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A sum rule from the shape function

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Abstract

We present a sum rule relating the electron energy spectrum to the hadron mass distribution in semileptonic $b \to u$ decays close to threshold. The relation found is free from non-perturbative effects and the theoretical error is expected to be $O(5\%)$. An experimental confirmation of this prediction can provide a check of the basic assumptions at the root of the theory of the shape function.

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In this note we present a sum rule which can be compared directly with data on the semi-leptonic decay

$$B \to X_u + l + \nu. \tag{1}$$

The comparison allows a verification of the theory of the structure function for the heavy flavors, usually called the shape function [1, 2]. The sum rule involves the electron spectrum and the integrated hadron mass distribution and reads

$$\frac{d\Gamma_B}{dx_e} = C\left(\alpha_S\right) \int_0^{m_B\sqrt{1-x_e}} \frac{d\Gamma_B}{dm_X}\, dm_X \;+\, O\left(\frac{\Lambda_{QCD}}{m_B}\right), \tag{2}$$

where the coefficient function is, to one-loop,

$$C\left(\alpha_S\right) = 2\left[1 + \frac{C_F\,\alpha_S}{2\pi}\,\frac{97}{72} + O\left(\alpha_S^2\right)\right]. \tag{3}$$

The adimensional electron energy is defined, as usual, as

$$x_e \equiv \frac{2E_e}{m_B} \qquad\qquad (0 \le x_e \le 1)\,. \tag{4}$$

Relation (2) holds in the region

$$1 - x_e \;\sim\; \frac{\Lambda_{QCD}}{m_B}. \tag{5}$$

Assuming $\Lambda_{QCD} \sim 300$ MeV, this means [1]

$$x_e \sim 0.94. \tag{7}$$

The condition (5) corresponds to a final invariant hadronic mass in the region [1, 2]

$$m_X \sim \sqrt{\Lambda_{QCD}\, m_B}, \tag{8}$$

[1]In practise, to kill the large $b \to c$ background, one has to satisfy the experimental constraint [3]

$$x_e > \frac{m_B^2 - m_D^2}{2m_B} \simeq 0.88. \tag{6}$$

i.e., $m_X \sim 1.3$ GeV for $\Lambda_{QCD} \sim 300$ MeV.

As eq.(2) states, the sum rule holds only if the upper invariant mass m_{cut} for the hadron distribution and the electron energy are related by

$$m_{cut} = m_B \sqrt{1 - x_e}. \tag{9}$$

A typical value for the experimental analysis is $m_{cut} = 1.6$ GeV, for which $x_e = 0.91$ or $E_e = 2.4$ GeV. One can actually decrease the cut mass to something like $m_{cut} = 1.3$ GeV, for which $x_e = 0.94$ or $E_e = 2.48$ GeV (the endpoint is at $E_e^{\max} = 2.64$ GeV).

The coefficient function has the numerical value

$$C\left(\alpha_S\right) \cong 2.114 \tag{10}$$

for $\alpha_S \equiv \alpha_S\left(m_B\right) = 0.2$. Taking instead for example, $\alpha_S \equiv \alpha_S\left(\mu = m_B/2\right) = 0.28$, the coefficient function rises to 2.16, a 2% variation: this can be taken as a crude estimate of the higher order terms, $\sim \left(\alpha_S/\pi\right)^2$. In general, the main corrections to eq.(2) originate from the so-called higher-twist effects, related to the matrix elements of power suppressed operators. Their size is [1, 2], as anticipated,

$$\text{(higher twist effects)} \sim \frac{\Lambda_{QCD}}{m_B} \sim 5\%. \tag{11}$$

The proof of eq.(2) is the following. Any distribution in the threshold region (8) satisfies the factorization formula (for a derivation see, for example, [2])

$$d\Gamma_B = \int_0^{m_B} dm_* \varphi\left(m_*\right) d\Gamma_* + O\left(\frac{\Lambda_{QCD}}{m_B}\right), \tag{12}$$

where $d\Gamma_*$ is the distribution for an hypothetical heavy quark with mass m_* and $\varphi\left(m_*\right)$ is the shape function in the notation of ref.[2].

The electron spectrum close to the endpoint is at tree-level

$$\begin{aligned} \frac{1}{\Gamma_0}\frac{d\Gamma_*}{dx} &= 2x^2\left(3 - 2x\right)\,\theta\left(1 - x\right) \\ &\cong 2\left[1 - 3\left(1 - x\right)^2\right]\,\theta\left(1 - x\right), \end{aligned} \tag{13}$$

where

$$x \equiv \frac{2E_e}{m_*} \qquad \left(0 \le x \le 1\right). \tag{14}$$

and[2]

$$\Gamma_0 \equiv \frac{G_F^2 m_b^5 |V_{ub}|^2}{192\pi^3}\,. \tag{15}$$

The term quadratic in $1-x$ in the last member in eq.(13) can be neglected because [3]

$$(1-x)^2 \sim \left(\frac{\Lambda_{QCD}}{m_B}\right)^2. \tag{16}$$

Inserting the r.h.s. of eq.(13) into eq.(12), one obtains

$$\begin{aligned}\frac{1}{\Gamma_0}\frac{d\Gamma_B}{dx_e} &= 2\int_{2E_e}^{m_B} dm_*\varphi\left(m_*\right)\frac{m_B}{m_*} \\ &= 2\int_{2E_e}^{m_B} dm_*\varphi\left(m_*\right) + O\left(\frac{\Lambda_{QCD}}{m_B}\right),\end{aligned} \tag{17}$$

where in the last line eq.(5) has been used. An analogous factorization of the hadron mass distribution gives

$$\frac{d\Gamma_B}{dm_X^2} = \int_0^{m_B} dm_*\varphi\left(m_*\right)\frac{d\Gamma_*}{dm_X^2}. \tag{18}$$

At the tree level, the parton distribution reads

$$\frac{1}{\Gamma_0}\frac{d\Gamma_*}{dm_X^2} = \delta\left(m_X^2 + 2E_X\left(m_* - m_B\right)\right), \tag{19}$$

where E_X is the final hadronic energy. The latter has a range, for fixed m_X^2,

$$m_X \le E_X \le \frac{m_B}{2}\left(1 + \frac{m_X^2}{m_B^2}\right). \tag{20}$$

Configurations with $E_X \gtrsim m_X$ correspond to a final hadronic system X essentially at rest and do not have the typical logarithmic enhancement in the infrared region [4]. Because of eq.(8), we then set

$$E_X \sim \frac{m_B}{2}. \tag{21}$$

[2]The actual value of the heavy mass entering inside Γ_0 is irrelevant, as this dependence cancels in taking the ratio of the widths (see later).

[3]An eventual linear term in $1-x$ could have been neglected as well, as this term would give a contribution $\sim \Lambda_{QCD}/m_B$.

[4]This reasoning is not very rigorous. The main justification for neglecting this region is that infrared logarithms turn out to cancel in the coefficient function $C\left(\alpha_S\right)$ (for a general discussion on this point see ref. [4]).

Integrating over m_X^2 we obtain for the cumulative hadron mass distribution

$$\frac{1}{\Gamma_0}\int_0^{m_{cut}}\frac{d\Gamma_B}{dm_X}dm_X=\int_{m_B\left(1-m_{cut}^2/m_B^2\right)}^{m_B}dm_*\varphi\left(m_*\right). \tag{22}$$

Comparing the expressions for the two distributions and assuming eq.(9), we obtain the tree-level approximation to eq.(2), i.e. the equation with $\alpha_S=0$ on the r.h.s.. The inclusion of the correction of order α_S is straightforward and can be done extracting the relevant formulas from ref.[5].

Let us now comment on the result represented by eq.(2). The dependence on the non-perturbative effects related to Fermi motion — described by the shape function — cancels in taking the ratio of the widths. Cancellation occurs also for the CKM matrix element $|V_{ub}|^2$ and for the heavy mass power m_b^5, both entering inside Γ_0. It is the cancellation of all these unknown or poorly known quantities which makes the sum rule rather accurate.

An equation similar to (2), with the replacement $m_B\rightarrow m_{\Lambda_b}$, applies also to the hyperion decay [5]

$$\Lambda_b\rightarrow X_u+l+\nu. \tag{23}$$

The experimental analysis is more difficult in this case because hyperion production cross sections are generally much smaller than the corresponding mesonic ones. The relevance of a combined analysis is that higher twist corrections are expected to be different in the two cases ((1) and (23)), because for example the B-meson has $1/m_B$ spin-dependent corrections, which vanish instead in the Λ_b case [6].

In general, we would like to stress the simplicity of the result (2). The result is however non-trivial, as the presence of non-vanishing perturbative corrections are higher-twist effects indicate. Using only a general parametrization of the hadronic tensor describing the decay (1), it does not seem possible to derive eq.(2). Let us remark that the prediction (2) does not involve neither a parametrization of the shape function nor an evaluation of the Mellin moments of the distributions — the latter requiring a knowledge of the spectra in the whole kinematical range. On the experimental side, both the rates entering eq.(2) can be easily measured — they are actually measured — because the background coming from $b\rightarrow c$ transitions can be eliminated [3, 7][6]. The sum rule (2) allows also a consistency check between the electron

[5]The shape function is different in the two cases, $\varphi_B\neq\varphi_{\Lambda_b}$.

[6]See footnote 1.

spectrum computed inside the AC^{2}M^2 model [3] and the hadron mass distribution computed inside the shape function theory [7]. Both these models are currently used for the experimental determination of $|V_{ub}|$.

To conclude, the experimental confirmation of eq.(2) can provide a check at the 5% level of the theory of the shape function and of its basic assumptions: infinite mass limit for the beauty quark, infinite energy limit for the light final quark and local parton-hadron duality. Finally, a comparison with accurate experimental data can provide an estimate of the higher-twist effects.

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