DEPENDENCE OF THE BEAM-BEAM SYNCHROTRON RADIATION ON THE TRANSVERSE DIMENSIONS FOR GAUSSIAN BEAMS.

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1. General formulae

The LEP note 122[1] gives the beam-beam synchrotron radiation for a round Gaussian beam and other simplified distributions. The calculations for the more general case of a real two-dimensional Gaussian involves finding an analytical solution for a 4-dimensional integral, which is obtained below, starting from two different but equivalent integral expressions for the electric field.

The general formulae would seem to be valid as the independent analytical expressions for the integral agreed and the correct value was found for the limiting case of a round beam.

The integral I_{2bb} is defined as follows :

$$I_{2bb} = \int \frac{ds}{\rho^2} \qquad 1.1$$

This differs from the definition of I_{2bb}[1] but the results are compatible (see Appendix).

Let us assume for the transverse distributions that :

$$K(x,y,\sigma_x,\sigma_y) = \frac{N}{2\pi\cdot\sigma_x\,\sigma_y}\, e^{-\left(\frac{x^2}{2\sigma_x^2} \quad \frac{y^2}{2\sigma_y^2}\right)} \qquad 1.2$$

for the general gaussian beam, and that

$$K(x,y,\sigma,\sigma) = \frac{N}{2\pi\,\sigma^2}\, e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{N\cdot e^{-\frac{r^2}{2\sigma^2}}}{2\pi\,\sigma^2} \qquad 1.3$$

for the round gaussian beam.

The result (see Appendix) is

$$I_{2bb}\,(\sigma_x,\sigma_y) = \frac{8\, r_e^{\,2}\, N^2}{\sqrt{\pi}\,\gamma^2\,\sigma_x\,\sigma_y\,\sigma_s}\,\frac{r}{\sqrt{D}}\,\arctan\left(\frac{\sqrt{D}}{Q}\right) \qquad 1.4$$

where

$$r = \frac{\sigma_y}{\sigma_x} \qquad 0 < r < \infty \qquad 1.5$$

$$D = 3r^4 - 10r^2 + 3 \qquad 1.6$$

$$Q = 3r^2 + 8r + 3 \qquad 1.7$$

Equation 1.4 is valid for

$$D > 0 \qquad 1.8$$

namely

$$r < \frac{1}{\sqrt{3}} \quad \text{or} \quad r > \sqrt{3} \qquad 1.9$$

For

$$\frac{1}{\sqrt{3}} < r < \sqrt{3}$$

the factor

$$g = \frac{1}{\sqrt{D}}\,\arctan\left(\frac{\sqrt{D}}{Q}\right)$$

in equation 1.4 is replaced by

$$g = \frac{1}{i\sqrt{|D|}}\,\arctan\frac{i\sqrt{|D|}}{Q} \qquad 1.10$$

This can be written as

$$g = \frac{1}{\sqrt{|D|}}\,\operatorname{arctanh}\frac{\sqrt{|D|}}{Q} \qquad 1.11$$

Using the relation

$$\operatorname{arctanh}(x) = \frac{1}{2}\,\ln\left(\frac{1+x}{1-x}\right)$$

equation 1.4 for $D < 0$ can be written as follows :

$$I_{2bb}\,(\sigma_x,\,\sigma_y) = \frac{4\, r_e^{\,2}\, N^2}{\sqrt{\pi}\,\gamma^2\,\sigma_x\,\sigma_y\,\sigma_s}\,\frac{r}{\sqrt{|D|}}\,\ln\left(\frac{1+\frac{\sqrt{|D|}}{Q}}{1-\frac{\sqrt{|D|}}{Q}}\right) \qquad 1.4^*$$

To show the influence of the r parameter we will consider the ratio η between the I_{2bb} values of the general and the round beam case. From 1.4, 1.4^*, 1.6, 1.7, we have :

$$\eta = \frac{I_{2bb}(\sigma_x, \sigma_y)}{I_{2bb}(\sigma, \sigma)} = \frac{4\ \sigma^2}{\sigma_x\ \sigma_y\ \ln(\frac{4}{3})} \frac{r}{\sqrt{D}} \arctan \frac{\sqrt{D}}{Q} \qquad 1.12$$

We will look at η for two different hypotheses.

2. Constant cross section

We suppose that r is variable and that

$$\sigma_x\ \sigma_y = \sigma^2 \qquad 2.1$$

This case is useful when the cross section, at the intersection point is given. We must adjust the values of β_x, β_y, the coupling parameter and the emittance in order to get it.

From equation 1.12 we can obtain easily

$$\eta_1\ (\sigma_x\ \sigma_y = \sigma^2) = \frac{4}{\ln\ (\frac{4}{3})}\ r\ \frac{\arctan\left(\frac{\sqrt{D}}{Q}\right)}{\sqrt{D}} \qquad 2.2$$

A graph of η_1 is shown in Figure 1.

From equations 1.6 and 1.7 we have for small r

$$\lim_{r \to 0} \eta\ (\sigma_x\ \sigma_y = \sigma^2) = \frac{\frac{4}{\sqrt{3}}}{\ln(\frac{4}{3})}\ r \arctan\left(\frac{1}{\sqrt{3}}\right) = 4.20325\ r \qquad 2.3$$

Equation 2.3 shows that with the same cross section, and therefore luminosity, a flat beam will give lower radiation.

Equation 2.2 is valid for all values of r. From the definitions given by 1.5, 1.6 and 1.7

$$\eta_1(r) = \eta_1\ (\frac{1}{r}) \qquad 2.4$$

The significance of equation 2.4 is that the synchrotron radiation does not change if the observer turns his head by $\frac{\pi}{2}$. Also the equation shows that η_1 must be flat around r=1 as can be seen in Figure 1.

3. Natural coupling cross section variation

This case applies when for fixed ε, β_x and β_y we would like to observe the influence of the coupling on the beam beam synchrotron radiation. If the dispersion vanishes the radial and vertical beam sizes are given by the well known formulae

$$\sigma_x = \sqrt{\frac{\varepsilon\beta_x}{1 + r_c^2}}$$
$$\sigma_y = \sqrt{\frac{\varepsilon\beta_y \cdot r_c^2}{1 + r_c^2}} \qquad 3.1$$

where ε is the uncoupled radial emittance and where r_c, less than 1, is the coupling parameter dependent on both the coupling strength (skew quadrupoles and solenoid fields) and the difference between the radial and vertical betatron tune. From 3.1 we obtain

$$r = \frac{\sigma_y}{\sigma_x} = r_c \sqrt{\frac{\beta_y}{\beta_x}} < \sqrt{\frac{\beta_y}{\beta_x}} \qquad 3.2$$

and

$$\sigma_x \sigma_y = \varepsilon \sqrt{\beta_x \beta_y} \frac{r_c}{1 + r_c^2} \qquad 3.3$$

Equation 3.3 has a maximum at $r_c=1$, given by

$$\max(\sigma_x \sigma_y) = \sigma_x(r_c=1) \cdot \sigma_y(r_c=1) = \sqrt{\frac{\varepsilon \cdot \beta_x}{2}} \sqrt{\frac{\varepsilon \cdot \beta_y}{2}} = \frac{\varepsilon}{2}\sqrt{\beta_x \beta_y} \qquad 3.4$$

To have a meaningful formula for η (1.12) we assume

$$\sigma^2 = \max(\sigma_x \sigma_y) = \frac{\varepsilon}{2}\sqrt{\beta x \beta y} \qquad 3.5$$

From equations 1.4, 3.2, 3.3 and 3.5 we have

$$\eta_2 = \frac{2\left(1 + \frac{\beta_x}{\beta_y} r^2\right)}{\ln\left(\frac{4}{3}\right)} \sqrt{\frac{\beta_y}{\beta_x}} \frac{\arctan\left(\frac{\sqrt{D}}{Q}\right)}{\sqrt{D}} \qquad 3.6$$

$$0 < r < \sqrt{\frac{\beta_y}{\beta_x}} \qquad 3.7$$

From 2.2, 3.6 it can be seen that at the maximum possible value of r_c, the η_1, η_2 values coincide (see fig. 1).

Also, when r_c and hence the cross section varies, η_2 gives the beam beam synchrotron radiation of the elliptical beam, as a fraction of the beam beam synchrotron radiation of the round beam, whose cross section is the maximum possible value of the elliptical beam.

In the limit of vanishing r we obtain :

$$\lim_{r \to 0} \eta_2 = \frac{\frac{2}{\sqrt{3}} \sqrt{\frac{\beta_y}{\beta_x}}}{\ell n \left(\frac{4}{3}\right)} \arctan\left(\frac{1}{\sqrt{3}}\right) = 2.10163 \sqrt{\frac{\beta_y}{\beta_x}} \qquad 3.8$$

The above equation shows that for small r, η_2 does not depend on r, that is on the vertical dimension.

This result is clearly related to the other limit of η_1 for vanishing r. In this case we had vanishing beam beam synchrotron radiation for a constant cross section and constant density, while now we have finite beam beam synchrotron radiation for vanishing cross section and infinite density.

The LEP case is :

$$\sqrt{\frac{\beta_y}{\beta_x}} = \frac{1}{4} \qquad 3.9$$

From equation 3.8 we have the limitation that

$$0 < r < 0.25 \qquad 3.10$$

At the maximum value of r we have from equation 3.6

$$\eta_2 \ (0.25) = \eta_1 \ (0.25) = 0.651$$

while from equation 3.7 the value for vanishing r is

$$\eta_2 \ (0) \ = \ 0.5254$$

There is a minimum value of

$$\eta_{2\ min} \ = \ 0.4887$$

around

$$r = 0.068 \qquad r_c = 0.272$$

We can conclude that for every coupling, I_{2bb} of LEP is between 49 % and 65 % of a value that a round beam with the same cross section of the full coupling would have.

References

1) A. Hofmann, E. Keil : LEP note 122 "Effects of the Beam-beam Synchrotron Radiation."

2) M. Bassetti, G.A. Erskine : CERN ISR-TH/80-06 "Closed Expression for the Electrical Field of a Two Dimensional Gaussian Charge."

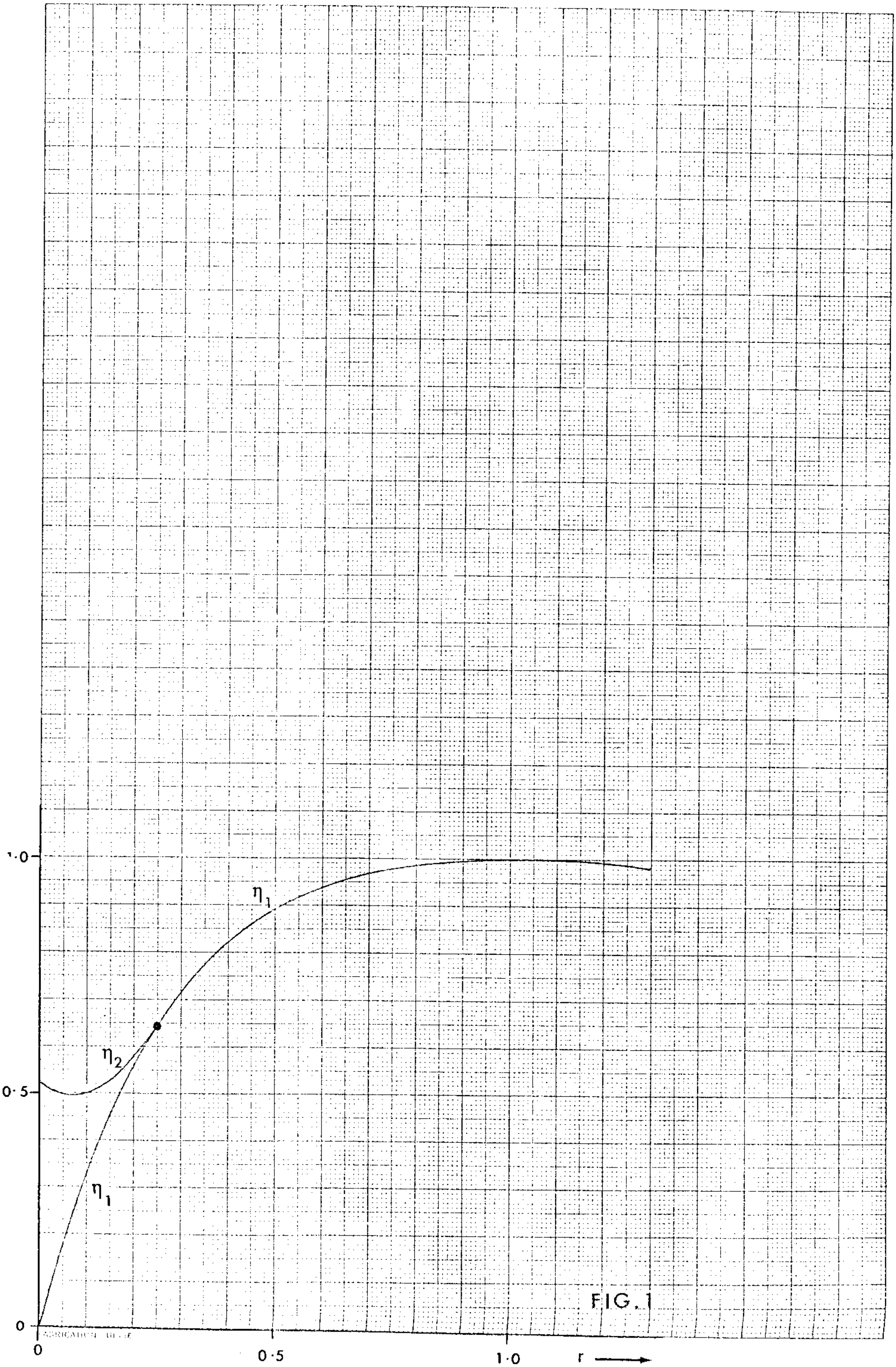

FIG. 1

APPENDIX

Introduction

The purpose of this calculation is to obtain the average value of the energy radiated by a particle in one beam passing through the other.

We suppose that both the beams have the same gaussian distribution in all three dimensions.

The effect of the longitudinal and transverse distributions can be computed separately (we assume that the beta function variations are negligible) and at first it is only necessary to calculate the average square kick undergone by a particle passing through the opposite beam.

The relation between the average square kick and the integral

$$I_{2bb} = \int \frac{ds}{\rho^2(s)}$$

can be found in the following way. The total Δ_x', Δ_y' undergone by a particle can be written as

$$\Delta_x' = \int_{-\infty}^{+\infty} \frac{d^2x}{ds^2}\, ds = \int_{-\infty}^{+\infty} \frac{ds}{\rho_x(s)} = \Delta_x' \cdot \int_{-\infty}^{+\infty} \frac{e^{-\frac{2s^2}{\sigma_s^2}}}{\sqrt{\frac{\pi}{2}}\;\sigma_s}\, ds$$

where σ_s has been replaced by $\frac{\sigma_s}{2}$ in order to take into account the fact that the two beams move against each other, and that each particle sees the length of the other beam reduced by a factor 2.

From this last relation we deduce that :

$$\frac{1}{\rho_x(s)} = \Delta_x' \cdot \frac{e^{-\frac{2s^2}{\sigma_s^2}}}{\sqrt{\frac{\pi}{2}}\;\sigma_s}$$

and therefore

$$I_2 = \int_{-\infty}^{+\infty} \frac{ds}{\rho^2(s)} = \int_{-\infty}^{+\infty} \left(\frac{ds}{\rho_x^{\,2}(s)} + \frac{ds}{\rho_y^{\,2}(s)} \right) = (\Delta_x'^2 + \Delta_y'^2) \int_{-\infty}^{+\infty} \frac{e^{-\frac{4s^2}{\sigma s^2}} ds}{\frac{\pi}{2}\sigma^2} = \frac{(\Delta x'^2 + \Delta y'^2)}{\sqrt{\pi}\,\sigma_s}$$

and for the average on the transverse plane

$$\left\langle \int \frac{ds}{\rho^2(s)} \right\rangle = \frac{\langle \Delta_x'^2 + \Delta_y'^2 \rangle}{\sqrt{\pi}\quad \sigma_s}$$

1. Definition of the 4-dimensional integral

We start from the formulae given in ref[2)] A 1.1

$$\Delta_x' = \frac{2r_e\ N}{\gamma\ \sigma_x^{\,2}} \cdot \left(\frac{1}{1 - r^2} \right) x \int_r^1 e^{\frac{1}{2\sigma_x^{\,2}(1-r^2)}\left[(t^2-1)x^2 + (1 - \frac{1}{t^2})\,y^2\right]} dt$$

$$\Delta_y' = \frac{2r_e N}{\gamma\ \sigma_x^{\,2}} \cdot \left(\frac{1}{1 - r^2} \right) y \int_r^1 e^{\frac{1}{2\sigma_x^{\,2}(1-r^2)}\left[(t^2-1)x^2 + (1 - \frac{1}{t^2})y^2\right]} \frac{dt}{t^2}$$

and from a normalized gaussian distribution

$$K(x,y) = \frac{e^{-\left(\frac{x^2}{2\sigma_x^{\,2}} + \frac{y^2}{2\sigma_y^{\,2}}\right)}}{2\pi\ \sigma_x\ \sigma_y} \qquad \text{A 1.2}$$

Using previous definitions we can write

$$I = \langle \Delta_x'^2 + \Delta_y'^2 \rangle = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\ K(x,y)\ (\Delta_x'^2 + \Delta_y'^2) \qquad \text{A 1.3}$$

From equations A1.1, A1.2 and A1.3 we obtain

$$I = \frac{2\, r_e^2\, N^2}{\pi\, \gamma^2\, \sigma_x^5\, \sigma_y (1-r^2)^2} \int_r^1 dz \int_r^1 dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left(x^2 + \frac{y^2}{z^2 t^2}\right) e^{\frac{1}{2\sigma_x(1-r^2)}\left\{x^2(z^2+t^2+r^2-3)+y^2\left[3-\left(\frac{1}{z^2}+\frac{1}{t^2}+\frac{1}{r^2}\right)\right]\right\}}$$

1st and 2nd integration

The x and y integrals can be done easily by using the following relationships

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}} \qquad \text{A2.1}$$

$$\int x^2\, e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \qquad \text{A2.2}$$

Hence

$$I = \frac{4 r_e^2\, N^2}{\gamma^2\, \sigma_x\, \sigma_y} \int_r^1 z\, dz \int_r^1 t\, dt \left\{ \frac{1}{\left[z^2+t^2+z^2t^2\left(\frac{1}{r^2}-3\right)\right]^{1/2} \left[3-r^2-z^2-t^2\right]^{3/2}} + \frac{1}{\left[z^2+t^2+z^2t^2\left(\frac{1}{r^2}-3\right)\right]^{3/2} \left[3-r^2-z^2-t^2\right]^{1/2}} \right\} \qquad \text{A2.3}$$

The above formula can be simplified by making the following substitutions:

$$\begin{cases} x = t^2 \\ y = z^2 \end{cases} \qquad \text{A3.1}$$

and by defining

$$\begin{cases} p = \frac{1}{r^2} - 3 \\ q = 3 - r^2 \end{cases} \qquad \text{A3.2}$$

Then we have

$$I = \frac{r_e^2 N^2}{\gamma^2 \sigma_x \sigma_y} \int_{r^2}^{1} dx \int_{r^2}^{1} dy \left\{ \frac{1}{[x+y+pxy]^{\frac{1}{2}} [q-(x+y)]^{3/2}} + \frac{1}{[x+y+pxy]^{3/2} [q-(x+y)]^{\frac{1}{2}}} \right\} \qquad \text{A3.3}$$

3rd integration

We are now going to integrate over y, by defining

$$\begin{aligned} a &= x \\ b &= px + 1 \\ f &= q - x \\ g &= -1 \end{aligned} \qquad \text{A4.1}$$

we obtain

$$I = \frac{r_e^2 N^2}{\gamma^2 \sigma_x \sigma_y} \int_{r^2}^{1} dx \int_{r^2}^{1} dy \left\{ \frac{1}{(a+by)^{\frac{1}{2}} (f+gy)^{3/2}} + \frac{1}{(a+by)^{3/2} (f+gy)^{\frac{1}{2}}} \right\} \qquad \text{A4.2}$$

Using the indefinite integral

$$\int \frac{dy}{(a+by)^{\frac{1}{2}} (f+gy)^{3/2}} = - \frac{2}{(ag-bf)} \left(\frac{a + by}{f + gy} \right)^{\frac{1}{2}} \qquad \text{A4.3}$$

we obtain

$$I = \frac{2r_e^2 N^2}{\gamma^2 \sigma_x \sigma_y} \int_{r^2}^{1} dx \frac{1}{(ag-bf)} \left[\left(\frac{f + g}{a + b} \right)^{\frac{1}{2}} - \left(\frac{a + b}{f + g} \right)^{\frac{1}{2}} \right] + \left[\left(\frac{a + br^2}{f + gr^2} \right)^{\frac{1}{2}} - \left(\frac{(f + gr^2)}{a + br^2} \right)^{\frac{1}{2}} \right] \qquad \text{A4.4}$$

If we put

$$I = I_1 + I_2 \qquad \text{A4.5}$$

and by using the same method for I_2 as for I_1 it is possible to show that

$$I_2 = I_1 \qquad \text{A4.6}$$

and therefore we have

$$I = 2I_1 = \frac{4r_e^2 N^2}{\gamma^2 \sigma_x \sigma_y} \int_{r^2}^{1} \frac{1}{(ag - bf)} \left[\left(\frac{f + g}{a + b} \right)^{\frac{1}{2}} - \left(\frac{a + b}{f + g} \right)^{\frac{1}{2}} \right] dx \qquad \text{A4.7}$$

Substituting back for a, b, g and f, we have

$$I = \frac{4r_e^2 N^2}{\gamma^2 \sigma_x \sigma_y} \int_{r^2}^{1} \frac{1}{px^2 - pqx - q} \left[\left(\frac{A - x}{Bx + 1} \right)^{\frac{1}{2}} - \left(\frac{Bx + 1}{A - x} \right)^{\frac{1}{2}} \right] dx \qquad \text{A4.8}$$

$$\begin{cases} A = q - 1 = 2 - r^2 \\ B = p + 1 = \dfrac{1}{r^2} - 2 \end{cases} \qquad \text{A4.9}$$

4th integration

At this point it is convenient to change the integration variable by putting

$$t^2 = \frac{A - x}{Bx + 1} \qquad \text{A5.1}$$

then

$$I = \frac{8r_e^2 N^2}{\gamma^2 \sigma_x \sigma_y} \int_{r}^{1} \frac{(1 - t^2)}{t^4 + bt^2 + 1} dt \qquad \text{A5.2}$$

where

$$b = 3\left(\frac{1}{r^2} + r^2 \right) - 8 \qquad \text{A5.3}$$

Defining

$$\begin{cases} h = (b^2 - 4)^{\frac{1}{2}} \\ g = \dfrac{b}{2} + \dfrac{h}{2} \\ f = \dfrac{b}{2} - \dfrac{h}{2} \end{cases} \qquad \text{A5.4}$$

we can write

$$I = \frac{1}{h}\left(\frac{16\pi\ r_e^2\ N^2}{\gamma^2\ \sigma_x\ \sigma_y}\right)\int_r^1 \left(\frac{1 - t^2}{t^2 + f} - \frac{1 - t^2}{t^2 + g}\right) dt \tag{A5.5}$$

This can be integrated. Recalling the fact that

$$I_{2bb} = \frac{I}{\sqrt{\pi} \cdot \sigma_s}$$

we have finally :

$$I_{2bb} = \frac{8\ r_e^2\ N^2}{\sqrt{\pi}\ \gamma^2\ \sigma_x\ \sigma_y\ \sigma_s} \cdot \frac{r}{\sqrt{D}}\ \arctan \frac{\sqrt{D}}{Q} \tag{A5.6}$$

where

$$D = 3\ r^4 - 10\ r^2 + 3 \tag{A5.7}$$

$$Q = 3\ r^2 + 8\ r + 3 \tag{A5.8}$$