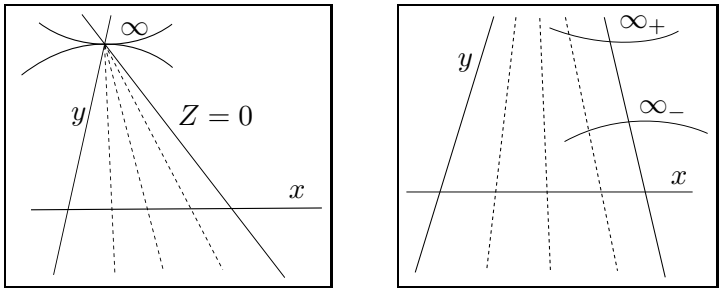
Figure 5 represents the two completions of C, in a neighborhood of ∞. The doted lines represent the projection onto the x-axis (projective line). Notice that $\widetilde{C}$ is smooth if and only if C is.

Notice that, as we plan to use the Jacobians of the curves in the family (the relative Jacobian) as a model for the Liouville tori in a 2-degrees of freedom system, we should prefer to use genus-2 curves. The singular curve $\bar{C}$ has, indeed, genus 2. When $\widetilde{C}$ is smooth, $\bar{C}$ has a generalized Jacobian[(4)] which is the extension

$$1 \longrightarrow \mathbf{C}^\star \longrightarrow \mathrm{Jac}(\bar{C}) \longrightarrow \mathrm{Jac}(\widetilde{C}) \longrightarrow 0$$

of the Jacobian of the elliptic curve $\widetilde{C}$ (the quotient of $\mathbf{C}$ by a lattice Λ) by a multiplicative group $\mathbf{C}^\star$. Recall that the period lattice Λ of the elliptic curve $\widetilde{C}$ is given by the integration of the holomorphic form dx/y on the cycles in $\widetilde{C}$. In the same way, the non compact subgroup $\mathbf{C}^\star$ created by the identification of ∞_- and ∞_+ into

(4) See § 4 and the references given there.



(a) The singular $\bar{C} \subset \mathbf{P}^2(\mathbf{C})$

(b) The normalized $\widetilde{C}$

FIGURE 5. The complete curves

a single point $\infty \in \bar{C}$ can be materialized by the integration of the *meromorphic* 1-form xdx/y on a small loop γ around ∞.

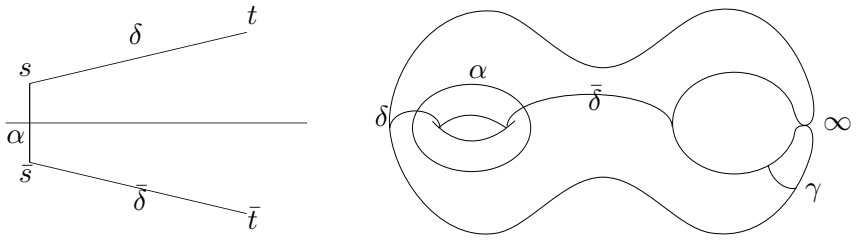
Look now at the real structure induced by that of C on the Jacobians. The singular affine curve C_{h_0,k_0} has equation $y^2 + (x^2 + ax + b)^2 = 0$ (with $a^2 - 4b < 0$), so that it has no real point, and the same is true for the smooth nearby curves. Then it is known that $\mathrm{Jac}(\widetilde{C}) = \mathrm{Pic}^0(\widetilde{C})$ (we must be specific with the degrees, or at least with their parity, here) is a real hyperelliptic curve whose real part has two connected components. On the other hand, the map $\mathbf{C}^\star \to \mathrm{Pic}(\bar{C})$ associates to the non zero complex number z the divisor of any function f on C such that $\dfrac{f(\infty_+)}{f(\infty_-)} = z$. Let us look at the real structure induced on $\mathbf{C}^\star$:

$$\frac{f(\infty_+)}{f(\infty_-)} = z \longmapsto \frac{\overline{f(\overline{\infty}_+)}}{\overline{f(\overline{\infty}_-)}} = \frac{\overline{f(\infty_-)}}{\overline{f(\infty_+)}} = \frac{1}{\bar{z}}$$

so that the real points are, indeed, the points of the unit circle in $\mathbf{C}^\star$.

Hence we see that the real part of the (non compact) Jacobian of the curve $\bar{C}$ is a compact 2-torus.

Let us now be more specific on the curves which generate the homology of this torus. Fix a neighbourhood $U_{\mathbf{R}}$ of the critical point (h_0, k_0) such that (h_0, k_0) is the unique critical point in $U_{\mathbf{R}}$. Choose a regular point (h, k) in $U_{\mathbf{R}}$. The polynomial has two pairs of conjugate (non real) roots. Call as above s, t, the two roots in the upper half plane and let δ be a cycle in $C = C_{h,k}$ above the segment $[s, t]$, let $\bar{\delta}$ be the conjugated cycle and let α be a cycle above the segment $[s, \bar{s}]$. Choose orientations so that $\alpha \cdot \delta = 1$.



(a) The roots of P

(b) The cycles on $\bar{C}$

FIGURE 6. The curve $\bar{C}$

Notice that the two double points of C_{k_0,k_0} are obtained when s and t come together, so that δ and $\bar{\delta}$ are indeed the vanishing cycles considered above.

The complex curve $\bar{C}$ is represented on Figure 6 with the cycles α, δ and $\bar{\delta}$, which we will use as a basis of a suitable version of its first homology group, namely of the group $H_1(C;\mathbf{Z})$.

Consider now the real structure S defined by complex conjugation of coordinates on $\widetilde{C}$. The holomorphic 1-form dx/y is real in the sense that $S^\star(dx/y) = dx/y$. Also, $S(\delta) \sim \delta$ and $S(\alpha) \sim -\alpha$. But the real structure induced by S on the Jacobian is a little more tricky, as $\widetilde{C}$ has no real point while $\mathrm{Pic}^0(\widetilde{C})_{\mathbf{R}}$ has two connected components. It turns out that the Abel-Jacobi map "defined" on divisors on $\widetilde{C}$ by

$$P - Q \longmapsto i\int_P^Q \frac{dx}{y}$$

defines an isomorphism of real curves

$$\left(\mathrm{Pic}^0(\widetilde{C}), S\right) \longrightarrow (\mathbf{C}/\Lambda, z \mapsto -\bar{z})$$

(see *e.g.* § 5.3 for a concrete example). The period lattice Λ is generated by

$$i\int_\delta \frac{dx}{y} \text{ and } i\int_\alpha \frac{dx}{y}$$

so that the real cycle is α.

Finally, notice that in $H_1(\widetilde{C},\mathbf{Z})$, $\delta = \bar{\delta}$ but that this is not the case in $H_1(\bar{C};\mathbf{Z})$, as $\delta - \bar{\delta}$ is represented by a small loop γ around ∞ and we have

$$\int_{\delta-\bar{\delta}} \frac{xdx}{y} = \int_\gamma \frac{xdx}{y} = \pm\int_\gamma \frac{dx}{ix} = \pm 2\pi \neq 0.$$

We have proved:

Proposition 2.2. *The relative Jacobian of the family of curves on the complement of the discriminant in $U_{\mathbf{C}}$ is a fibration in complex tori*

$$1 \longrightarrow \mathbf{C}^\star \longrightarrow \mathrm{Pic}^0(\bar{C}) \longrightarrow \mathrm{Pic}^0(\widetilde{C}) \longrightarrow 0$$

with real structure such that the connected components of the real part of the fiber on the complement of (h_0,k_0) in $U_{\mathbf{R}}$ are fibrations in compact real tori

$$1 \longrightarrow S^1 \longrightarrow T^2 \longrightarrow \mathbf{R}/T\mathbf{Z} \longrightarrow 0.$$

The homology of the real torus T^2 is generated by two elements $\widetilde{\alpha}$, γ, such that

- *the image of $\widetilde{\alpha}$ in $H_1(\mathbf{R}/T\mathbf{Z};\mathbf{Z})$ is the generator α coming from $H_1(C;\mathbf{Z})$ and satisfying $\displaystyle\int_\alpha \frac{dx}{y} = iT$*
- *γ is the image of the generator of $H_1(S^1;\mathbf{Z})$ coming from $H_1(C;\mathbf{Z})$, that is, the difference $\delta - \bar{\delta}$ of the two vanishing cycles and satisfying $\displaystyle\int_\gamma \frac{xdx}{y} = \pm 2\pi$.* □

Now, apply the Picard-Lefschetz formula (1) to get:

Corollary 2.3. *The monodromy of the fibration in tori over $U_{\mathbf{R}} - \{(h_0,k_0)\}$ is given by*

$$\widetilde{\alpha} \longmapsto \widetilde{\alpha} + \varepsilon\gamma \text{ and } \gamma \longmapsto \gamma.$$

□

Remark 2.4. Notice that the fact that there is, indeed, real monodromy in this situation reflects the fact that the complex monodromy respects the real structure on the Jacobian. This is certainly not the case turning around a single branch of the discriminant.

3. Hamiltonian monodromy

The two families of curves $C_{h,k}$ and $D_{h,k}$ used so far are the families of spectral curves for certain Lax equations. We will apply the previous study to the determination of the Hamiltonian monodromy of the corresponding integrable systems.

Both are integrable systems with two degrees of freedom. The regular levels are (unions of) 2-dimensional tori. In both cases, the set of regular values of the momentum mapping contains a punctured disk (see Figures 7 and 8).

According to a general theorem of Nguyen Tien Zung [**17**], if the singular point inside this disk is a "focus-focus" point, the fibration by Liouville tori should have non trivial monodromy. I will use the previous results

- to prove that this is, indeed, the case for the geodesic flow on $\mathrm{SO}(4)$, using Pryms
- and to prove that the well known Hamiltonian monodromy [**10, 9**] of the spherical pendulum, usually determined by direct computation, is indeed given by the Picard-Lefschetz formula as above, using generalized Jacobians.

3.1. Hamiltonian monodromy for geodesic flows on $\mathrm{SO}(4)$. The geodesic flow of an invariant metric on $\mathrm{SO}(n)$ gives rise to the differential system (Euler-Arnold equations [**2**])

$$\dot{M} = [M, \Omega]$$

where M and Ω are skew-symmetric matrices related by

$$M = \Omega J + J\Omega$$

for some diagonal constant matrix J. It also describes the motion of a free rigid body in a three dimensional space when M and Ω are 3×3-matrices. For this reason, the case of 4×4-matrices, that we will study now, is sometimes called the "4-dimensional rigid body".

A Lax equation. The Euler-Arnold equations have been put in Lax form by Manakov [**16**]:

$$\frac{d}{dt}(M + J^2\lambda) = [M + J^2\lambda, \Omega + J\lambda].$$

Call b_i ($0 \leq i \leq 3$) the diagonal entries of the matrix J and assume that they are distinct, satisfy $b_0^2 < b_1^2 < b_2^2 < b_3^2$. Without loss of generality, assume $b_0 = 0$ and write $a_i = b_i^2$. As is usual, write the skew-symmetric matrix M as

$$M = (x, y) = \begin{pmatrix} 0 & -x_3 & x_2 & y_1 \\ x_3 & 0 & -x_1 & y_2 \\ -x_2 & x_1 & 0 & y_3 \\ -y_1 & -y_2 & -y_3 & 0 \end{pmatrix}.$$

The spectral curve, given by the characteristic polynomial $\det\left(M + \lambda J^2 - \lambda\mu\,\mathrm{Id}\right)$ of the Lax matrix $A_\lambda = M + J^2\lambda$ is

$$\lambda^4\mu\prod_{i=1}^{3}(\mu - a_i) + \lambda^2\left(\mu^2 f_1(x,y) - \mu H(x,y) + K(x,y)\right) + f_2(x,y)^2$$

where

$$\begin{aligned} f_1(x,y) &= \sum x_i^2 + \sum y_i^2 \text{ is the norm of } M, \\ f_2(x,y) &= \sum x_i y_i \text{ is its Pfaffian,} \\ H(x,y) &= \sum_{i=1}^{3} a_i x_i^2 + \frac{1}{2}\sum_{\{i,j,k\}=\{1,2,3\}} (a_i + a_j) y_k^2, \\ K(x,y) &= \frac{1}{2}\sum_{\{i,j,k\}=\{1,2,3\}} a_i a_j y_k^2. \end{aligned}$$

We fix $f_1 > 0$ and f_2 such that $|2f_2| < f_1$. Then the set of skew-symmetric matrices $M = (x, y)$ that have norm f_1 and Pfaffian f_2 is a coadjoint orbit $\mathcal{O}_f$ of $\mathrm{SO}(4)$ diffeomorphic to $S^2 \times S^2$. The two functions H and K are then in involution on this symplectic manifold, the Euler-Arnold equations being the Hamiltonian system associated with H.

The algebro-geometric features of this integrable system were investigated in the classical and beautiful paper of Haine [**14**] and some of its topological aspects were described by Oshemkov [**18**].

As in § 2.1, call E the elliptic curve, quotient of D by the involution $\tau(\mu, \lambda) = (\mu, -\lambda)$. The spectral curve D and the curve E are of the type considered in § 2.1. The points A_i are indeed the points (a_i, ∞) (with $a_0 = 0$). The parameters in the equations of E and D are all real and satisfy:

- The orbit invariants f_1 and f_2 are two fixed real numbers such that $f_2 \neq 0$, $f_1^2 - 4f_2^2 = 4\beta^2 > 0$.
- The values h, k of H and K are positive parameters.

Then, it can be shown (see § 5.1 for a justification) that, for β in a suitable interval $(\varepsilon_1, \varepsilon_2)$, the point

$$(h_0, k_0) = \left(\frac{1}{2}(f_1 + 2\beta)(a_1 + a_2) + \frac{1}{2}(f_1 - 2\beta)a_3\right)$$

is a point at which

$$\Delta_{h_0,k_0}(x) = (ax^2 + bx + c)^2 \text{ with } b^2 - 4ac < 0,$$

thus two non real branches of the discriminant intersect at this point.

Eigenvector mapping. We consider now the eigenvectors of the Lax matrix

$$A_\lambda = M + J^2\lambda.$$

This will give us a very precise dictionary between the topology and the algebraic geometry. Fix a value (h, k) of (H, K). This fixes a curve $D = D_{h,k}$. Consider a point $(\lambda, \mu) \in D$. The eigenvectors of the Lax matrix for the eigenvalue $\lambda\mu$ form a line bundle on the complement of the branch points of λ, sub-bundle of the trivial bundle $D \times \mathbf{C}^4$. When the curve D is smooth, there is a unique way to extend this line bundle to the whole of D as a sub-line bundle of $D \times \mathbf{C}^4$ (an easy exercise, see [**13**]). Thus, we have a line bundle, the eigenvector bundle L, on the complete curve D.

Let $\mathcal{T}_D$ be the common level set of the first integrals H and K corresponding to the curve D. This is a subset of the phase space $S^2 \times S^2$. Letting the point (x, y) vary on the level $\mathcal{T}_D$, we get a map (recall we assume D to be smooth):

$$\varphi_D : \mathcal{T}_D \longrightarrow \mathrm{Pic}(D)$$

where the group $\mathrm{Pic}(D)$ of isomorphism classes of line bundles over D is identified with the set of linear equivalence classes of divisors on D.

Remark 3.1. Eigenvector mappings turn out to be very useful in the investigation of algebraically integrable systems as they allow to understand the geometry of the system with the help of the algebraic geometry of a curve: determination of Liouville tori[(5)], determination of the regular levels[(6)], construction of action coordinates[(7)]. Here we will use it to get information on the monodromy.

Proposition 3.2. *Assume the elliptic curve E is smooth. Then the tangent mapping to φ_D maps the Hamiltonian vector fields of H and K to independent vectors in $H^1(\mathcal{O}_D) = T.\,\mathrm{Pic}(D)$.*

(5) See [**7**] for the Kowalevski top.

(6) See [**3**] for the case of geodesics of quadrics, Corollary 3.4 for the geodesic flow on $\mathrm{SO}(4)$ and Corollaries 3.10 and 3.13 for the spherical pendulum.

(7) See [**5, 6**].

Proof. Assume E is smooth. Then, D is smooth. We use the covering

$$D = \mathcal{U}_+ \cup \mathcal{U}_-$$

for $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) is the open set where $\lambda \neq \infty$ (resp. $y \neq 0$). The cohomology group $H^1(\mathcal{O}_D)$ is isomorphic with the first cohomology group of this covering. We are going to express the images of the Hamiltonian vector fields in $H^1(\mathcal{O}_D$ as cocycles of this covering.

Notice that the Hamiltonian vector fields X_H and X_K generate the same subspace in $T\mathcal{O}_f$ as the Hamiltonian vector fields of the functions

$$H_3(M) = 3\,\mathrm{tr}(J^2M) \text{ and } H_4(M) = 6\,\mathrm{tr}(J^4M).$$

Now these two functions have the form

$$H_k(M) = \mathrm{Res}\left(\lambda\left(\lambda^{-1}(M + J^2\lambda)\right)^k d\lambda\right)$$

for $k = 3,\ 4$. Using the standard linearization theorem of [**1**] and [**19**], it is seen that the corresponding Hamiltonian vector fields are mapped to the cocycles

$$\lambda\left(\lambda^{-1}(\lambda\mu)\right)^k = \lambda\mu^k, \quad k = 3, 4$$

(recall our eigenvalue is $\lambda\mu$). It is proved that these two cocycles are independent by a residue computation (see § 5.2). □

Remark 3.3. Notice that these two cocycles are anti-invariant (with respect to τ) so that the eigenvector mapping maps $\mathcal{T}_D$ to a subvariety of $\mathrm{Pic}(D)$ parallel to $\mathrm{Prym}(D|E)$.

Notice that this Proposition implies that the eigenvector mapping is a covering of its image, a result of Haine that we will state more precisely below.

Corollary 3.4. *If (h, k) is such that the elliptic curve E is smooth, then this is a regular value of the momentum mapping.*

Proof. If the images of the Hamiltonian vector fields X_H and X_K are independent, the two vectors themselves are independent. □

A complete picture of the discriminant of this family of curves can be found in Chapter IV of [**4**]. We will only need the case, considered in Example 1.1, where $\varepsilon_1 < \beta < \varepsilon_2$ (see § 5.1). In this case, the discriminant has the form depicted on Figure 7, which, thanks to our Corollary 3.4, is coherent with Oshemkov's pictures [**18**].

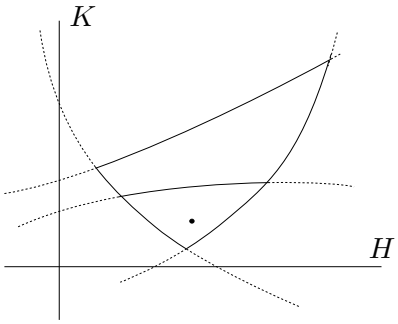


FIGURE 7. The regular values

Proposition 3.5 **([14]).** *If E is smooth, the eigenvector mapping*

$$\varphi_D : \mathcal{T}_D \longrightarrow \mathrm{Pic}(D)$$

is a covering of degree 4 of its image. This image is isomorphic to an open subset of $\mathrm{Prym}(D|E)$.

Proof. We have already mentioned that φ_D was a covering map and that its image was a subvariety parallel to $\mathrm{Prym}(C|E)$. The pairs (x, y) such that all the corresponding $M + J^2\lambda$ belong to the same conjugacy class modulo $\mathrm{GL}(4;\mathbf{C})$ (and are in the same $\mathrm{SO}(4)$-orbit) are all the

$$\begin{pmatrix} \varepsilon x_1 \\ \eta x_2 \\ \varepsilon\eta x_3 \end{pmatrix}, \begin{pmatrix} \varepsilon y_1 \\ \eta y_2 \\ \varepsilon\eta y_3 \end{pmatrix} \text{ with } \varepsilon^2 = \eta^2 = 1.$$

□

Hamiltonian monodromy. We now deduce the Hamiltonian monodromy from the algebro-geometric description in Theorem 2.1.

Theorem 3.6. *Assume $\varepsilon_1 < \beta < \varepsilon_2$. Then the set of regular values of the momentum mapping (H, K) for the geodesic flow on $\mathrm{SO}(4)$ has a connected component which is not simply connected. The monodromy of the fibration in Liouville tori along a loop generating the fundamental group is given by the matrix $\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$.*

Proof. We relate the relative Prym and the fibration in Liouville tori by the eigenvector mapping. According to Oshemkov [**18**], the regular levels close to the singular point (h_0, k_0) have two connected components. We concentrate on the monodromy of the fibration for one of them. According to Haine, the eigenvector mapping is a 4-fold covering map. This is of course a real map, so that it maps our Liouville tori to the real part of the Prym variety by a degree 2 covering map. In each fiber, we thus have a map $T^2 \to T^2$ inducing, at the level of fundamental groups

$$\begin{array}{ccc} \mathbf{Z}^2 & \longrightarrow & \mathbf{Z}^2 \\ (a, b) & \longmapsto & (a, 2b). \end{array}$$

The only possibility to have monodromy $\begin{pmatrix} 1 & 0 \\ 2\varepsilon & 1 \end{pmatrix}$ on the right is to have monodromy $\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ on the left. □

3.2. Hamiltonian monodromy for the spherical pendulum. Both the Hamiltonian systems for the spinning top and the spherical pendulum can be described by Lax equations whose spectral curves are of the type considered in Example 1.2 and § 2.2. To be more specific, I concentrate here on the case of the spherical pendulum.

The spherical pendulum is the mechanical system described on the unit sphere in $\mathbf{R}^3$ by the Hamiltonian

$$H = \frac{1}{2}\|p\|^2 - \Gamma \cdot q,$$

q denoting the position of the ball on the unit sphere, Γ the (constant) gravity and p the momentum. The phase space is

$$TS^2 = \left\{(q, p) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid \|q\|^2 = 1 \text{ and } p \cdot q = 0\right\}$$

and the Hamiltonian system

$$\left\{ \begin{array}{rcl} \dot{q} & = & p \\ \dot{p} & = & \Gamma - (q \cdot \Gamma + \|p\|^2)q. \end{array} \right.$$

The system is invariant under the rotations around the vertical (*i.e.* Γ) axis, so that the momentum

$$K = (q \times p) \cdot \Gamma$$

is a first integral: the spherical pendulum is a completely integrable system.

Remark 3.7. The S^1-action by rotations around the vertical is generated by the flow of K. Considering q, p as vectors in $\mathbf{C}^3$, this action complexifies into a $\mathbf{C}^\star$-action. This is the reason why, in this example as in the spinning top case, one should expect to need non compact tori (as are the generalized Jacobians).

The elliptic curve, down-to-earth approach. It is more or less obvious (and in any case well known) that there should be an elliptic curve present. Choose an orthonormal basis of $\mathbf{R}^3$ such that $\Gamma = -e_3$ and eliminate q_1, q_2, p_1 and p_2 using H and K to get

$$p_3^2 = 2(H - q_3)(1 - q_3^2) - K^2,$$

so that $x = q_3$ satisfies a differential equation

$$\dot{x}^2 = f(x)$$

where f is a degree-3 polynomial. The elliptic curve X of equation

$$y^2 = 2x^3 - 2Hx^2 - 2x + 2H - K^2$$

plays a role.

It allows to solve the equations, using the Weierstrass $\wp$-function it defines. Even more directly, it shows, as the polynomial f must have two real roots between -1 and 1[(8)], that the ball will oscillate between two horizontal parallel circles on the unit sphere, according to everyday's experience.

A Lax equation. The best way to understand why there should be elliptic curves around is to re-write the Hamiltonian system as a Lax equation with a spectral parameter.

A Lax equation for the spherical pendulum involving matrices in the Lie algebra $\mathfrak{so}(3,1)$ appears in [**20**] as an example of the beautiful general constructions explained in this very recommendable paper. Here I will use something simpler[(9)]. As usual, identify vectors in $\mathbf{R}^3$ with skew-symmetric 3×3-matrices (and the vector cross product with the matrix bracket). The Lax equation

$$\frac{d}{dt}\left(q + \lambda(p \times q) - \lambda^2\Gamma\right) = \left[q + \lambda(p \times q) - \lambda^2\Gamma, p \times q - \lambda\Gamma\right]$$

is equivalent to the Hamiltonian system on TS^2. Then there is a spectral curve given by the characteristic polynomial of the Lax matrix:

$$\begin{aligned}\det\left(q + \lambda(p \times q) - \lambda^2\Gamma - \mu\,\mathrm{Id}\right) &= -\mu\left(\mu^2 + \left\|q + \lambda(p \times q) - \lambda^2\Gamma\right\|^2\right)\\ &= -\mu\left(\mu^2 + \lambda^4 - 2(p \times q)\cdot\Gamma + 2\lambda^2(\|p \times q\|^2 - \Gamma \cdot q) + \|q\|^2\right)\\ &\text{... where we recover the first integrals}\\ &= -\mu\left(\mu^2 + \underbrace{\lambda^4 + 2K\lambda^3 + 2H\lambda^2 + 1}_{P_{H,K}(\lambda)}\right).\end{aligned}$$

We will thus use the curve[(10)] of equation

$$\mu^2 + P_{H,K}(\lambda) = 0.$$

Remark 3.8. It is also possible to replace $\mathfrak{so}(3)$ by $\mathfrak{su}(2)$ to avoid the extra factor μ as is done in [**12**] for the Lagrange top. I have preferred to use real matrices in order to keep an eye on the reality questions, especially when I will use eigenvectors.

Now we have a family of the type we have discussed in § 2.2. I will use the notation C (affine), $\widetilde{C}$ (elliptic, normalized) and $\bar{C}$ (genus 2, singular)[(11)] as above.

[(8)] It always has a real root in $]1, +\infty[$; for a real motion it must also have roots in $[-1, 1]$.

[(9)] The Lax equation I use here is so simple that it probably belongs to folklore. I learned it from Alexei Reyman.

[(10)] The curve C defined by this equation is isomorphic to the elliptic curve X defined above. See § 5.3 for a more precise statement.

[(11)] The analogue of the singular curve $\bar{C}$ is used by Gavrilov and Zhivkov [**12**] to describe the "complex geometry" of the spinning top. Here we will use it to describe the "real monodromy". See also [**21**], where the monodromy of the top is determined, still by a direct computation, but using the singular curves.

Eigenvector mapping (classical). We now come to the eigenvectors of the Lax matrix

$$A_\lambda = q + \lambda(p \times q) - \lambda^2\Gamma.$$

They will give us a very precise dictionary between the topology of the Liouville tori and the geometry of the relative Jacobian. Fix a value (h, k) of (H, K). This fixes a curve C. Consider a point $(\lambda, \mu) \in C$. The eigenvectors of the Lax matrix for the eigenvalue μ form a line bundle on the complement of the branch points of λ, sub-bundle of the trivial bundle $C \times \mathbf{C}^3$. As above, when the curve C is smooth, there is a unique way to extend this line bundle to the whole of $\widetilde{C}$ as a sub-line bundle of $\widetilde{C} \times \mathbf{C}^3$. Thus, we have a line bundle, the eigenvector bundle L, on the complete curve $\widetilde{C}$.

Let $\mathcal{T}_C$ be the common level set of the first integrals H and K corresponding to the curve C. This is a subset of the phase space TS^2. Letting the point (q, p) vary on the level $\mathcal{T}_C$, we get a map (recall we assume C to be smooth):

$$\varphi_C : \mathcal{T}_C \longrightarrow \mathrm{Pic}(\widetilde{C})$$

where the group $\mathrm{Pic}(\widetilde{C})$ of isomorphism classes of line bundles over $\widetilde{C}$ is identified with the set of linear equivalence classes of divisors on $\widetilde{C}$. This mapping relates the geometry of the spherical pendulum with the algebraic geometry of $\widetilde{C}$.

Eigenvector mapping (modified). We can also define an eigenvector bundle on the singular curve[(12)] $\bar{C}$. Let us look at what happens when (λ, μ) tends to infinity in C. Write

$$(q + \lambda(p \times q) - \lambda^2\Gamma) \cdot v = \mu v.$$

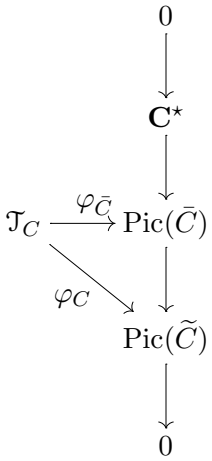
The eigenvalue μ is equivalent to $\pm i\lambda^2$, so that we are interested in the eigenvectors of the matrix $-\Gamma$ (the infinitesimal rotation around the vertical axis) with respect to the eigenvalues $\pm i$, namely in the two vectors ${}^t(1, \pm i, 0)$—one for each branch.

We have found a preferred basis of L_{∞_+} and L_{∞_-}, so that we have an isomorphism $L_{\infty_+} \to L_{\infty_-}$, which allows to define a line bundle over the singular curve $\bar{C}$.

Letting the point (q, p) vary on the level $\mathcal{T}_C$, we now get a map (recall we assume C to be smooth):

$$\varphi_{\bar{C}} : \mathcal{T}_C \longrightarrow \mathrm{Pic}(\bar{C})$$

where the group $\mathrm{Pic}(\bar{C})$ of isomorphism classes of line bundles over $\bar{C}$, is identified with the set of linear equivalence classes of divisors on $\bar{C}$. The neutral component is a generalized Jacobian, a dimension-2 complex algebraic group. The exact sequence relating it to the usual Picard group of the smooth curve $\widetilde{C}$



allows to relate the two "eigenvector mappings". We do not need to compute the map φ_C explicitly. But we will use its tangent mapping.

[(12)] See § 4 and the references given there for the generalized Jacobian and the Picard group of the singular curve $\bar{C}$.

Tangent map to $\varphi_{\bar C}$. The next aim is to show that, under $\varphi_{\bar C}$, the flows of H and K are mapped to linear and independent flows on $\mathrm{Pic}(\bar C)$. Here we really need the generalized Jacobian: the level set $\mathcal{T}_C$ is 2-dimensional while the Jacobian $\mathrm{Pic}(\widetilde C)$ of the genus-1 curve $\widetilde C$ is only 1-dimensional (either over $\mathbf{C}$ or over $\mathbf{R}$—but over the same field for both!). Let us begin by the usual vector mapping φ_C.

Proposition 3.9. *The eigenvector mapping φ_C is constant on the orbits of the vector field X_K. Its tangent mapping maps the Hamiltonian vector field X_H to the class in $H^1(C;\mathcal{O}_{\widetilde C})$ of the 1-cocycle $\lambda^{-1}\mu$ of the covering of $\widetilde C$ by U_0 (where $\lambda\neq\infty$) and U_∞ (where $\lambda\neq 0$).*

Proof. The flow of K is the flow of rotations around the vertical axis. The first assertion can of course be checked directly. For further use, I will prove it in an indirect way. The differential system associated with X_K is

$$\left\{\begin{array}{rcl}\dot q &=& \Gamma\times q\\ \dot p &=& \Gamma\times p,\end{array}\right.$$

a system that has an obvious Lax form using our matrix A_λ:

$$\frac{d}{dt}\left(q+\lambda(p\times q)-\lambda^2\Gamma\right)=\left[q+\lambda(p\times q)-\lambda^2\Gamma,-\Gamma\right].$$

The matrix B_λ in this Lax equation is

$$-\Gamma=\left[\lambda^{-2}(q+\lambda(p\times q)-\lambda^2\Gamma)\right]_+$$

(the notation $_+$ denotes the polynomial part of a Laurent polynomial). Using as above the linearization theorem (for this case, see also *e.g.* [**6**, theorem IV.2.5]), we get that the image of X_K in $H^1(\widetilde C;\mathcal{O}_{\widetilde C})=T.\,\mathrm{Pic}(\widetilde C)$ is the class of the cocycle $\lambda^{-2}\mu$.

In the same way, the matrix B_λ in the Lax equation describing the Hamiltonian system associated with X_H is

$$p\times q-\lambda\Gamma=\left[\lambda^{-1}(q+\lambda(p\times q)-\lambda^2\Gamma)\right]_+$$

so that the image of X_H is the class of the cocycle $\lambda^{-1}\mu$.

Now, as we have already mentioned it, μ is equivalent to $\pm i\lambda^2$ when λ goes to infinity, so that the function $\lambda^{-2}\mu$ on $U_0\cap U_\infty$ extends to a holomorphic function on U_∞. Thus the cocycle $\lambda^{-2}\mu$ is a coboundary and its class is zero.

On the opposite, the cocycle $\lambda^{-1}\mu$ is non zero in $H^1(\widetilde C;\mathcal{O}_{\widetilde C})$. An easy way to prove this is to compute its residue at 0 (or at ∞) against a holomorphic form on $\widetilde C$. We take the form $d\lambda/\mu$ and compute

$$\mathrm{Res}_{\lambda=0}\left(\lambda^{-1}\mu\frac{d\lambda}{\mu}\right)=\mathrm{Res}_{\lambda=0}\left(\lambda^{-1}d\lambda\right)=2.$$

Eventually, the tangent map to φ_C sends X_K to zero and X_H to a generator of $H^1(\widetilde C;\mathcal{O}_{\widetilde C})$. □

Notice that Proposition 3.9 readily has the following consequence:

Corollary 3.10. *If the complex curve C is smooth, the corresponding value of (H,K) in $\mathbf{C}^2$ or $\mathbf{R}^2$ is regular.*

Proof. Note that the eigenvector mapping φ_C is well-defined as soon as the curve C is smooth. Assume this is the case. Then X_K is mapped to zero and X_H to a nonzero element. The only way in which X_H and X_K can be dependent is then to have $X_K=0$. This means that q and p are vertical. As they are orthogonal and q is nonzero, p must be zero. Thus $H=\pm1$ and $K=0$, the equation of C writes

$$\mu^2+\left(\lambda^2\pm1\right)^2=0$$

an obviously singular curve (it has two double points)... and a contradiction. Thus X_H and X_K were independent and the value is regular. □

Let us now look more closely at the map $\varphi_{\bar C}$, to get:

Proposition 3.11. *Assume the curve C is smooth. Then the tangent mapping to the eigenvector map $\varphi_{\bar{C}}$ maps X_H and X_K to independent vectors in $H^1(\bar{C};\mathcal{O}_{\bar{C}})$.*

Proof. It is based on the considerations used in the proof of Proposition 3.9. A 1-cocycle of holomorphic functions on $\widetilde{C}$ for the covering

$$\widetilde{C} = U_0 \cup U_\infty$$

is just a holomorphic function

$$f_{0,\infty} : U_0 \cap U_\infty \longrightarrow \mathbf{C}.$$

We can also consider such an $f_{0,\infty}$ as a cocycle $\bar{f}_{0,\infty}$ of holomorphic functions on $\bar{C}$, now for the covering

$$\bar{C} = U_0 \cup U_{\infty_+} \cup U_{\infty_-}$$

(with an obvious notation, in which $U_{\infty_+} \cap U_{\infty_-} = \varnothing$). On $\widetilde{C}$, coboundaries are differences $f_0 - f_\infty$ of functions that are holomorphic on U_0, U_∞ resp. On $\bar{C}$, these are also the differences of functions $f_0 - f_{\infty_\pm}$ on $U_0 \cap U_{\infty_\pm}$. But now, the functions $f_{\infty_\pm}$ must take the same value at the two points at ∞ (see § 4).

As soon as we have taken care of the fact that the eigenvector bundle is well-defined on $\bar{C}$, the proof of the linearization theorem works to give us, as in the proof of Proposition 3.9, that

$$T_{(q,p)}\varphi_{\bar{C}}\left(X_H(q,p)\right) \text{ is the class of } \lambda^{-1}\mu$$

and that

$$T_{(q,p)}\varphi_{\bar{C}}\left(X_K(q,p)\right) \text{ is the class of } \lambda^{-2}\mu.$$

The big difference now is that these two cocycles give *independent* classes in $H^1(\bar{C};\mathcal{O}_{\bar{C}})$. For this, consider the two *holomorphic* forms $d\lambda/\mu$ and $\lambda d\lambda/\mu$ on $\bar{C}$ and compute the residues,

– for X_H:

$$\mathrm{Res}_{\lambda=0}\left(\lambda^{-1}\mu\frac{d\lambda}{\mu}\right) = \mathrm{Res}_{\lambda=0}\left(\lambda^{-1}d\lambda\right) = 2$$

as above and

$$\mathrm{Res}_{\lambda=0}\left(\lambda^{-1}\mu\frac{\lambda d\lambda}{\mu}\right) = \mathrm{Res}_{\lambda=0}\left(\lambda^{-2}d\lambda\right) = 0,$$

– and for X_K:

$$\mathrm{Res}_{\lambda=0}\left(\lambda^{-2}\mu\frac{d\lambda}{\mu}\right) = \mathrm{Res}_{\lambda=0}\left(\lambda^{-2}d\lambda\right) = 0$$

once again, as now

$$\mathrm{Res}_{\lambda=0}\left(\lambda^{-2}\mu\frac{\lambda d\lambda}{\mu}\right) = \mathrm{Res}_{\lambda=0}\left(\lambda^{-1}d\lambda\right) = 2.$$

Thus the two cocycles give independent cohomology classes. □

Remark 3.12. Let us show that the fact that spectral curve has no real point is related to the periodicity of the flow of K. Look at the diagram

$$\begin{array}{ccc} S^1 & & \mathbf{C}^\star \\ \downarrow & & \downarrow \\ \mathcal{T}_C & \xrightarrow{\varphi_{\bar{C}}} & \mathrm{Pic}(\bar{C}) \\ \downarrow & & \downarrow \\ \mathcal{T}_C/S^1 & \xrightarrow{\varphi_C} & \mathrm{Pic}(\widetilde{C}) \end{array}$$

where, on the left, we factor out the regular real level $\mathcal{T}_C$ by the flow of K (rotations around the vertical axis) and on the right, we look at the generalized Jacobian as above. As $\mathcal{T}_C$ is compact, the map $\varphi_{\bar{C}}$, whose differential is always injective, according to Proposition 3.11, is a covering of its image. This re-proves that the real part of the right side should also be an S^1-fibration (see § 2.2).

Critical values vs discriminant. We have shown that, if the value (H, K) corresponds to a smooth curve C, it is a regular value. Of course, non-smooth curves C correspond to polynomials with double roots

$$\lambda^4 + 2K\lambda^3 + 2H\lambda^2 + 1 = (\lambda - u)^2(\lambda - v)(\lambda - w)$$

that is

$$\begin{cases} H &= \dfrac{1}{2}u^2 - \dfrac{3}{2u^2} \\ K &= -u + \dfrac{1}{u^3} \end{cases} \quad u \in \mathbf{C}^\star.$$

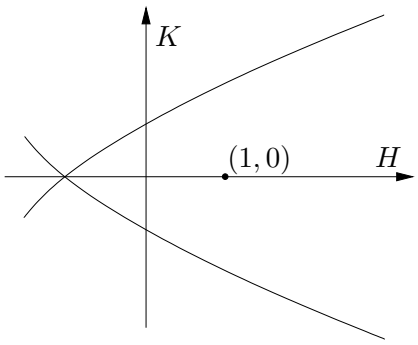


FIGURE 8. The regular values for the spherical pendulum

The real part of the discriminant curve (meaning that H and K are real, which does not mean that u is real) is shown on Figure 8. The point $(1, 0)$ is part of the discriminant (a real double point with imaginary branches). Obviously, for real (q, p), H must be greater than or equal to -1, so that the only connected component of the complement of the discriminant that are actual values for real (q, p) is the one containing the positive H-axis. These points must be regular values (according to Corollary 3.13) except for the point $(1, 0)$ which we have seen to correspond to a singular level. The boundary points of the image of (H, K) must be critical values. Thus we have proved the converse of Corollary 3.13 for real (H, K), namely:

Corollary 3.13. *The value of (H, K) in $\mathbf{R}^2$ is regular if and only if the corresponding complex curve C is smooth.* □

The points of the discriminant curve actually correspond to the trajectories of the pendulum on horizontal circles, namely are the images of points

$$(q, p) = \left(\frac{1}{u^2}\Gamma + q', uq' \times \Gamma\right) \text{ with } q' \perp \Gamma$$

for which $X_H(q, p) = -uX_K(q, p)$—critical points indeed, a result of Huygens (see [**10**]).

Hamiltonian monodromy. We have seen that the eigenvector mapping is a covering of its image, so that $H_1(\mathcal{T}_C; \mathbf{Z})$ is a sublattice of $H_1(\mathrm{Pic}^2(\bar{C}_{\mathbf{R}}); \mathbf{Z})$. We have more precisely:

Proposition 3.14. *The eigenvector mapping $\mathcal{T}_C \to \mathrm{Pic}^2(\bar{C})$ embeds $H_1((\mathcal{T}_C)_{\mathbf{R}}; \mathbf{Z})$ as the sublattice*

$$2H_1(\mathrm{Pic}^2(\bar{C})_{\mathbf{R}}; \mathbf{Z})) \subset H_1(\mathrm{Pic}^2(\bar{C})_{\mathbf{R}}; \mathbf{Z}).$$

Thus, we have a basis in $H_1(\mathcal{T}_C; \mathbf{Z})$ which is sent to $(2\widetilde{\alpha}, 2\gamma) \in H_1(\mathrm{Pic}^0(\bar{C}); \mathbf{Z})$ so that the monodromy of the fibration in Liouville tori is the same as that of the fibration by the relative Jacobian.

Corollary 3.15 **(Cushman [10]).** *The monodromy of the fibration in Liouville tori for the spherical pendulum is given by the matrix* $\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$. □

Remark 3.16. According to Proposition 2.2, the Hamiltonian monodromy for the spherical pendulum is thus, eventually, given by the integration of a meromorphic 1-form. See [**8**] for a proof of this fact by a direct computation.

Proof of the proposition. Let us now compute the degree of the eigenvector mapping $\varphi_{\bar{C}}$. We have seen that the classical eigenvector mapping φ_C defines a covering map

$$\varphi_C : \mathcal{T}_C/\mathbf{C}^\star \longrightarrow \mathrm{Pic}^2(\widetilde{C}).$$

It is well known (see *e.g.* [**4**] or [**12**]) that φ_C maps the real level set $\mathcal{T}_C/S^1$ to one of the two components of $\mathrm{Pic}^2(\widetilde{C})_{\mathbf{R}} = X_{\mathbf{R}}$ by a degree-2 map. To compute the degree of $\varphi_{\bar{C}}$, we just need to understand what happens in the direction of the field X_K.

Lemma 3.17. *The restriction of $\varphi_{\bar{C}}$ to the orbits of X_K is a covering of degree 2. It maps a real orbit to a circle via a degree-2 map.*

Proof. It obviously suffices to show this for one (well chosen) orbit in a well chosen level set. Fix $q = \Gamma$, then p is horizontal ($p_3 = 0$), $K = 0$ and $\|p\|$ is determined by

$$\frac{1}{2}\|p\|^2 - \|\Gamma\|^2 = H, \text{ that is } \|p\|^2 = 2(H+1).$$

Fix H (not equal to ± 1) and look at the orbit of X_K through (Γ, p) in the level $(H, 0)$.

The Lax matrix A_λ has the form

$$\begin{aligned} A_\lambda &= q + (p \times q)\lambda - \Gamma\lambda^2 \\ &= (1-\lambda^2)\Gamma + (p \times \Gamma)\lambda, \end{aligned}$$

this is the skew-symmetric matrix corresponding to the vector ${}^t(-\lambda p_2, \lambda p_1, \lambda^2 - 1)$. The vector

$$V = \begin{pmatrix} \left(1 - \frac{1}{\lambda^2}\right) p_2 - \frac{\mu}{\lambda^2} p_1 \\ -\left(1 - \frac{1}{\lambda^2}\right) p_1 - \frac{\mu}{\lambda^2} p_2 \\ (p_1^2 + p_2^2)\frac{1}{\lambda} \end{pmatrix}$$

is an eigenvector of A_λ for the eigenvalue μ, that never vanishes and has neither a zero nor a pole at infinity.

Let us look at the pole divisor of V in $\mathrm{Pic}(\bar{C})$. There is a pole when a component of V tends to infinity. This happens only for $\lambda = 0$. The divisor obtained does not depend on p. This is no surprise, since we know (Proposition 3.9) that the classical eigenvector mapping φ_C is constant on the orbits of X_K.

We must thus concentrate on what happens at infinity.

– at ∞_+, we have $\dfrac{\lambda}{\mu^2} = i$ and

$$V(\infty_+) = \begin{pmatrix} p_2 - ip_1 \\ -p_1 - ip_2 \\ 0 \end{pmatrix} = (p_2 - ip_1) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix},$$

– at ∞_-, similarly, $\dfrac{\lambda}{\mu^2} = -i$ and

$$V(\infty_-) = \begin{pmatrix} p_2 + ip_1 \\ -p_1 + ip_2 \\ 0 \end{pmatrix} = (p_2 + ip_1) \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.$$

The vectors ${}^t(1,\pm i,0)$ are the reference vectors we have used to define the eigenvector bundle as a bundle over $\bar C$. The mapping $\varphi_{\bar C}$ thus sends the point (Γ,p) to the ratio $V(\infty_+)/V(\infty_-)$, namely to

$$\frac{p_2-ip_1}{p_2+ip_2}=\frac{(p_2-ip_1)^2}{p_1^2+p_2^2}=\frac{(p_2-ip_1)^2}{2(H+1)}.$$

Thus $\varphi_{\bar C}$ is a degree-2 map, as the solutions of

$$\left\{\begin{array}{rcl}(p_2-ip_1)^2&=&2a(H+1)\\ p_1^2+p_2^2&=&H+1\end{array}\right.$$

for $a=\alpha^2\in\mathbf{C}^\star$ are the two points

$$p_1=\frac{\varepsilon\sqrt{H+1}}{2i}\left(\frac{1}{\alpha}-\alpha\right),\quad p_2=\frac{\varepsilon\sqrt{H+1}}{2}\left(\frac{1}{\alpha}+\alpha\right)$$

for $\varepsilon=\pm1$. Notice that they are real when $|\alpha|=1$. □

□

4. The generalized Jacobian, a few words

Let me recall very briefly a few notions related to the geometry of the singular curve $\bar C$ that I have used in this paper. See [**11**] for detail.

The curves. Consider a smooth[(13)] curve $\widetilde C$ of genus $g-1$. Choose two distinct points ∞_+ and ∞_- on $\widetilde C$, call C their complement in $\widetilde C$ and $\bar C$ the curve obtained by the identification of ∞_+ and ∞_- to a single point, that we call $\infty\in\bar C$. Notice that the natural map

$$H_1(C;\mathbf{Z})\longrightarrow H_1(\widetilde C;\mathbf{Z})$$

is surjective, its kernel being the infinite cyclic group generated by the homology class of a small loop turning once around ∞_+ (or ∞_-).

Functions, vector fields, forms. We start from germs of holomorphic functions on $\bar C$: these are germs of holomorphic functions on $\widetilde C$ which take the same value at ∞_+ and ∞_-. Note that a germ at ∞ on $\bar C$ is a germ at both ∞_+ and ∞_- on $\widetilde C$. Then, vector fields must be derivations, we must thus have

$$(X\cdot f)(\infty_+)=(X\cdot f)(\infty_-)$$

for all functions f, which forces $X(\infty_+)=X(\infty_-)=0$, thus the holomorphic vector fields on $\bar C$ are the holomorphic vector fields on $\widetilde C$ satisfying this condition.

Now, the holomorphic 1-forms are obtained by duality: a holomorphic form on $\bar C$ is a meromorphic form on $\widetilde C$ with at worst simple poles at ∞_+ and ∞_- (with opposite residues). According to the Riemann-Roch theorem, the complex vector space of meromorphic forms on $\widetilde C$ that have a simple pole at ∞_+, a simple pole at ∞_- and no other pole, has dimension 1. Hence the cokernel of the natural injection $H^0(\Omega^1_{\widetilde C})\to H^0(\Omega^1_{\bar C})$ has dimension 1. The dimension of $H^0(\Omega^1_{\bar C})$ is g, that is, $\bar C$ has *arithmetic genus* g.

Examples 4.1

- Consider, on the curve $\widetilde C=\mathbf{P}^1(\mathbf{C})=\mathbf{C}\cup\{\infty\}$, the two points $\infty_+=\infty$, $\infty_-=0$. Then $C=\mathbf{C}-\{0\}$ and $\bar C$ is a sphere with two points identified (or a pinched torus). The meromorphic form dz/z over C can be considered as a holomorphic form on $\bar C$.
- On the curve $\bar C$ considered in this paper, the form xdx/y is equivalent to $\pm idx/x$ near $\infty_\pm$, so that it defines a holomorphic form on $\bar C$.

(13) Starting from a curve with double points, we see that there is no difficulty to make, by induction, more complicated examples.

Divisors, line bundles, Picard group. In order to be able to integrate "holomorphic" forms over paths, it is necessary that the paths do not go through the points ∞_+ and ∞_-. Hence divisors on $\bar{C}$ must be divisors on $\widetilde{C}$ not involving ∞_+ or ∞_-. Linear equivalence of divisors on $\bar{C}$ is defined by $D \sim D'$ if and only if there exists a meromorphic function f on $\bar{C}$, which has neither a zero nor a pole at ∞ and such that $D - D' = (f)$. We can then assume that $f(\infty) = 1$.

Consider now a line bundle over $\widetilde{C}$. In order that it defines a line bundle on $\bar{C}$, there should be an isomorphism between the fibers of ∞_+ and ∞_-. This can be achieved if a meromorphic section with neither a zero nor a pole at ∞_+ and at ∞_- is given. This way, we get the generalized version of the equivalence between line bundles and divisors we need.

The Picard group $\mathrm{Pic}(\bar{C})$ is defined either as the group of isomorphism classes of line bundles over $\bar{C}$ or as the group of linear equivalence classes of divisors. There is a projection $\mathrm{Pic}(\bar{C}) \to \mathrm{Pic}(\widetilde{C})$, the forgetful map which sends the class of D as a divisor on $\bar{C}$ to its class as a divisor on $\widetilde{C}$. The kernel is a set of divisors of functions, isomorphic to $\mathbf{C}^\star$ under the map $z \mapsto (f)$ where f is any function on $\widetilde{C}$ such that $f(\infty_+)/f(\infty_-) = z$.

Generalized Jacobian. The Jacobian can be defined by holomorphic forms. The map

$$\begin{array}{ccc} H_1(C;\mathbf{Z}) & \longrightarrow & H^0(\Omega_{\bar{C}})^\star \\ \alpha & \longmapsto & \left(\omega \mapsto \int_\alpha \omega\right) \end{array}$$

is an embedding $\mathbf{Z}^{2g-1} \subset \mathbf{C}^g$, the Jacobian is the quotient, a non compact g-dimensional complex torus. More precisely, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{C} & \longrightarrow & H^0\left(\Omega^1_{\bar{C}}\right)^\star & \longrightarrow & H^0\left(\Omega^1_{\widetilde{C}}\right)^\star & \longrightarrow & 0 \\ & & \uparrow u_\infty & & \uparrow u & & \uparrow \widetilde{u} & & \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & H_1(C;\mathbf{Z}) & \longrightarrow & H_1(\widetilde{C};\mathbf{Z}) & \longrightarrow & 0 \end{array}$$

in which the usual Abel-Jacobi map $\widetilde{u}$ embeds $H_1(\widetilde{C};\mathbf{Z}) \cong \mathbf{Z}^{2g-2}$ as a lattice Λ into the complex $(g-1)$-dimensional vector space $H^0\left(\Omega^1_{\widetilde{C}}\right)^\star$. The integration map u_∞ sends $\mathbf{Z}$ to $2i\pi\mathbf{Z} \subset \mathbf{C}$ so that $H_1(C;\mathbf{Z})$ is mapped to a "lattice"[14] $\bar{\Lambda}$. The exact sequence of quotients

$$0 \longrightarrow \mathbf{C}/2i\pi\mathbf{Z} \longrightarrow \mathrm{Jac}\,\bar{C} \longrightarrow \mathrm{Jac}\,\widetilde{C} \longrightarrow 0$$

is isomorphic to the exact sequence

$$0 \longrightarrow \mathbf{C}^\star \longrightarrow \mathrm{Pic}^0\,\bar{C} \longrightarrow \mathrm{Pic}^0\,\widetilde{C} \longrightarrow 0$$

of Picard groups.

Example 4.2. Consider again the case of the sphere with two points identified. This is a curve of arithmetic genus 1 (and it is easy to embed it in the plane as a rational cubic with a double point). The Jacobian of $\mathbf{P}^1(\mathbf{C})$ is trivial, so that the sequences above reduce to the inclusion $H_1(\mathbf{C} - \{0\}\,;\mathbf{Z}) \to \mathbf{C}$ given by integration of the form dz/z. The generalized Jacobian is $\mathbf{C}/2i\pi\mathbf{Z} \cong \mathbf{C}^\star$.

5. Complements

5.1. The family of curves $E_{h,k}$. In this §, we prove the assertions on the family $E_{h,k}$ that we have used in § 3.1. Recall the equation

$$y^2 x \prod_{i=1}^{3}(x - a_i) + y(f_1 x^2 - hx + k) + f_2^2 = 0$$

[14] I use the quotation marks because this is a $\mathbf{Z}^{2g-1}$ in $\mathbf{C}^g$, and thus not quite a lattice.

and the assumptions we have made on the a_i's

$$0 < a_1 < a_2 < a_3,$$

on the f_i's:

$$f_2 \neq 0, \quad f_1^2 - 4f_2^2 = 4\beta^2 > 0$$

and that h, k are positive parameters. Write

$$\Delta(x) = (f_1x^2 - hx + k)^2 - 4f_2^2x\prod_{i=1}^{3}(x - a_i)$$

and define σ_1, σ_2, σ_3 for the elementary symmetric functions of the a_i's.

We are looking for values of the parameters such that

$$\Delta(x) = (ax^2 + bx + c)^2 \text{ with } b^2 - 4ac < 0.$$

Identifying terms,

- using constant terms, we get $k^2 = c^2$, we choose $c = k$,
- using coefficients of x^4, we get $f_1^2 - 4f_1^2 = a^2$, so that $a = \pm\beta$. As $c = k > 0$, we must take $a = 2\beta$,
- from the first and third degree terms, we get h and b as functions of k:

$$h = \frac{1}{2}(f_1 + 2\beta)\left(\sigma_1 - 2\beta\frac{\sigma_3}{k}\right), \qquad b = \frac{1}{2}(f_1 + 2\beta)\left(\sigma_1 - f_1\frac{\sigma_3}{k}\right).$$

Replacing in the x^2 term gives an equation in k:

$$8k^3 - 4(f_1 + 2\beta)\sigma_2k^2 + 2(f_1 + 2\beta)^2\sigma_1\sigma_3k - (f_1 + 2\beta)^3\sigma_3^2 = 0.$$

This is easy to solve: as the roots of

$$\alpha^3 - \sigma_2\alpha^2 + \sigma_3\sigma_1\alpha - \sigma_3^2 = 0$$

are the a_ia_j ($i < j$), we get

$$k = \frac{1}{2}(f_1 + 2\beta)a_ia_j \text{ and } h = -2\beta a_k + \frac{1}{2}(f_1 + 2\beta)\sigma_1$$

(in these formulas, $i < j$ and $\{i, j, k\} = \{1, 2, 3\}$). Moreover, in terms of $u = \frac{1}{2}(f_1 + 2\beta)$,

$$a = 2u - f_1, \qquad b = -f_1a_k + u\sigma_1, \qquad c = ua_ia_j$$

so that

$$b^2 - 4ac = (\sigma_1^2 - 8a_ia_j)u^2 + 2f_1(2a_ia_j - \sigma_1a_k)u + f_1^2a_k^2.$$

Notice that this is positive at $u = 0$. In order that there exists values of u for which this expression is negative, we must have

$$f_1^2((2a_ia_j - \sigma_1a_k)^2 - a_k^2(\sigma_1^2 - 8a_ia_j) > 0,$$

that is

$$4a_ia_jf_1^2(a_k - a_i)(a_k - a_j) > 0.$$

Thus we must have $i = 1$ and $j = 2$ or $i = 2$ and $j = 3$. In the second case, there are no values of β such that $b^2 - 4ac < 0$ and $0 < \beta < 2\,|f_1|$. The first case gives the desired interval $(\varepsilon_1, \varepsilon_2)$.

5.2. Holomorphic forms and cocycles on D. The equation of the elliptic curve E of Example 1.1 and § 2.1 can be put in the form

$$t^2 + 2a(x)t + b(x) = 0, \text{ with } t = \frac{1}{y}.$$

Hence the form

$$\omega = \frac{dx}{t + a(x)} = \frac{-2dt}{2a'(x)t + b'(x)}$$

is holomorphic (and has no zero!). Let η be its pull back to D, thus

$$\eta = \frac{dx}{u^2 + a(x)} = \frac{-4udu}{2a'(x)u^2 + b'(x)}, \text{ with } u^2 = \frac{1}{z^2} = t,$$

is a holomorphic form on D. Its divisor is

$$(\eta) = (u)_0 = (z)_\infty = \sum_{i=0}^{3} A_i.$$

The divisor of the meromorphic form $z\eta$ is

$$(z\eta) = \sum_{i=0}^{3} A_i + (z)_0 - \sum_{i=0}^{3} A_i = (z)_0$$

thus $z\eta$ is also holomorphic. Now, on D, z and x both have degree 4 and the zeroes of z are the poles of x so that $(z)_0 = (x)_\infty$ and

$$(xz\eta) = (x)_0 - (x)_\infty + (z)_0 = x_0.$$

Thus we have exhibited three holomorphic forms on the genus-3 curve[(15)] D, the forms η, $z\eta$ and $xz\eta$. Notice that the first is τ-invariant while the two others are anti-invariant.

Consider now, as in the proof of Proposition 3.2, the cocycles zx^k of the covering $D = \mathfrak{U}_+ \cup \mathfrak{U}_-$ and let us prove the lemma that ends the proof of this proposition.

Lemma 5.1. *The cocycles zx^3 and zx^4 define independent elements of $H^1(\mathcal{O}_D)$.*

Proof. We show that the images of these cocycles under the linear map

$$\begin{array}{rcl} H^1(\mathcal{O}_D) & \longrightarrow & \mathbf{C}^2 \\ f & \longmapsto & (\mathrm{Res}_{z=\infty}(fz\eta), \mathrm{Res}_{z=\infty}(fxz\eta)) \end{array}$$

are independent vectors. Let us thus compute

$$\begin{aligned} R_m = \mathrm{Res}_{z=\infty}(z^2 x^m \eta) &= \mathrm{Res}_{u=0} \frac{-4ux^m du}{u^2(2a'(x)u^2 + b'(x))} \\ &= -2\,\mathrm{Res}_{t=0} \frac{2x^m dt}{t(2a'(x)t + b'(x))}. \end{aligned}$$

The points of E at which $t = 0$ are the four points A_i's. Near such a point, t is a local coordinate on E and x can be expressed as a function of t,

$$x = a_i + O(t),$$

hence

$$R_m = -2 \sum_{i=1}^{3} \frac{a_i^m}{b'(a_i)}$$

(notice that the residue at A_0 is zero as $m \geq 3$). Now

$$b(x) = \frac{\prod_{j=0}^{3}(x - a_j)}{f_2^2}, \text{ so that } b'(a_i) = \frac{\prod_{j \neq i}(a_i - a_j)}{f_2^2} \text{ and}$$

(15) The computation below will show that they are independent in $H^0(\Omega^1_D)$.

$$R_m=\frac{2}{\prod_{1\leq i<j\leq 3}(a_i-a_j)}\left(a_1^{m-1}(a_2-a_3)+a_2^{m-1}(a_3-a_1)+a_3^{m-1}(a_1-a_2)\right).$$

To prove that zx^3 and zx^4 are independent, we only need to check that

$$\begin{vmatrix}R_3 & R_4\\ R_4 & R_5\end{vmatrix}\neq 0,$$

which is true. □

5.3. An equivalence of elliptic curves. Let us investigate the relations between the two elliptic curves that appeared in the spherical pendulum problem:

- the curve X
$$y^2=2x^3-2Hx^2-2x+2H-K^2$$
obtained by the "naïve" method
- and the genus-1 spectral curve C
$$\mu^2+\underbrace{\lambda^4+2K\lambda^3+2H\lambda^2+1=0}_{P(\lambda)}.$$

Let us check firstly that they are isomorphic as complex curves. The method is classical and can be found *e.g.* in [**22**, page 453]. We send a root λ_0 of P to ∞ to transform it into a degree-3 polynomial. Expand P by the Taylor formula

$$P(\lambda)=4A_3(\lambda-\lambda_0)+6A_2(\lambda-\lambda_0)^2+4A_1(\lambda-\lambda_0)^3+(\lambda-\lambda_0)^4$$

and put $u=\dfrac{1}{\lambda-\lambda_0}$, so that

$$P(\lambda)=(\lambda-\lambda_0)^4\underbrace{\left[4A_3u^3+6A_2u^2+4A_1u+1\right]}_{Q(u)}$$

and

$$\mu^2+P(\lambda)=(\lambda-\lambda_0)^4\left\{\left(\frac{\mu}{(\lambda-\lambda_0)^2}\right)^2+Q(u)\right\}.$$

Let us now transform $Q(u)$ to eliminate the u^2 term. Write $s=A_3u+\frac{1}{2}A_2$ to get

$$\begin{aligned}Q(u)&=\frac{1}{A_3^2}\left[4s^3-\underbrace{(3A_2^2-4A_1A_3)}_{g_2}s-\underbrace{(2A_1A_2A_3-A_3^2-A_2^3)}_{g_3}\right]\\&=\frac{1}{A_3^2}(4s^3-g_2s-g_3).\end{aligned}$$

Putting $v=\dfrac{iA_3\mu}{(\lambda-\lambda_0)^2}$, we get a complex isomorphism

$$\begin{array}{ccc}C&\xrightarrow{\ \varphi\ }&Y\\(\lambda,\mu)&\longmapsto&(s,v)\end{array}$$

from C to the elliptic curve Y of equation

$$v^2=4s^3-g_2s-g_3.$$

The whole point of the computation now is that g_2 and g_3 do not depend on λ_0. Using

$$4A_3=P'(\lambda_0),\qquad 12A_2=P''(\lambda_0),\qquad 24A_1=P'''(\lambda_0)$$

we get after a few computation

$$g_2=1+\frac{1}{3}H^3,\text{ and }g_3=\frac{1}{3}H-\frac{1}{27}H^3-\frac{1}{4}K^2.$$

The next step is to put the equation of X in the same form, which is (more) easily done, putting

$$s = -2\left(x - \frac{1}{3}H\right) \text{ and } v = \frac{iy}{2\sqrt{2}},$$

a change of variable which gives an isomorphism between X and Y. The reason why I have given such explicit formulas is that they allow to give a more precise result.

Proposition 5.2. *As complex curves, X and $\widetilde{C}$ are isomorphic. As real curves, X and $\mathrm{Pic}^{2k}(\widetilde{C})$ are isomorphic for any integer k.*

Proof. The "complex" statement has already been proved. Let us look at the reality questions. Firstly, we use the Abel-Jacobi mapping to identify Y with $\mathbf{C}/\Lambda$:

$$\begin{array}{ccc} Y & \xrightarrow{u} & \mathbf{C}/\Lambda \\ P & \longmapsto & \displaystyle\int_\infty^P \frac{ds}{v}. \end{array}$$

Look at X and at the composition:

$$X \xrightarrow{\psi} Y \xrightarrow{u} \mathbf{C}/\Lambda$$

where ψ is the change of variable above

$$\psi(x,y) = \left(s = -2\left(x - \frac{H}{3}\right), v = \frac{iy}{2\sqrt{2}}\right).$$

Hence we have

$$u(\psi(P)) = \int_\infty^{\psi(P)} \frac{ds}{v} = \frac{i}{\sqrt{2}} \int_\infty^P \frac{dx}{y},$$

so that, calling S the real structure on X ($S(x,y) = (\bar{x}, \bar{y})$), we have

$$u \circ \psi(S(P)) = -\overline{u \circ \psi(P)}.$$

Thus, the real curve (X, S) is isomorphic with the real curve $(\mathbf{C}/\Lambda, z \mapsto -\bar{z})$.

Let us now look at $\mathrm{Pic}^0(\widetilde{C})$ (recall that $\widetilde{C}$ has no real point, so that there is no natural base point to identify the real curve $\widetilde{C}$ with $\mathbf{C}/\Lambda$, we must use the parity of the degree here) and at the composition

$$\mathrm{Pic}^0(\widetilde{C}) \xrightarrow{\varphi} Y \xrightarrow{u} \mathbf{C}/\Lambda.$$

Here I call φ the change of variables

$$\varphi(\lambda, \mu) = \left(s = \frac{A_3}{\lambda - \lambda_0} + \frac{1}{2}A_2, v = \frac{iA_3\mu}{(\lambda - \lambda_0)^2}\right)$$

and for the map it defines on $\mathrm{Pic}^0(C)$, so that:

$$\begin{aligned} u \circ \varphi(\sum P_i - \sum Q_i) &= \sum \int_\infty^{\varphi(P_i)} \frac{ds}{v} - \sum \int_\infty^{\varphi(Q_i)} \frac{ds}{v} \\ &= \sum \int_{\varphi(Q_i)}^{\varphi(P_i)} \frac{ds}{v} \\ &= i \sum \int_{Q_i}^{P_i} \frac{d\lambda}{\mu}. \end{aligned}$$

Thus, the real curve $(\mathrm{Pic}^0(\widetilde{C}), S)$ is—also—isomorphic with $(\mathbf{C}/\Lambda, z \mapsto -\bar{z})$. □

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