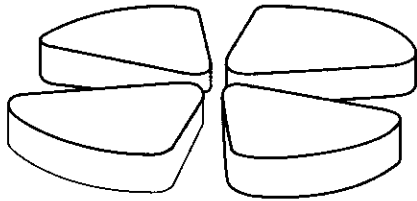


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Abstract

A quantal version of Goldhaber's fragmentation model is presented. The basic assumption of sudden dichotomy is retained initially but the effect of the nuclear mean field is taken into account by placing nucleons in a harmonic oscillator potential. It is shown that, if the oscillator constants of the initial nucleus and of the final fragments are identical and if zero-point motion is neglected, Goldhaber's distribution is recovered *exactly*. An alternative to this simple model which imposes conservation of the number of oscillator quanta is proposed. This condition is equivalent to conservation of energy only in the case of equal oscillator constants but can be readily generalised to unequal ones.

I. INTRODUCTION

In a classic paper on nucleus-nucleus fragmentation processes [1], Goldhaber showed that the momentum distribution of fragments produced in a peripheral nuclear collision followed the rule

$$\langle \bar{P}_{A_1}^2 \rangle = \langle \bar{P}_{A_2}^2 \rangle = \frac{A_1 A_2}{A-1} \langle \bar{p}^2 \rangle. \quad (1)$$

This expression assumes the break-up of an initial compound system with A nucleons into two fragments with A_1 and A_2 nucleons with $A_1 + A_2 = A$, and gives the momentum

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distribution of the fragments as a function of their mass in terms of the mean squared nucleon momentum $\langle \bar{p}^2 \rangle$. (The expression can be generalised to the case of fragmentation into k pieces.) The derivation of Goldhaber’s distribution rests on an assumption of “sudden dichotomy” whereby the nucleons of the initial system are randomly assigned to one or the other fragment without a change in the individual nucleon momenta. The derivation is entirely classical. The nucleons are non-interacting and no nuclear mean-field potential enters the discussion.

In this contribution a quantum analogue of Goldhaber’s model is considered. Initially, the quantal version is constructed in closest possible analogy with the classical model, and under such assumptions it is shown to give identical results. These points are developed in Sects. II and III. Although there are no practical advantages in using the quantal model, the exercise nevertheless indicates some conceptual difficulties in the application of Goldhaber’s model. Specifically, the assumption of equal oscillator constants for the initial nucleus and the final fragments, necessary to establish equivalence with classical Goldhaber, appears unrealistic. An alternative to Goldhaber’s model which imposes conservation of the number of oscillator quanta is proposed in Sect. IV and in the final section some conclusions are offered.

II. EXPECTATION VALUE OF THE CENTRE-OF-MASS MOMENTUM OPERATOR

We derive here the expression of the expectation value of the centre-of-mass operator $(\sum_i \bar{p}_i)^2$ in a Slater determinant of harmonic oscillator wavefunctions of the form

$$\Phi_A(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_A) = \frac{1}{\sqrt{A!}} \det |\phi_{\alpha_1}(\bar{r}_1) \phi_{\alpha_2}(\bar{r}_2) \dots \phi_{\alpha_A}(\bar{r}_A)|, \quad (2)$$

where $\phi_\alpha(\bar{r})$ is a single-particle wavefunction characterised by the quantum numbers $\alpha \equiv nlm$, $\phi_\alpha(\bar{r}) = R_{nl}(r)Y_{lm}(\theta, \varphi)$ [2]. Alternatively, single-particle wavefunctions of the three-dimensional harmonic oscillator can be expressed in terms of cartesian coordinates, $\phi_\nu(\bar{r}) =$

$H_{n_x}(x)H_{n_y}(y)H_{n_z}(z)$, where $H_n(x)$ is the Hermite polynomial of order n . A single-particle state is now characterised by the quantum numbers $\nu \equiv n_x n_y n_z$ denoting the numbers of oscillator quanta in the x , y and z directions, respectively, which are thus conserved quantities in this basis. The sum of quanta in the three directions corresponds to the major oscillator quantum number of the spherical basis, $n_x + n_y + n_z = n$, and hence n is a good quantum number in both bases.

The expectation value of the centre-of-mass operator contains a one-body and a two-body part,

$$\langle \Phi_A | (\sum_i \bar{p}_i)^2 | \Phi_A \rangle = \langle \Phi_A | \sum_i \bar{p}_i^2 + 2 \sum_{i < j} \bar{p}_i \cdot \bar{p}_j | \Phi_A \rangle. \quad (3)$$

The expectation value of the one-body part follows from the virial theorem for the harmonic oscillator:

$$\langle \Phi_A | \sum_i \bar{p}_i^2 | \Phi_A \rangle = m_n \hbar \omega_A \sum_{nlm} \left(n + \frac{3}{2} \right) k_{nlm}, \quad (4)$$

with m_n the mass of a single particle and $\hbar \omega_A$ the energy required for a single quantal excitation in nucleus A . The summation in (4) extends over *occupied* states nlm only and requires the knowledge of the Slater configuration. A particular Slater configuration is specified by the occupation numbers $\{k_{nlm}\}$ which can be either 0 (empty) or 1 (occupied).

The expectation value of the two-body part between spherical-basis states equals

$$\langle \Phi_A | 2 \sum_{i < j} \bar{p}_i \cdot \bar{p}_j | \Phi_A \rangle = -m_n \hbar \omega_A \sum_{nl n' l'} \langle n' l' || a^\dagger || nl \rangle^2 \sum_{mm' \mu} \begin{pmatrix} l & l' & 1 \\ m & m' & \mu \end{pmatrix}^2 k_{nlm} k_{n' l' m'}, \quad (5)$$

where the bracketed symbol is a Wigner $3j$ -symbol [4]. In (5) also appears the reduced matrix element of a quantum a^\dagger between harmonic oscillator states which is known to be [3]

$$\langle n' l' || a^\dagger || nl \rangle^2 = \begin{cases} (n + l + 3)(l + 1) & \text{if } n' = n + 1, l' = l + 1 \\ (n - l + 2)l & \text{if } n' = n + 1, l' = l - 1 \end{cases}, \quad (6)$$

and zero otherwise.

The final result for the expectation value of the centre-of-mass operator in a Slater determinant of spherical oscillator wavefunctions then follows from (4) and (5):

$$\begin{aligned} \langle \Phi_A | (\sum_i \bar{p}_i)^2 | \Phi_A \rangle &= m_n \hbar \omega_A \left[\sum_{nlm} (n + \frac{3}{2}) k_{nlm} \right. \\ &\quad \left. - \sum_{nl'n'l'} \langle n'l' || a^\dagger || nl \rangle^2 \sum_{mm'\mu} k_{nlm} k_{n'l'm'} \begin{pmatrix} l & l' & 1 \\ m & m' & \mu \end{pmatrix}^2 \right]. \end{aligned} \quad (7)$$

The expectation value of the centre-of-mass operator in a Slater determinant of cartesian oscillator wavefunctions can be derived similarly and yields

$$\begin{aligned} \langle \Phi_A | (\sum_i \bar{p}_i)^2 | \Phi_A \rangle &= m_n \hbar \omega_A \sum_{n_x n_y n_z} \left[n + \frac{3}{2} - (n_x + 1) k_{(n_x+1)n_y n_z} \right. \\ &\quad \left. - (n_y + 1) k_{n_x(n_y+1)n_z} - (n_z + 1) k_{n_x n_y(n_z+1)} \right] k_{n_x n_y n_z}, \end{aligned} \quad (8)$$

where now the Slater configuration is specified in terms of the occupation numbers $k_{n_x n_y n_z}$.

For a ground-state Slater configuration it can be shown that both (7) and (8) reduce to $\frac{3}{2} m_n A \hbar \omega_A$, that is, except for the contribution due to zero-point motion, the one-body part of $(\sum_i \bar{p}_i)^2$ is exactly cancelled by its two-body part. This result is well-known from the nuclear shell model: no spurious centre-of-mass motion occurs in a $0\hbar\omega_A$ valence space of the harmonic oscillator [4]. For an excited Slater configuration the one- and two-body parts of $(\sum_i \bar{p}_i)^2$ do not cancel and a non-zero result is obtained for (7) or (8). The resulting momentum distribution is the main focus of interest in the following. The use of the two different bases (spherical and cartesian) will allow the conservation of either angular momentum projection M or the impuls momentum \bar{P} during the fragmentation process.

III. GOLDBABER'S HYPOTHESIS OF SUDDEN DICHOTOMY

The quantal equivalent of Goldhaber's model is now obtained as follows. Start from an initial nucleus A in its ground-state Slater configuration and split A into two by distributing particles over the fragments A_1 and A_2 , $A_1 + A_2 = A$, *without changing the orbits of the particles*. It is assumed in this section that the oscillator constants of the initial nucleus and the final fragments are the same, $\omega_A = \omega_{A_1} = \omega_{A_2} \equiv \omega$. This assumption is unjustified and will be elaborated upon in the next section. If two fragments with mass A_1 and A_2 are formed, the number of final-state configurations is given by

$$C(A, A_1) = C(A, A_2) = \frac{A!}{A_1!A_2!}, \quad (9)$$

just as it is in the classical case. The symbol $C(A, A_1)$ stands for the number of configurations which are possible if, from the initial A -particle ground state, A_1 are retained and $A - A_1$ are removed. Note that to each possible configuration of fragment A_1 corresponds only one configuration of fragment A_2 and *vice versa*. The momentum distribution of fragment A_1 is then found by averaging over all possible configurations:

$$\langle\langle \Phi_{A_1} | (\sum_i \bar{p}_i)^2 | \Phi_{A_1} \rangle\rangle_{C(A, A_1)} = \frac{1}{C(A, A_1)} \sum_{q=1}^{C(A, A_1)} \langle \Phi_{A_1}^q | (\sum_i \bar{p}_i)^2 | \Phi_{A_1}^q \rangle, \quad (10)$$

where the index q runs over all possible Slater configurations $\Phi_{A_1}^q$.

A delicate problem arises with regard to zero-point motion. It is clear that, in order to recover Goldhaber's classical result, its contribution should not be included since it is a purely quantum mechanical effect. But, even if quantum mechanics is taken into account, there is the following argument not to consider zero-point motion. It can be shown [4] that a ground-state Slater determinant $\Phi_A(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_A)$ can be written as the product of $\Phi'_A(\bar{r}'_1, \bar{r}'_2, \dots, \bar{r}'_A)$, depending on the relative coordinates $\bar{r}'_i \equiv \bar{r}_i - \bar{R}$, and an oscillator wavefunction $\phi_{N=0, L=M=0}(\bar{R})$ in its ground state associated with the centre-of-mass coordinate \bar{R} . The translational invariance of the true nuclear hamiltonian can then simply be restored by replacing $\phi_{N=0, L=M=0}(\bar{R})$ by a plane wave $\exp(i\vec{K} \cdot \bar{R})$ which has no zero-point energy. This shows that the zero-point energy $\frac{3}{2}A\hbar\omega_A$ is an artefact of the use of an oscillator basis which disappears when translational invariance is restored. Although a generalisation of this argument to an excited Slater configuration is not obvious, the argument pleads in favour of the omission of the zero-point energy throughout. Therefore all results shown in the following are obtained after subtraction of the contribution stemming from zero-point motion. We should point out, however, that for the particle numbers considered here [necessarily rather small to limit the number of configurations (9) over which has to be summed] the effect of the zero-point energy is sizeable and its inclusion would have modified the results considerably.

In Fig. 1 the fragment momentum distribution (10) is shown for mass numbers $A = 4, 10$ and 20 of the initial nucleus as a function of the fragment's mass A_1 . The mass numbers of the initial nucleus correspond to closed-shell configurations of identical spinless particles in a harmonic oscillator [5]. These results are compared to Goldhaber's distribution (1), showing a perfect agreement. In fact, one may deduce the relation (valid for closed shells)

$$\langle\langle\Phi_{A_1} | (\sum_i \bar{p}_i)^2 | \Phi_{A_1} \rangle\rangle_{C(A,A_1)} = m_n \hbar \omega \cdot \frac{3}{4} n_{\max} \cdot \frac{A_1 A_2}{A-1}, \quad (11)$$

where n_{\max} is the major oscillator quantum number of the last filled oscillator shell ($n_{\max} = 1, 2, 3$ for $A = 4, 10, 20$, respectively). This results can be understood in the following manner. The mean squared nucleon momentum $\langle\bar{p}^2\rangle$ for a closed-shell oscillator configuration is given by

$$\langle\bar{p}^2\rangle = \frac{1}{A} \sum_{n=0}^{n_{\max}} \frac{1}{2} (n+1)(n+2) \langle\bar{p}^2\rangle_n = \frac{1}{A} \times \begin{cases} \frac{1}{8} (n_{\max}+1)(n_{\max}+2)^2(n_{\max}+3) \\ \frac{1}{8} n_{\max}(n_{\max}+1)(n_{\max}+2)(n_{\max}+3) \end{cases}, \quad (12)$$

where the upper result is exact and the lower one is valid if the zero-point motion contribution to $\langle\bar{p}^2\rangle_n$ is neglected. On the other hand, the particle number A and n_{\max} are related through

$$A = \sum_{n=0}^{n_{\max}} \frac{1}{2} (n+1)(n+2) = \frac{1}{6} (n_{\max}+1)(n_{\max}+2)(n_{\max}+3). \quad (13)$$

This shows that (11) is obtained from the classical result (1) if the mean squared momentum distribution of a nucleon in a harmonic oscillator is taken without its zero-point motion contribution. This establishes complete equivalence between the classical and quantal versions of Goldhaber's fragmentation model.

Figure 2 shows similar results but now for the open-shell configurations $A = 8$ and 13 . Again, a Goldhaber distribution is obtained (dots). The curve is calculated from (11) with a (non-integer) n_{\max} derived from (13). The small deviations are an indication of a shell effect in the sense that an analytic continuation of the relation (13) towards non-integer values of n_{\max} and its use in (11) does not coincide exactly with the numerically calculated distribution (10). However, the deviations are too small to be of any consequence in an application of Goldhaber's fragmentation law to heavy-ion reactions.

IV. CONSERVATION OF ENERGY

Goldhaber's hypothesis of sudden dichotomy implies conservation of number of quanta, $N_1 + N_2 = N$, where N is the total number of quanta of the initial nucleus and N_i are the total numbers of quanta of the fragments. The converse is not necessarily true in the sense that the condition $N_1 + N_2 = N$ is much weaker and allows many more final-state configurations. For a fragmentation $A \rightarrow A_1 + A_2$ this number is given by

$$\sum_{N_1+N_2=N} C_3(A_1, N_1)C_3(A_2, N_2), \quad (14)$$

where $C_3(A, N)$ is the number of Slater determinants for A particles in a harmonic oscillator with a total of N oscillator quanta. This can be computed with a recursive algorithm [6]. Table I illustrates the number of final-state configurations in the specific example of the fragmentation of a nucleus with $A = 56$ particles for a few combinations $A_1 + A_2 = 56$ and compares it to the corresponding number found with Goldhaber's sudden-dichotomy hypothesis. It is clear that many more configurations are sampled with the condition $N_1 + N_2 = N$.

If the mean-field potentials of the initial nucleus A , and of the fragments A_1 and A_2 are the same, $\omega_A = \omega_{A_1} = \omega_{A_2}$, the condition of conservation of the number of quanta is equivalent to that of conservation of energy since all quanta have the same energy. As a corollary one finds that Goldhaber's hypothesis is consistent with conservation of energy (as it is a subcase of $N_1 + N_2 = N$) but only if equal oscillator constants are taken, that is, the sudden-dichotomy hypothesis cannot be generalised to unequal oscillator constants without violation of energy conservation.

This should be considered as a serious drawback of Goldhaber's model. Indeed, a more realistic assumption is to scale the harmonic oscillator constant with the size of the nuclei as $\omega \propto A^{-1/3}$ [7]. This immediately leads to unequal oscillator constants for initial nucleus and final fragments and, under Goldhaber's hypothesis, to energy non-conservation. On the other hand, unequal oscillator constants can be easily accommodated in the $N_1 + N_2 = N$ scenario in which case this condition should be replaced by

$$N_1 \sqrt[3]{\frac{A}{A_1}} + N_2 \sqrt[3]{\frac{A}{A_2}} = N, \quad (15)$$

which implies a number of final-state configurations given by

$$\sum_{N_1 \sqrt[3]{\frac{A}{A_1}} + N_2 \sqrt[3]{\frac{A}{A_2}} = N} C_3(A_1, N_1) C_3(A_2, N_2). \quad (16)$$

Preliminary calculations of the momentum distributions obtained by averaging over all final-state configurations implied by (14) or (16) indicate substantial deviations from Goldhaber's law (1). It should be noted, however, that, even if all final-state configurations in either (14) or (16) are consistent with conservation of energy, they do not necessarily conserve momentum. It will thus be necessary to limit the final-state configurations to those that conserve momentum as well, and to make the corresponding averages of the momentum distribution.

V. CONCLUSION

The objective of this contribution was mainly of a pedagogical kind. It was, first and foremost, the purpose to establish an exact quantum mechanical analogue of Goldhaber's fragmentation model. This was achieved by distributing particles, initially confined by a harmonic oscillator, over two harmonic oscillators. It followed that Goldhaber's momentum distribution law is recovered *exactly* if zero-point motion is neglected. Furthermore, it was shown that Goldhaber's hypothesis of sudden dichotomy is consistent with energy conservation if, *and only if*, the oscillator constants of the initial nucleus and of the final fragments are all equal. Departures from this assumption—which are called for on the basis of standard nuclear physics considerations—cannot be made compatible with the sudden-dichotomy hypothesis.

A possible alternative to Goldhaber's model was formulated which does allow for different oscillator constants while being consistent with energy conservation. However, more work is required to impose conservation of momentum in this case. At the moment it is too early to tell whether these ideas can be usefully applied in the analysis of heavy-ion reaction data.

REFERENCES

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- [4] I. Talmi, *Simple Models of Complex Nuclei* (Harwood, Chur, Switzerland, 1993).
- [5] The expressions (7) and (8) can be readily generalised to particles with spin which would lead to similar conclusions. For the sake of limiting the number of possible configurations $C(A, A_1)$, this is not done here.
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TABLES

TABLE I. Number of final-state configurations in the two-fragment break-up of an $A = 56$ nucleus.

| (A_1, A_2) | (4,52) | (10,46) | (20,36) | (28,28) |
|-----------------|---------------------|---------------------|---------------------|---------------------|
| Goldhaber | $3.6 \cdot 10^5$ | $3.5 \cdot 10^{10}$ | $7.8 \cdot 10^{14}$ | $7.6 \cdot 10^{15}$ |
| $N_1 + N_2 = N$ | $3.4 \cdot 10^{21}$ | $4.9 \cdot 10^{32}$ | $1.3 \cdot 10^{41}$ | $8.1 \cdot 10^{42}$ |

FIGURES

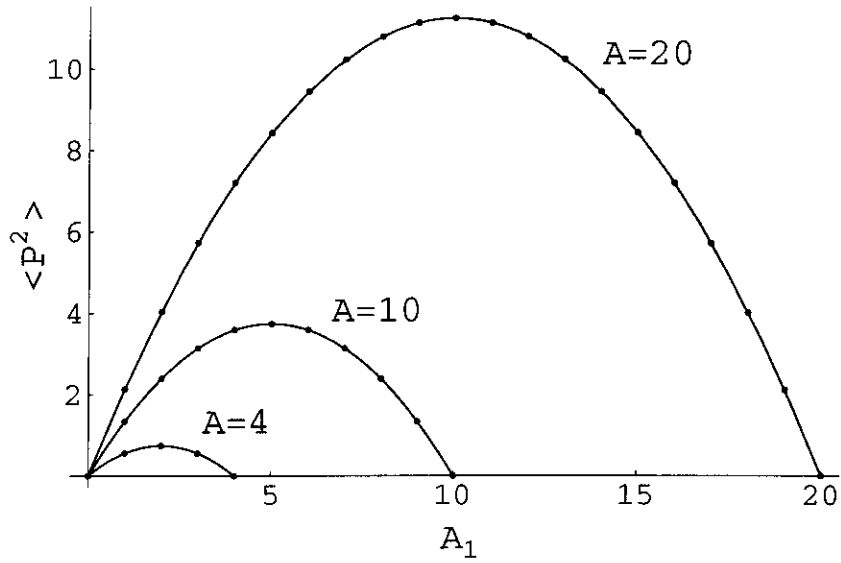


FIG. 1. Fragment momentum distributions (in units $m_n \hbar \omega$) for mass numbers $A = 4, 10$ and 20 of the initial nucleus as a function of the fragment's mass A_1 . The dots are calculated from (10) and the curve is obtained from (11) with $n_{\max} = 1, 2$ and 3 , respectively.

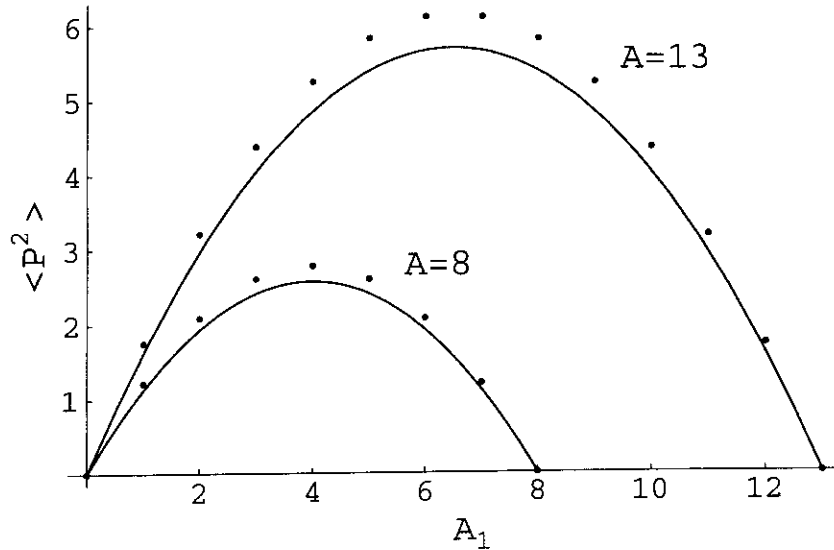


FIG. 2. Fragment momentum distributions (in units $m_n \hbar \omega$) for mass numbers $A = 8$ and 13 of the initial nucleus as a function of the fragment's mass A_1 . The dots are calculated from (10) and the curve is obtained from (11) with $n_{\max} = 1.72594$ and 2.35067 , respectively.

