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Depilating Global Charge From Thermal Black Holes

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Abstract

At a formal level, there appears to be no difficulty involved in introducing a chemical potential for a globally conserved quantum number into the partition function for space-time including a black hole. Were this possible, however, it would provide a form of black hole hair, and contradict the idea that global quantum numbers are violated in black hole evaporation. We demonstrate dynamical mechanisms that negate the formal procedure, both for topological charge (Skyrmions) and complex scalar-field charge. Skyrmions collapse to the horizon; scalar-field charge fluctuates uncontrollably.

1 Introduction

The classic no-hair theorems of black hole physics [1] are commonly interpreted as implying that (non-gauge) global conservation laws are inevitably violated in the process of black hole evaporation [2], even in the absence of any explicit microscopic mechanism for such violation [3, 4, 5, 6, 7, 8]. If black hole evaporation, and more generally quantum gravity does violate global conservation laws then this has many implications for the use of global symmetries in fundamental physics, including baryon and lepton-number violation, axion physics, the use of very light scalars in cosmology, and models of low-scale quantum gravity [3, 4, 6, 7, 9]. But since the no-hair theorems are essentially classical, and they are derived strictly only for stationary geometries, which contain unresolved singularities, perhaps some doubt remains possible.

An interesting alternative perspective is afforded by passing to imaginary time, and considering Euclidean black holes. The Euclidean Schwarzschild metric

$$ds^2 \;=\; \left(1-\frac{2M}{r}\right)d\tau^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2d\Omega^2 \qquad (1)$$

appears to have a singularity at $r = 2M$, but upon substituting $v = \sqrt{8M}\sqrt{r-2M}$ one finds that near this limit

$$ds^2 \;\rightarrow\; \frac{1}{16M^2}v^2d\tau^2+dv^2+(2M)^2d\Omega^2, \qquad (2)$$

and by enforcing the periodicity $\tau/4M = \tau/4M + 2\pi$ we match non-singular polar coordinates. Two consequences of this construction are that the region behind the horizon, $r < 2M$, does not appear; and that we are describing an appropriate, non-singular background for quantum field theory at temperature $T = (8\pi M)^{-1}$ [10].

Indeed, if one considers formally the thermal partition function

$$Z \;=\; \int \mathcal{D}g\mathcal{D}\varphi e^{-S_\beta} \qquad (3)$$

including gravity and generic matter fields φ, and integrating over fields periodic in the imaginary time period β, the Euclidean Schwarzschild solution appears as a stationary point.

Superficially it appears innocuous to add a chemical potential term, for any microscopically conserved global charge, into this definition. But if by doing so we found a meaningful partition function, depending on the chemical potential, we would be able to define a conserved charge on the black hole, and contravene the above-mentioned conventional wisdom [5]. In the remainder of this note we shall consider two different sorts of global charge, topological and ordinary, and demonstrate that in both cases something goes wrong with this attempt – quite different things in the two cases.

2 Skyrmion Collapse

Let us briefly recall the construction of Skyrmions[11]. Consider the non-linear σ model defined by the four-component scalar field φ^a, $a = 0, \ldots, 3$ with $\varphi^2 = 1$; this defines a target space S^3. At spatial infinity the field approaches a uniform value, say $\varphi^0 = 1$, defining the normal vacuum. Field configurations approaching a constant at spatial infinity define maps $S^3 \to S^3$. There is a topological current density

$$j^\mu \;=\; \frac{1}{3\pi}\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{dabc}\varphi^d\partial_\alpha\varphi^a\partial_\beta\varphi^b\partial_\gamma\varphi^c \tag{4}$$

which is identically conserved. The charge obtained by integrating the zero component of this current over space is the degree of the associated mapping. The static Skyrme textures

$$\begin{aligned}\varphi^0(\vec{r}) &= \cos f(r),\\ \vec{\varphi}(\vec{r}) &= \sin f(r)\,\hat{r},\end{aligned} \tag{5}$$

define a class of symmetrical mappings with

$$j^0(r) \;=\; \frac{2}{\pi}f'(\sin f)^2 \;=\; \partial_r\frac{1}{\pi}\left(f - \frac{1}{2}\sin(2f)\right). \tag{6}$$

Regularity at the origin implies $f(0) = n\pi$. If $f(r) \to 0$ at spatial infinity, arriving at the normal vacuum, then the charge is $-n$.

Now let us consider the energetics, first with reference to flat space. If we use the standard non-linear σ model Lagrangian $\mathcal{L} = \sqrt{g}\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^a$ then the energy E_λ of a re-scaled Skyrme texture $f_\lambda(r) = f(\lambda r)$ transforms as $E(\lambda) = \lambda E(1)$. Thus we have charged configurations with arbitrarily small energy, using singular maps whose structure is concentrated near the origin (Derrick's theorem). The standard remedy is to supplement the Lagrangian with the higher-derivative Skyrme term

$$\mathcal{L}_{\rm Skyrme} \propto -\sqrt{g}g^{\mu\rho}g^{\nu\sigma}F^{ab}_{\mu\nu}F^{ab}_{\rho\sigma} \tag{7}$$

with $F^{ab}_{\mu\nu} \equiv \partial_\mu\varphi^a\partial_\nu\varphi^b - \partial_\nu\varphi^a\partial_\mu\varphi^b$. A special property of this term is that it contains no higher than two powers of the time derivative, so that the action including it continues to define a normal dynamical system. The energy associated with this term scales as $E_{\rm Skyrme}(\lambda) = \lambda^{-1}E_{\rm Skyrme}(1)$, oppositely to the minimal term. Thus it is no longer favorable for field configurations to collapse. Indeed, in flat space one finds a non-trivial function f with $n = 1$ that satisfies the equation of motion and minimizes the energy.

In the Euclidean Schwarzschild background, both the topology and the energetics of the situation are altered. The topology of spatial sections (constant τ) is $S^2 \times R^+$ (a sphere times a half-line), and of the entire space-time is $S^2 \times R^2$; in neither case is there a quantized degree defined. Indeed, in the Skyrme texture, the restriction $f(0) = n\pi$ is no longer required by continuity nor (for physical purposes, decisive) by demanding finite energy. This is because the factor multiplying the angular part of the metric does not degenerate at the origin, which is effectively the horizon $r = 2M$.

In more detail, the energy near the horizon for a Skyrme texture parametrized by $f(r) = h(v)$ behaves as

$$E \sim \int_0 dv\; v\;\left(c_1(h')^2 + c_2(\sin h)^2 + c_3(\sin h)^2(h')^2 + c_4(\sin h)^4\right), \tag{8}$$

with numerical constants c_i. (Here and below we refer to the energy conjugate to Schwarzschild time, i.e. the action per unit τ.) The first two terms arise from the minimal σ-model Lagrangian, the second two from the Skyrme term, for radial and angular derivatives respectively. To analyze this, it is convenient to switch to the tortoise coordinate $s \equiv \log 1/v$, so

$$E \sim \int^{\infty} ds \left(c_1 (h')^2 + c_2 e^{-2s} (\sin h)^2 + c_3 (\sin h)^2 (h')^2 + c_4 e^{-2s} (\sin h)^4 \right) . \tag{9}$$

No divergence arises when h approaches an arbitrary constant value at $s \to \infty$, so requiring finite energy does not quantize the charge [12]. Much more is true. Consider a configuration with h varying uniformly from 0 to ζ over the interval $[a, b]$. The energy behaves as

$$E \leq c_1 \frac{\zeta^2}{b-a} + c_2 e^{-2a} + c_3 \frac{\zeta^2}{b-a} + c_4 e^{-2a}. \tag{10}$$

Since this can be made arbitrarily small by taking a and b suitably to infinity, the mass/charge ratio for Skyrmion charge is minimized at zero, by collapsing charge toward the horizon. Equivalently, defining $u = r - 2M$, and taking $h(u) \sim (\log u)^p$ then gives a most singular contribution of the form

$$E \sim \int du \frac{(\log u)^{2p-2}}{u} \tag{11}$$

which is convergent for $p < 1/2$. Thus a logarithmic divergence in the ansatz function $h(r)$ is can occur with finite action. This implies that the mass/charge ratio for Skyrmion charge is minimized at zero.

By way of contrast, in flat space the behavior of the energy near the origin is

$$E \sim \int_0 dr\, r^2 \left(\kappa_1 (f')^2 + \kappa_2 \frac{1}{r^2} (\sin f)^2 + \kappa_3 \frac{1}{r^2} (f')^2 (\sin f)^2 + \kappa_4 \frac{1}{r^4} (\sin f)^4 \right) . \tag{12}$$

Quantization of the charge and non-triviality of the mass/charge ratio are implicit in this form. Indeed, finiteness of the κ_4 term requires $\sin f(0) = 0$; and some simple arguments using Schwarz's inequality with the κ_3 term for small r and the κ_1 term for large r allow us to bound the charge in terms of the energy. It seems worth recording that without the Skyrme term there would be no physical quantization of charge, since the κ_1 and κ_2 terms permit non-zero $\sin f(0)$ at finite energy.

The possible accumulation of charge with zero energy near the horizon reminds us of the increasingly red-shifted image that remains, in real space-time, as a record of whatever has fallen into the hole. On the other hand, such charge leaves no long-time residue at any finite distance from the horizon, and in this sense it does not provide hair.

3 Scalar Field – Limiting Mass/Charge Ratio

Now let us consider the conserved charge associated with the phase symmetry of a complex scalar field φ. For simplicity we specialize to factorized s-wave configurations

$\varphi(\vec{r},\tau) = \eta(v)e^{-in\tau/4M}$, taking account of the periodicity in imaginary time. Near the horizon the charge behaves as

$$
\begin{aligned}
Q &= \int \sqrt{g} g^{\tau\tau}\mathrm{Im}(\varphi^*\partial_\tau\varphi) \\
&= (4\pi(2M)^2)\frac{n}{4M}\int_0 dv\ \frac{1}{v}(\eta(v))^2 \\
&= (4\pi(2M)^2)\frac{n}{4M}\int^\infty ds\ (\eta(s))^2.
\end{aligned} \tag{13}
$$

Near the horizon the energy ($\equiv$ action/unit τ) behaves as

$$
\begin{aligned}
E &= \int \sqrt{g}(g^{\tau\tau}\partial_\tau\varphi^*\partial_\tau\varphi + g^{vv}\partial_v\varphi^*\partial_v\varphi + m^2\varphi^*\varphi) \\
&= (4\pi(2M)^2)\int_0 dv\ \left(\frac{1}{v}\eta^2\left(\frac{n}{4M}\right)^2 + v(\eta')^2 + vm^2\eta^2\right) \\
&= (4\pi(2M)^2)\int^\infty ds\ \left(\left(\frac{n}{4M}\right)^2\eta^2 + (\eta')^2 + e^{-2s}m^2\eta^2\right).
\end{aligned} \tag{14}
$$

By taking $\eta \to \mathrm{const}.s^{-\frac{1}{2}+\varepsilon}$ as $s \to \infty$, with small ε, we can enhance the first term in the integrand relative to the other two. In the limit, the energy and charge become proportional, in the form

$$
\frac{E}{Q} = \frac{1}{4M}, \tag{15}
$$

using $n = 1$ (the most favorable). The peculiar appearance of this equation arises from the fact that we have put the Planck mass $M_{\mathrm{Pl}} = 1$. Restoring units, we have $E/Q = M_{\mathrm{Pl}}^2/4M$. For large black holes, this becomes small. Indeed, the ratio is of order the Hawking temperature. Thus E/Q is equal to the electron mass for $M \sim 2 \times 10^{17}$ gm; for a solar-mass black hole, the ratio is $E/Q \approx 6 \times 10^{-16} m_e$. So for macroscopic holes it becomes energetically favorable to hide quantum numbers in a singular manner near the horizon, similar to what we found in the Skyrme model. In any case, the charge near the horizon undergoes significant fluctuations due to the Hawking temperature. Without further analysis, however, it is not clear how this phenomenon addresses our problem of principle for small holes and heavy charge quanta.

4 Scalar Field – Charge Veto

Referring again to the energy (or action) expression, we see that there is a qualitative difference between the behavior of the $n = 0$ and the $n = 1$ (or higher) sectors near the horizon. In both cases the mass terms become negligible, and we are left to compare actions of the form $\int^\infty ds\kappa_2(\eta')^2$ versus $\int^\infty ds(\kappa_1\eta^2 + \kappa_2(\eta')^2)$. The first form permits asymptotics $\eta \to s^{+\frac{1}{2}-\varepsilon}$ while the second requires the much more stringent condition $\eta \to s^{-\frac{1}{2}-\varepsilon}$, so we might expect that the class of configurations it supports has relatively small measure. To quantify this, consider the functional determinants accompanying integration over these sectors, concentrating on the contribution to the action from an interval $[a, a + L]$ in s. For simplicity, take $\kappa_1 = \kappa_2 = 1$, remove the constant mode, and assume periodic boundary conditions. This is only a reasonable approximation to the contribution for short-wavelength modes, which decouple from other intervals. The relevant determinants involve the product of inverse square roots of the eigenvalues, so for their ratio r we have the infinite product

$$r = \prod_{1}^{\infty} \frac{(\frac{2\pi n}{L})^2}{1 + (\frac{2\pi n}{L})^2}. \tag{16}$$

This evaluates to $\frac{L}{2\pi}(\sinh(\frac{L}{2\pi}))^{-1}$. The low eigenvalues are not to be taken seriously, as already mentioned. But the exponential falloff in L arises from the high eigenvalues, and it indicates that the $n \neq 0$ sectors have zero measure relative to the $n = 0$ sector. As we have seen in the previous section, the charge operator itself contains a term of the same asymptotic form as the κ_1 contribution. Thus a chemical potential can decrease the net numerical value of κ_1, or even reverse its sign. Therefore we arrive at two simple possibilities. If the chemical potential is less than the critical value $1/4M$, it has no effect. If it is greater than this value, the action becomes unbounded below. Neither of these possibilities yields a clear realization of a hairy black hole. Projection on charge eigenstates goes through the intermediary of partition functions with chemical potential [5], and so encounters the same singularity. We cannot, from this analysis, entirely preclude the possibility of some more delicate construction, somehow working with chemical potentials infinitesimally near the critical value, but it would require some additional ideas.

5 Conclusion

Straightforward attempts find thermal hair in canonical realizations of topological and non-topological global charges appear to go awry, for different and perhaps not entirely straightforward reasons. In both cases, special properties of the Euclidean black hole metric allow one to store charge in a singular fashion near the horizon of a large hole, with small cost in energy.

Throughout our discussion we have treated the background geometry as fixed, thus neglecting possibility of "back-reaction". The crucial configurations for our arguments, in the Skyrmion case, involved fields varying rapidly near the horizon. Although their integrated action is small, they induce large local values of the energy-momentum tensor. Thus there is no guarantee that they will correspond to valid configurations of the full quantum gravity theory. Indeed, if they did we would appear to be in danger of providing an infinite entropy for the black hole, since there is at least one low-energy state for each value of the charge. (This is related to the phenomena discussed by 't Hooft [13].)

Finally, let us venture a heuristic interpretation of the charge veto. The Schwarzschild temperature $(8\pi M)^{-1}$ is appropriate for static frames far from the hole, but at finite distances the effective local temperature is $\sqrt{g^{\tau\tau}}$ larger, which diverges near the horizon. Thus the chemical potential becomes negligible compared to the local temperature, and loses its influence upon configurations concentrated near the horizon, unless it triggers an instability.

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References

[1] J. D. Bekenstein, Phys. Rev. D **5**, 1239 (1972); Phys. Rev. D **5**, 2403 (1972); Phys. Rev. Lett. **28**, 452 (1972).
R. H. Price, Phys. Rev. D **5**, 2419 (1972).
C. Teitelboim, Phys. Rev. D **5**, 2941 (1972).
S. L. Adler and R. B. Pearson, Phys. Rev. D **18**, 2798 (1978).

[2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[3] D. Toussaint, S. B. Treiman, F. Wilczek and A. Zee, Phys. Rev. D **19**, 1036 (1979).

[4] L. M. Krauss and F. Wilczek, Phys. Rev. Lett. **62**, 1221 (1989);
M. G. Alford, J. March-Russell and F. Wilczek, Nucl. Phys. B **337**, 695 (1990).

[5] S. R. Coleman, J. Preskill and F. Wilczek, Phys. Rev. Lett. **67**, 1975 (1991);
Nucl. Phys. B **378**, 175 (1992) [arXiv:hep-th/9201059].

[6] M. Kamionkowski and J. March-Russell, Phys. Rev. Lett. **69**, 1485 (1992) [arXiv:hep-th/9201063]; Phys. Lett. B **282**, 137 (1992) [arXiv:hep-th/9202003].

[7] R. Holman, *et al.*, Phys. Lett. B **282**, 132 (1992) [arXiv:hep-ph/9203206];
Phys. Rev. Lett. **69**, 1489 (1992).

[8] J. A. Wheeler, "Black Hole Chemical Potential And Particle Conservation," *In *Batavia 1977, Proceedings, Unification Of Elementary Forces and Gauge Theories*, 505-507.*

[9] P. C. Argyres, S. Dimopoulos and J. March-Russell, Phys. Lett. B **441**, 96 (1998) [arXiv:hep-th/9808138]. N. Arkani-Hamed, *et al.*, Nucl. Phys. B **567**, 189 (2000) [arXiv:hep-ph/9903224].

[10] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).

[11] T. H. Skyrme, Proc. Roy. Soc. Lond. A **260**, 127 (1961);
Nucl. Phys. **31**, 556 (1962).

[12] H. Luckock and I. Moss Phys. Lett. B **176**, 341 (1986).

[13] G. 't Hooft, Nucl. Phys. Proc. Suppl. **43**, 1 (1995).