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An interesting property of the beam-beam interaction transverse kick.

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1. Introduction

Let us consider the head-on beam-beam interaction of two bunches and the phase space perturbation experienced by each particle in a single bunch.

We assume that the bunch length σ_s is short, that $x'\sigma_s \ll \sigma_x$ and that $y'\sigma_s \ll \sigma_y$, thus we can simplify the calculations by adopting the longitudinal thin lens approximation.

We would like to look at the average values $\langle(\Delta x')^2\rangle$ and $\langle(\Delta y')^2\rangle$ of the kicks $\Delta x'$ and $\Delta y'$

2. Is $(\Delta x')^2\rangle$ equal to $\langle\Delta y')^2\rangle$?

Let us further assume that the transverse density distribution of each bunch is gaussian with a given ratio "r" between the axes σ_x, σ_y, where $r=\sigma_y/\sigma_x<1$. For small values of x and y we have the well known formulae.

$$\Delta'_x = \frac{Nr_e}{\gamma(\sigma_x + \sigma_y)} \left(\frac{x}{\sigma_x}\right)$$

$$\Delta'_y = \frac{Nr_e}{\gamma(\sigma_x + \sigma_y)} \left(\frac{y}{\sigma_y}\right) \qquad (1.1)$$

As the distribution function of particles is:

$$\rho(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x{}^2} + \frac{y^2}{\sigma_y{}^2}\right)\right] \qquad (1.2)$$

it is easy to deduce the relation:

$$\langle \Delta x'^2 \rangle = \langle \Delta y'^2 \rangle \qquad (1.3)$$

Let us now assume that we add all the higher order terms to equation (1.1). One would expect that equation (1.3) could no longer be verified. Surprisingly it remains true even when considering the exact kick.

The interesting property of equation (1.3) has already been mentioned, see [2]. The proof of the equation, which will be demonstrated in the next paragraph, goes back to the calculations made in [1] but repeats them in rather more detail.

Unfortunately, this rather formal proof, does not suggest any intuitive interpretation of this elegant property.

3. The proof of equation (1.3)

The proof was hidden in [1] where the integral of the total beam-beam synchrotran radiation

$$I_{2bb} = \int \frac{ds}{\rho^2(s)} = \frac{\langle \Delta x'^2 \rangle + \langle \Delta y'^2 \rangle}{\sqrt{\pi}\,.\,\sigma_s} \qquad (1.4)$$

was computed.

The formula A4.4 of [1] for I_{2bb} can be written:

$$I_{2bb} = \frac{2r_e^2 N}{\sqrt{\pi}\gamma^2\sigma_x\sigma_y\sigma_s}\left[I_x + I_y\right] = \frac{2r_e^2 N}{\sqrt{\pi}\gamma^2\sigma_x\sigma_y\sigma_s}\left[(I_{x_2} - I_{x_1}) + (I_{y_1} - I_{y_2})\right] \qquad (2.1)$$

Each term of the last square bracket is of the form:

$$I = \int_r^1 \frac{\left[(Au + B)/(Cu + D)\right]^{1/2}}{(r^{-2}-3)u^2 + \left[10-3(r^2+ r^{-2})\right]u + (r^2-3)}\,du \qquad (2.2)$$

with the following definitions:

$$\left[\begin{array}{l} A_{y_1} = C_{x_1} = -1 \\ B_{y_1} = D_{x_1} = (2-r^2) \\ C_{y_1} = A_{x_1} = (r^{-2}-2) \\ D_{y_1} = B_{x_1} = 1 \end{array}\right. \qquad (2.3)$$

$$\left[\begin{array}{l} A_{x_2} = C_{y_2} = (2-3r^2) \\ B_{x_2} = D_{y_2} = r^2 \\ C_{x_2} = A_{y_2} = -1 \\ D_{x_2} = B_{y_2} = (3-2r^2) \end{array}\right. \qquad (2.4)$$

Every integral (2.2) can be greatly simplified by making the following substitutions:

$$\frac{Au + B}{Cu + D} = t^2 \qquad u = \frac{B - Dt^2}{Ct^2 - A} \qquad du = \frac{2(AD - BC)t\,dt}{(Ct^2 - A)^2} \qquad (2.5)$$

for I_{y_1} and I_{x_2} and:

$$\frac{Cu + D}{Au + B} = t^2 \qquad u = \frac{D - Bt^2}{At^2 - C} \qquad du = \frac{2(BC - AD)t\,dt}{(At^2 - c)^2} \qquad (2.6)$$

for I_{x_1} and I_{y_2}

After many algebraic manipulations we get:

$$I_{y_1} = I_{x_2} = -2 \int_r^1 \frac{t^2\, dt}{t^4 + bt^2+1} \tag{2.7}$$

$$I_{x_1} = I_{y_2} = -2 \int_r^1 \frac{dt}{t^4 + bt^2+1} \tag{2.8}$$

where $b = 3(r^2 + r^{-2}) - 8.$ (2.9)

It follows from (2.1), (2.7) and (2.8) that:

$$I_x = I_{x_2} - I_{x_1} = I_y = I_{y_1} - I_{y_2} = 2 \int_r^1 \frac{(1-t^2)}{t^4+bt^2+1}\, dt \tag{2.10}$$

hence

$$\langle \Delta x'^2 \rangle = \langle \Delta y'^2 \rangle \tag{1.3}$$

Finally we note, for convenience, the result obtained for I_{2bb} in [2]

$$I_{2bb} = I_x + I_y = \frac{8 r_e^2 N^2}{\sqrt{\pi}\gamma^2 \sigma_x \sigma_y \sigma_s} \quad \frac{r}{\sqrt{D}} \arctan \frac{\sqrt{D}}{Q} \tag{2.11}$$

where

$$D = 3r^4 - 10r^2 + 3 \tag{2.12}$$

$$Q = 3r^2 + 8r + 3 \tag{2.13}$$

References

1) M. Bassetti, M.Gygi-Hanney: LEP Note 221, "Dependance of the beam-beam synchrotron radiation on the transverse dimensions for gaussian beams".

2) M. Basssetti, J. Bosser, M.Gygi-Hanney, A. Hofmann and R. Coisson, "Properties and possible use of beam-beam synchrotron radiation", LEP Divisional Report CERN-LEP/TH/83-24.

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