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1 Introduction

The **Plateau problem for surfaces of prescribed mean curvature** H is to find a surface with given boundary Γ and with mean curvature $H(a)$ at each point a on the surface where H is a given function. For 2-dimensional parametric surfaces of disc type in Euclidean space $\mathbb{R}^3$ this problem has been treated by several authors, e.g. [He1], [Hi1], [Hi3], [GS1], [GS2], [Ste1], [Ste2], [We], and various optimal theorems have been obtained in the early seventies which essentially settle the existence problem in $\mathbb{R}^3$. To give one prominent example we recall the result of Hildebrandt that the Plateau problem can be solved in a Euclidean ball B_R of radius R in $\mathbb{R}^3$ whenever Γ is a rectifiable closed Jordan curve contained in B_R and the prescribed mean curvature function satisfies $|H| \leq R^{-1}$ on B_R.

In a Riemannian 3-manifold M the Plateau problem for parametric surfaces of prescribed mean curvature has been treated by Gulliver [Gu1] and independently by Hildebrandt & Kaul [HK]. They considered the problem in a normal geodesic ball B_R^M of radius R in M with sectional curvature not exceeding K and they proved that a solution exists whenever $\Gamma \subset B_R^M$ and the prescribed mean curvature function satisfies $|H| \leq h(K,R)$ on B_R^M, where $h(K,R)$ is the constant mean boundary curvature of a normal geodesic ball of radius R in a 3-manifold of constant sectional curvature K. This extended Hildebrandt's theorem to the case of an ambient Riemannian manifold. Gulliver and Hildebrandt & Kaul proved in fact more general results and it was clear that their method could be used to extend various other existence theorems like [GS1], [GS2], [Hi3] from the Euclidean to the Riemannian situation.

However, the method was restricted to boundary curves and surfaces in certain subregions of M, like normal geodesic balls or geodesically starshaped domains, and there were existence theorems known in $\mathbb{R}^3$ for which clearly no such restriction should be assumed in the hypothetical Riemannian analogue. For example, Wente's theorem [We] in the improved form [Ste2] states that a closed rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ bounds disc-type parametric surfaces of prescribed mean curvature H for all functions H on $\mathbb{R}^3$ with $\sup|H| < \sqrt{2\pi/3a_\Gamma}$ where a_Γ is the minimal area of discs spanning Γ. In particular, Γ bounds surfaces of large prescribed mean curvature whenever the least spanning area a_Γ is small. It is intuitively clear that a similar statement should hold in a complete Riemannian 3-manifold M (assumed compact or satisfying some uniformity condition at infinity) without any condition on the shape of the boundary curve Γ except the smallness of a_Γ. Another example is the theorem of the second author [Ste2] that $\Gamma \subset \mathbb{R}^3$ bounds a disc-type surface of prescribed mean curvature H whenever the integral of $|H|^3$ over $\mathbb{R}^3$ is smaller than $9\pi/2$. One would expect a Riemannian version of this result to hold with a suitable integral condition on $|H|^3$ on M, but without any assumption restricting the geometry of the boundary curve Γ.

The present paper is the result of our attempt to establish Riemannian generalizations of those existence theorems for the Plateau problem with prescribed mean curvature in $\mathbb{R}^3$ which are not covered by the work of Gulliver [Gu1] and Hildebrandt & Kaul [HK]. In particular, we did not want to assume that the boundary Γ is contained in a normal geodesic subdomain of the 3-manifold M. One of our main results presented in Section 5 is an existence theorem of Wente type asserting that the problem admits a solution whenever

$$a_\Gamma < \alpha(M) \quad \text{and} \quad \sup_M |H| < \gamma(M) a_\Gamma^{-1/2},$$

where $\alpha(M)$, $\gamma(M)$ are positive numbers computable in terms of certain isoperimetric constants of M (Theorem 5.4, Corollary 5.5, and Corollary 5.6). The example of the standard sphere $M = S^3$ with $\Gamma \subset M$ a great circle shows that the upper bound $\alpha(M)$ we must require for the minimal spanning area a_Γ to secure existence of solutions for $\sup_M |H|$ sufficiently small cannot be improved, in general. (However, the constant $\gamma(M)$ we obtain is certainly not optimal; even in the Euclidean case the best known constant $\sqrt{2\pi/3}$ [Ste2] is very likely not the optimal one which is conjectured to be $\gamma(\mathbb{R}^3) = \sqrt{\pi}$.)

From the general theorem we prove in Section 4 (Theorem 4.4) it follows that also the other known existence theorems for the Plateau problem with prescribed mean curvature in $\mathbb{R}^3$ can be extended to the Riemannian case under assumptions on M, Γ, H not covered by [Gu1] or [HK]. We work this out in one further instance by generalizing a result of Gulliver & Spruck [GS2] to C^2 domains A in a Riemannian 3-manifold M (using a completely different method of proof): If $|H|$ does not exceed pointwise on A the (inward) mean curvature of the local parallel surfaces to ∂A (where they exist) and A satisfies a certain homology condition, then the Plateau problem with prescribed mean curvature H is solvable in A for all contractible boundary curves $\Gamma \subset A$ (Theorem 5.3).

Even in the geometrical situation treated by Gulliver [Gu1] and Hildebrandt & Kaul [HK] our results give substantial improvements. For example (Theorem 5.1): If Γ is contained in a normal geodesic ball $B = B_R^M$ of radius R and sectional curvature $\leq K$, then we have a solution of the Plateau problem in B if the prescribed mean curvature modulus $|H|$ does not exceed, pointwise on ∂B, the (inward) mean boundary curvature of B and satisfies

$$\sup_B |H| < h(k, R),$$

where

$$h(k,R) = \begin{cases} \dfrac{2\sqrt{K}\sin^2(\sqrt{K}R)}{2\sqrt{K}R - \sin(2\sqrt{K}R)} & \text{in the case } K > 0, \\ \dfrac{3}{2R} & \text{in the case } K = 0, \\ \dfrac{2\sqrt{|K|}\sinh^2(\sqrt{|K|}R)}{2\sqrt{|K|}R - \sinh(2\sqrt{|K|}R)} & \text{in the case } K < 0. \end{cases}$$

The bound $h(K,R)$ required for $\sup_B |H|$ here is less restrictive than the bound $h(K,R)$ in [Gu1], [HK] which is $\sqrt{K}\cot(\sqrt{K}R)$, R^{-1}, or $\sqrt{|K|}\coth(\sqrt{|K|}R)$ in the respective cases. Moreover, if K is nonpositive, then instead of $\sup_B |H| < h(K,R)$ the following weaker condition is sufficient for existence (Theorem 5.2):

$$\int_{\{a\in B:|H(a)|\geq h(K,R)\}} |H|^3 dvol_M < \frac{9\pi}{2}.$$

This gives a simultaneous generalization of the results of [Gu1], [HK] for manifolds of nonpositive curvature and of the existence theorem [Ste2] in $\mathbb{R}^3$ which as hypothesis requires the bound $9\pi/2$ for the integral of $|H|^3$.

Our method is a combination of the approach adopted by the second author [Ste1] in the Euclidean situation and the isoperimetric inequalities used in our joint work [DS2] on the Plateau problem for hypersurfaces of prescribed mean curvature in Riemannian manifolds in the setting of geometric measure theory. We work in the Sobolev class $W^{1,2}(U,M)$ of mappings x from the unit disc U to the 3-manifold M and we set up an energy functional $\mathbf{E}_H(x)$ which is the sum of the surface energy $\mathbf{D}(x)$ (Dirichlet's integral) and a geometrically defined (in Section 3) volume $2\mathbf{V}_H(x)$ with respect to the prescribed mean curvature function H as weight function. We then want to minimize $\mathbf{E}_H$ in suitable subclasses of $W^{1,2}(U,M)$ to obtain the solutions to our Plateau problem which are referred to as H-surfaces of least energy in the title. The emphasis is on deriving reasonable geometric conditions for the data M, H, Γ which are sufficient for the existence of a weak solution to the Plateau problem. (In Section 6 we indicate how one proves that the energy minimizing weak solutions are actually smooth.)

The principal difference between the geometric measure theory setting in [DS2] and the present context of parametric surfaces is that we cannot work here in a fixed homology class of surfaces, because weak $W^{1,2}$ convergence does not preserve homology, in general. One may imagine this as a bubbling phenomenon: A (spherical) bubble can splitt off in the limiting process and this bubble can also carry away homology. Therfore, in order to define the H-volume $\mathbf{V}_H(x) = \mathbf{V}_H(x,y)$ enclosed by x and a fixed reference surface y with the same boundary Γ we must require homological triviality of the closed surface composed by x and y. This leads us to the notions of spherical 2-currents and spherical homological triviality in Section 3. For the proof of lower semicontinuity of $\mathbf{E}_H$ with respect to weak $W^{1,2}$ convergence of surfaces we use isoperimetric inequalities (for spherical 2-currents in M) in order to dominate the possible jump of H-volume by the jump of Dirichlet's integral on a minimizing sequence. For this it is essential to have a precise description of the bubbling process. We provide this with a new construction in Section 4 since the arguments of [Ste1] are restricted to Euclidean space $\mathbb{R}^3$ at this point.

To treat the Plateau problem for surfaces of prescribed mean curvature in a given subdomain of M we establish a geometric inclusion principle (maximum principle) in Section 2 which implies that minimizers for the H-energy under the constraint of being contained in the closure A of the domain do in fact avoid the boundary ∂A, provided this is true for the curve $\Gamma \subset A$ and $|H|$ is strictly smaller along ∂A than the inward mean boundary curvature of A. Similar statements can be found in the literature, e.g. [Hi3], [Hi4], [Hi5], [Gu1], [GS2], [HK], but the strong form given in Proposition 2.4 below, which applies to weak $W^{1,2}$ solutions of the variational inequality, appears to be new.

The results of this work, in weaker form and without complete proofs, have been announced in our expository article [DS3] on the Plateau problem for parametric surfaces of prescribed mean curvature. We want to mention that Toda has recently studied closed parametric H-surfaces in a Riemannian 3-manifold and has obtained interesting existence results [To1], [To2] of a different nature. The second author acknowledges the hospitality of the first author, of Humboldt University and the Sonderforschungsbereich 288 Berlin which helped to prepare this article and put the results in the final form.

2 The variational problem

We begin with a formulation of the Plateau problem for parametric surfaces in a Riemannian manifold (M, g). We always assume that M is of dimension 3, connected, oriented, complete, without boundary, and sufficiently smooth. Riemannian inner products and norms are denoted $(\tau, \tau')_M = g_a(\tau, \tau')$ and $|\tau|_M = (\tau, \tau)_M^{1/2}$ for $a \in M$ and tangent vectors $\tau, \tau' \in T_aM$. The Riemannian metric and the orientation determine a volume form Ω on M. Since M is of dimension 3 we have the exterior vector product $\tau \wedge_M \tau'$ which is characterized by $(\tau \wedge_M \tau', \tau'')_M = \langle \Omega_a, \tau \wedge \tau' \wedge \tau'' \rangle$ for $a \in M$ and $\tau, \tau', \tau'' \in T_aM$.

We consider parametric surfaces $x: U \to M$ which are defined on the unit disc $U = \{w = (u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$. Lebesgue measure on U will be denoted $\mathcal{L}^2$, dw or $dudv$. If the parametrization is conformal and of maximal rank, i.e. $|x_u|_M^2 = |x_v|_M^2 > 0$ and $(x_u, x_v)_M = 0$ holds for the tangent vector fields x_u, x_v along x, then we can describe the mean curvature H_x of x by the equation $\Delta^M x = 2H_x x_u \wedge_M x_v$ on U, where Δ^M is the Euler operator associated with **Dirichlet's integral**

$$(2.1)\quad \mathbf{D}(x) = \tfrac{1}{2} \int_U (|x_u|_M^2 + |x_v|_M^2)\,dudv.$$

This assertion is easily verified with the variational characterization of mean curvature (see below), but one can, of course, also use the definition of H_x as one half the trace of the second fundamental form of x in M with respect to the normal field $|x_u|_M^{-1}|x_v|_M^{-1} x_u \wedge_M x_v$ along x.

The (in general) nonlinear elliptic differential operator of second order Δ^M is well known from the theory of harmonic mappings (cf. [EL], [Jo1], [Jo2], [Jo3]) and called the **tension field operator** for mappings from the Euclidean disc U to the Riemannian manifold M. Note that $\Delta^M x$ is a tangent vector field of M along the mapping x, i.e. $\Delta^M x(w) \in T_{x(w)}M$ for $w \in U$, which is orthogonal to x, i.e. $(\Delta^M x) \cdot x_u = 0$ and $(\Delta^M x) \cdot x_v = 0$ on U (assuming x is a C^2 mapping of U into M).

We want to find surfaces $x: U \to M$ with *prescribed* mean curvature and given boundary in M. For this we assume that a bounded continuous function $H: M \to \mathbb{R}$, which we call the **prescribed mean curvature**, and a rectifiable oriented Jordan curve Γ in M are given. (Of course, Γ must be contractible in M or in the subdomain of M where we want to find the surfaces.) The problem to find a parametric surface $x: U \to M$ with prescribed mean curvature H and with boundary Γ, known as the **Plateau problem** $\mathcal{P}(H, \Gamma)$, then has the following formulation:

(2.2) $\Delta^M x = 2(H \circ x) x_u \wedge_M x_v \quad \text{on } U,$

(2.3) $|x_u|_M^2 - |x_v|_M^2 = 0 = (x_u, x_v)_M \quad \text{on } U,$

(2.4) $x|_{\partial U}$ is a weakly monotonic parametrization of Γ.

Here (2.2) is called the **H-surface-equation**, (2.3) are the **conformality relations**, and nonconstant C^2 solutions x to (2.2), (2.3) will be termed **H-surfaces** in M. The **Plateau boundary condition** (2.4) means precisely that $x|_{\partial U}$ is the uniform limit of orientation preserving homeomorphisms from ∂U onto Γ. This is a free boundary condition with one degree of freedom, in contrast with a Dirichlet boundary condition where $x|_{\partial U}$ would be a given map from ∂U into M. For the existence theory it is convenient to require only that $x|_{\partial U}$ is a weakly monotonic mapping from ∂U onto Γ, but the H-surfaces produced below will in fact map ∂U homeomorphically onto Γ. (In writing $x|_{\partial U}$ we understand that x is continuous on the closed disc $\overline{U}$ and we take the restriction of x to the boundary ∂U, or that x is of Sobolev class $W^{1,2}$ on U and $x|_{\partial U}$ is its boundary trace.)

Introducing local coordinates on M we can write equations (2.2), (2.3), on each subset of U which is mapped into the coordinate domain by x, as follows (see e.g. [HK], [Gu1], [Jo3]:

(2.5) $x_{uu}^\ell + x_{vv}^\ell + (\Gamma_{ij}^\ell \circ x)(x_u^i x_u^j + x_v^i x_v^j) = 2(H \circ x)\sqrt{\gamma \circ x}\, \varepsilon_{ijk}(g^{\ell k} \circ x)(x_u^i x_v^j - x_v^i x_u^j),$

(2.6) $(g_{ij} \circ x)(x_u^i x_u^j - x_v^i x_v^j) = 0 = (g_{ij} \circ x) x_u^i x_v^j,$

where $x^\ell, g_{ij}, \Gamma_{ij}^\ell$ are the coordinate components of x, the Riemannian metric g, and the Levi-Civita connection (Christoffel symbols of g), and we have set $(g^{\ell k}) = (g_{ij})^{-1}$, $\gamma = \det(g_{ij}), \varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$; summation over repeated indices i, j, k from 1 to 3 is understood.

As we want to solve $\mathcal{P}(H, \Gamma)$ with a variational method we need a weak formulation of the H-surface-equation. This is no problem if one restricts the considerations to surfaces x with image in a fixed coordinate domain in M, because we can then multiply (2.5) with a test function and integrate by parts. Such a point of view, which is equivalent to studying parametric surfaces in $\mathbb{R}^3$ equipped with a general Riemannian metric, was adapted by Hildebrandt & Kaul [HK] and by Gulliver [Gu1] in their first treatment of the Plateau problem with prescribed mean curvature in a Riemannian manifold. However, as we have pointed out in the introduction, we do not want to assume that our surfaces are contained in a fixed coordinate domain on M, because we want to prove existence of solutions for *all* curves Γ in M bounding a surface of sufficiently small area. Hence we must introduce the Sobolev space $W^{1,2}(U, M)$ and give a meaning to the H-surface-equation for $W^{1,2}$ mappings $x: U \to M$.

Following the usual procedure in the theory of harmonic mappings we therefore assume that M is isometrically embedded in some Euclidean space $\mathbb{R}^m$ as a closed subset. (This is not really necessary, but it is convenient and no essential loss of generality, by the well-known theorems of Nash and Gromov & Rohlin on isometric embedding.) We then define $W^{1,2}(U, A)$ for $A \subset M$ as the subclass of $W^{1,2}(U, \mathbb{R}^m)$ consisting of surfaces x which map ($\mathcal{L}^2$ almost every point of) U into A. Interpreting $x: U \to M$ as a mapping into $\mathbb{R}^m$ we can replace Riemannian inner products and norms of tangent vectors to M by their Euclidean inner products and norms in $\mathbb{R}^m$. Thus, the Riemannian Dirichlet integral (2.1) becomes the Euclidean one, the same is true for the conformality relation (2.3), and we henceforth omit the subscripts refering to M in formulas of this kind.

By a general principle from the calculus of variations, the Euler operator Δ^M associated with Dirichlet's integral and the constraint $x(w) \in M \subset \mathbb{R}^m$ is just the tangential projection of the Euler operator for the unconstrained problem which here is the Euclidean Laplacean $\Delta x = x_{uu} + x_{vv}$ (applied to each of the m components of x). Denoting by Π^M the field of orthogonal projectors $\Pi^M(a) : \mathbb{R}^m \to T_a M$ and by $A^M(a) : T_a M \times T_a M \to (T_a M)^\perp$ the second fundamental form of M we have

$$(2.7) \quad \Delta^M x = (\Pi^M \circ x)\Delta x = \Delta x - (A^M \circ x)(x_u, x_u) - (A^M \circ x)(x_v, x_v),$$

and (2.2) in this setting becomes

$$(2.8) \quad (\Pi^M \circ x)\Delta x = 2(H \circ x) x_u \wedge_M x_v \quad \text{on } U.$$

A weak formulation of (2.8), meaningful for $x \in W^{1,2}(U, M)$ is then the following:

$$(2.9) \quad \begin{array}{l} \int_U [x_u \cdot ((\Pi^M \circ x)\xi)_u + x_v \cdot ((\Pi^M \circ x)\xi)_v + 2(H \circ x)\xi \cdot x_u \wedge_M x_v] dudv = 0 \\ \text{for all vector fields } \xi \in W_0^{1,2}(U, \mathbb{R}^m) \cap L^\infty(U, \mathbb{R}^m), \end{array}$$

or equivalently,

$$(2.10) \quad \begin{array}{l} \int_U [x_u \cdot \nabla_u^M \zeta + x_v \cdot \nabla_v^M \zeta + 2(H \circ x)\zeta \cdot x_u \wedge_M x_v] dudv = 0 \\ \text{for all } \zeta \in W_0^{1,2}(U, \mathbb{R}^m) \cap L^\infty(U, \mathbb{R}^m) \text{ tangential to } M \text{ along } x, \end{array}$$

the latter conditon meaning that $\zeta(w) \in T_{x(w)}M$ for (almost all) $w \in U$. We call (2.9), (2.10) the **weak H-surface-equation** in M and any nonconstant and (almost everywhere on U) conformal solution $x \in W^{1,2}(U, M)$ a **weak H-surface** in M, or a **weak solution to the Plateau problem** $\mathcal{P}(H, \Gamma)$ if also (2.4) is satisfied.

The covariant derivatives appearing in (2.10) are defined by $\nabla_u^M \zeta = (\Pi^M \circ x)\zeta_u$. Since x_u, x_v are tangential to M along x we could, of course, simply write ζ_u, ζ_v in (2.10) instead of $\nabla_u^M \zeta, \nabla_v^M \zeta$; we have preferred the latter expressions because of their intrinsic geometric meaning. Note that in (2.9) we can equivalently use smooth $\xi \in C^1_{\text{cpt}}(U, \mathbb{R}^m)$ while in (2.10) we may not, in general, assume that ζ is smooth. The tension field operator is expressed in terms of covariant derivatives simply by $\Delta^M x = \nabla_u^M x_u + \nabla_u^M x_v$, and in local coordinates on M as above one has the representation $(\nabla_u^M \zeta)^\ell = \zeta_u^\ell + (\Gamma_{ij}^\ell \circ x) x_u^i \zeta^j$ at almost all points $w \in U$ with image $x(w)$ in the coordinate domain.

To set up a functional with (2.10) as variational equation we consider a deformation x_t of $x = x_0$ in M with initial vector field $\zeta = \frac{d}{dt}|_{t=0} x_t$. Then $\int_U (H \circ x)\zeta \cdot x_u \wedge_M x_v dudv$

is the initial rate of change of H-weighted volume swept out by the deformation while $\int_U \zeta \cdot \Delta x dudv$ describes the initial change of the **area** $\mathbf{A}(x) = \int_U |x_u \wedge_M x_v| dudv$, because we have $\mathbf{A}(x_t) \leq \mathbf{D}(x_t)$ with equality at $t = 0$ and hence, assuming $x_u \wedge_M x_v \neq 0$, the equation $\frac{d}{dt}|_{t=0}\mathbf{A}(x_t) = \frac{d}{dt}|_{t=0}\mathbf{D}(x_t)$ holds. Since the mean curvature is half the ratio of the initial rates of change of area and volume under a normal deformation of x in M, we recognize that (2.2) and (2.10) express that x has prescribed mean curvature $H \circ x$ wherever x is of maximal rank. We also see that as a variational functional we should use an H-**energy** of the form

(2.11) $\mathbf{E}_H(x) = \mathbf{D}(x) + 2\mathbf{V}_H(x)$,

where the H-**volume** $\mathbf{V}_H(x)$ is, in a sense, the oriented volume enclosed by the surface x (and a fixed comparison surface with the same boundary) in M measured with weight function H.

We will give precise definitions of the H-volume lateron, but we note here that with any of these definitions we have **independence of parametrization**

(2.12) $\mathbf{V}_H(x) = \mathbf{V}_H(x \circ \varphi)$ for orientation preserving C^1 diffeomorphisms $\varphi: \overline{U} \to \overline{U}$,

and the **homotopy formula** (with Ω the volume form on M)

(2.13) $\mathbf{V}_H(x_t) - \mathbf{V}_H(x) = \int_U \int_0^t (H \circ X)\langle \Omega \circ X, X_s \wedge X_u \wedge X_v \rangle dsdudv$

for the variations $X(s,u,v) = x_s(u,v)$ of $x(u,v) = x_0(u,v)$ we need. We call X a **sufficiently regular variation** in $M \subset \mathbb{R}^m$ if $x_t \in W^{1,2}(U,M)$ for sufficiently small $|t|$, if the initial vector field $\zeta = \frac{d}{ds}\Big|_{s=0} x_s$ is of class $W^{1,2}(U,\mathbb{R}^m) \cap \mathrm{L}^\infty$, and if formal differentiation under the integral with respect to t at $t = 0$ is valid for $\mathbf{D}(x_t)$ and for $\mathbf{V}_H(x_t) - \mathbf{V}_H(x)$.

2.1 Proposition (first variation).

(i) For sufficiently regular variations x_t of $x \in W^{1,2}(U,M)$ with initial field $\zeta \in W^{1,2}(U,\mathbb{R}^m) \cap L^\infty$ we have

$$\frac{d}{dt}\Big|_{t=0}\mathbf{E}_H(x_t) = \int_U [x_u \cdot \zeta_u + x_v \cdot \zeta_v + 2(H \circ x)\zeta \cdot x_u \wedge_M x_v]\, dudv =: \delta\mathbf{E}_H(x;\zeta).$$

(ii) If $x \in W^{1,2}(U,M)$ and φ_t is the flow of a C^1 vector field η on $\overline{U}$ which is tangential to ∂U along ∂U, then

$$\frac{d}{dt}\Big|_{t=0}\mathbf{E}(x \circ \varphi_t) = \int_U \mathrm{Re}\left[(|x_u|^2 - |x_v|^2 - 2\mathbf{i}x_u \cdot x_v)\bar\partial\eta\right] dudv =: \partial\mathbf{E}_H(x;\eta)$$

where we have used complex notation $\bar\partial\eta = \frac{1}{2}(\eta_u + \mathbf{i}\eta_v)$ identifying $\mathbb{R}^2 = \mathbb{C}$.

Proof. (i) Formal differentiation of $\mathbf{D}(x_t)$ gives the integrand $x_u \cdot \zeta_u + x_v \cdot \zeta_v$ on U, and formal differentiation of (2.13) leads to the integrand $(H \circ x)\langle \Omega \circ x, \zeta \wedge x_u \wedge x_v \rangle = (H \circ x)\zeta \cdot x_u \wedge_M x_v$.

(ii) In view of (2.12) we only have to compute $\frac{d}{dt}\Big|_{t=0} \mathbf{D}(x \circ \varphi_t)$ and this is well known. Using the transformation $w = \varphi_t(\tilde{w})$ one gets

$$\int_U |D(x \circ \varphi_t)|^2 d\tilde{w} = \int_U |Dx(D\varphi_t) \circ \varphi_t^{-1}|^2 \det(D\varphi_t^{-1}) dw,$$

and since $D\varphi_t(\tilde{w}) = 1 + tD\eta + o(t)$ one obtains

$$\frac{d}{dt}\Big|_{t=0} \mathbf{D}(x \circ \varphi_t) = \int_U [Dx \cdot (DxD\eta) - \tfrac{1}{2}|Dx|^2 \text{trace } D\eta] dw.$$

The claim then follows by appropriately collecting terms in the integrand. ∎

The integral $\delta \mathbf{E}_H(x;\zeta)$ appearing in (i) is called the **first variation of energy** at x in the direction of ζ and $\partial \mathbf{E}_H(x;\eta)$ from (ii) the **first variation of independent variables** (inner first variation) for $\mathbf{E}_H$ at x in the direction η.

2.2 Corollary.

(i) $x \in W^{1,2}(U, M)$ *is a solution to the weak H-surface equation if and only if* $\delta\mathbf{E}_H(x;\zeta) = 0$ *for all vector fields* $\zeta \in W_0^{1,2}(U, \mathbb{R}^m) \cap L^\infty(U, \mathbb{R}^m)$ *which are tangential to M along x.*

(ii) $x \in W^{1,2}(U, M)$ *satisfies the conformality relations almost everywhere on U if and only if* $\partial \mathbf{E}_H(x;\eta) = 0$ *for all vector fields* $\eta \in C^1(\overline{U}, \mathbb{R}^2)$ *which are tangential to ∂U along ∂U.*

(iii) $x \in W^{1,2}(U, M)$ *is weak H-surface if and only if x is nonconstant with* $\delta\mathbf{E}_H(x;\zeta) = 0$ *and* $\partial\mathbf{E}(x;\eta) = 0$ *for all ζ and η as in (i) and (ii).*

Proof: (i) and (iii) are clear, and (ii) is well known: $\partial\mathbf{E}_H(x;\eta) = 0$ for $\eta \in C^1_{\text{cpt}}(U, \mathbb{R}^2)$ is equivalent with the equation $\bar{\partial} h = 0$ in the distributional sense for the function $h = |x_u|^2 - |x_v|^2 - 2\mathbf{i}x_u \cdot x_v$. This means that h is (weakly, hence also classically) holomorphic on U. With a formal integration by parts on sees that $w^2 h(w)$ is real on ∂U in the weak sense if $\partial \mathbf{E}_H(x;\eta) = 0$ also holds for all $\eta \in C^1(\overline{U}, \mathbb{R}^2)$ which are tangential to ∂U on ∂U. Such holomorphic functions can be extended by reflection to (weakly) holomorphic functions on $\mathbb{C}$ which are bounded and hence constant. Evaluation at $w = 0$ then shows that the constant must be zero and h vanishes on U. ∎

The meaning of (i) is that the weak H-surface equation (2.10) is the Euler equation associated with the energy functional $\mathbf{E}_H$. The equation $\bar{\partial} h = 0$ is sometimes called the second Euler equation. The reality condition for $w^2 h(w)$ on ∂U is the natural boundary condition for variations of the independent variables which is associated with the free Plateau boundary condition (2.4). (It can be expressed invariantly by stating that the quadratic holomorphic differential $h(w)(dw)^2$, known as Hopf differential, is real on the boundary). It has become customary to call a mapping stationary for a variational problem if the first variation and also the first variation of independent variables vanish for all initial fields which are admissible in the problem. Thus, in the present context stationarity of a nonconstant parametric surface is equivalent to being a weak H-surface.

Various types of variations have been used in the theory of harmonic mappings to deduce the Euler equation for minimizers of the Dirichlet integral in subclasses of $W^{1,2}(U,M)$ (see [DS3], (3.15)-(3.17)). For our purpose here we find variations of the form

(2.14) $x_t(w) = \Phi^Y(t\eta(w), x(w))$,

most convenient, where Φ^Y is the flow of a smooth tangent vector field Y on M and η a smooth function on U so that $\eta(Y \circ x)$ is the initial field of the variation (cf. [Du], [DS2]). With these variations we obtain:

2.3 Proposition (variational equation/inequality).

(i) Suppose that $x \in W^{1,2}(U,M)$ is $\mathbf{E}_H$ minimizing with respect to the variation x_t, $|t|<<1$, from (2.14) for each smooth compactly supported tangent vector field Y on M and for each smooth real function η on U with compact support in U. Then x is a solution to the weak H-surface equation (2.10).

(ii) Suppose A is the closure of a C^2 domain in M. If the $\mathbf{E}_H$ minimizing property of x is known only for one-sided variations x_t, $0 \le t << 1$, as in (i) and only with the restrictions $\eta \ge 0$ and $Y(a) = 0$ or $Y(a)$ directed strictly to the interior of A at all points $a \in \partial A$, then x is a solution to the variational inequality.

(2.15) $\delta \mathbf{E}_H(x;\zeta) = \int_U [x_u \cdot \zeta_u + x_v \cdot \zeta_v + 2(H \circ x)\zeta \cdot x_u \wedge_M x_v] du dv \ge 0$

for all vector fields $\zeta \in W_0^{1,2}(U,\mathbb{R}^m) \cap L^\infty(U,\mathbb{R}^m)$ which are tangential to M along x and satisfy $\zeta \cdot (\tilde{\nu} \circ x) \ge 0$ almost everywhere on $x^{-1}V$ for some neighbourhood V of ∂A in M and some C^1 vector field $\tilde{\nu}$ on M extending the inner unit normal field ν of ∂A.

(iii) If $x \in W^{1,2}(U,M)$ is $\mathbf{E}_H$ minimizing with respect to all variations $x_t = x \circ \varphi_t$, $|t| << 1$, of the independent variables where φ_t is the flow of a smooth vector field on $\overline{U}$ which is tangential to ∂U along ∂U, then x satisfies the conformality relations (2.4) almost everywhere on U.

In (i) and (ii) we also assume implicitely that the homotopy formula (2.13) is valid for the volume functional $\mathbf{V}_H$ and the variations considered.

Proof. (i) One readily verifies that the variation x_t is sufficiently regular in the sense described after (2.13) so that (2.10) follows for the vector fields $\zeta = \eta(Y \circ x)$. By approximation we can then admit $\eta(Y \circ x)$ in (2.10) with Y as before, but $\eta \in W_0^{1,2}(U,\mathbb{R}) \cap L^\infty(U,\mathbb{R})$. Given a general vector field ζ as allowed in (2.10), we consider smooth and compactly supported functions $\psi: M \to \mathbb{R}$ and vector fields Y_1, Y_2, Y_3 on M such that Y_1, Y_2, Y_3 are linearly independent on a neighbourhood of sptφ. We then have $(\psi \circ x)\zeta = \sum_{i=1}^3 \eta_i\,(Y_i \circ x)$ with $\eta_i \in W_0^{1,2}(U,\mathbb{R}) \cap L^\infty(U,\mathbb{R})$ and we deduce that $(\psi \circ x)\zeta$ is admissible in (2.10). With a partition of unity we conclude that this is true for each smooth function ψ with compact support in M, and approximating the constant 1 suitably with such functions ψ we obtain (2.10) for ζ.

(ii) As in (i) we now have $\delta\mathbf{E}_H(x;\zeta) \geq 0$ for $\zeta = \eta(Y \circ x)$ with $0 \leq \eta \in W_0^{1,2}(U,\mathbb{R}) \cap L^\infty(U,\mathbb{R})$ and Y satisfying the additional restriction on ∂A. We first observe that we may, by approximation, admit also vector fields Y which are directed weakly to the interior of A along ∂A. Moreover, $\delta\mathbf{E}_H(x;\eta(Y \circ x)) = 0$ holds if Y is tangential to ∂A along ∂A, because then $-Y$ is also admissible. Clearly, the condition $\eta \geq 0$, may be dropped in this tangential case. For general ζ as allowed in (ii) and smooth functions $\psi \geq 0$ on M with sufficiently small support intersecting ∂A we can now use the previous argument with $Y_1 = \tilde{\nu}$ and Y_2, Y_3 tangential to ∂A to deduce $\eta_1 \geq 0$ and $\delta\mathbf{E}(x;(\psi \circ x)\zeta) \geq 0$. Since, on the other hand, $\delta\mathbf{E}_H(x;(\psi \circ x)\zeta)$ vanishes in the case $\partial A \cap \operatorname{spt}\psi = \emptyset$, we can proceed with a partition of unity argument and conclude $\delta\mathbf{E}_H(x;\zeta) \geq 0$ as in the proof of (i).

(iii) is clear with Proposition 2.1(ii) and Corollary 2.2(ii). ∎

If we minimize the H-energy in a subclass of $W^{1,2}(U,A)$, where A is the closure of a smooth proper subdomain of M, then we have an obstacle problem ($M \backslash A$ is the obstacle) and we cannot expect more than the variational inequality (2.15) for a solution x. It is well known (see Remark 2.5(3) below), however, that with a suitable inequality relating the prescribed mean curvature H on M to the mean boundary curvature $H_{\partial A}$ of A one can prove the variational *equation* for x and, in fact, the inclusion of the image of x in the interior of A. We present our version of this geometric inclusion principle in the following

2.4 Proposition (variational inequality and strong inclusion). *Suppose A is the closure of a C^2 domain in M, ν is the inner unit normal along ∂A, the mean curvature $H_{\partial A}$ of ∂A with respect to ν is bounded from below, and $x \in W^{1,2}(U,A)$ is a conformal solution to the variational inequality (2.15) for all vector fields ζ allowed there. Then the following statements hold:*

(i) There exists a nonnegative smooth Radon measure λ on U which is absolutely continuous with respect to Lebesgue measure $\mathcal{L}^2$ on U and concentrated on the coincidence set $x^{-1}\partial A$ such that

$$(2.16)\quad \delta\mathbf{E}_H(x;\zeta) = \int_{x^{-1}\partial A} \zeta \cdot (\nu \circ x)\, d\lambda$$

for all $\zeta \in W_0^{1,2}(U,\mathbb{R}^m) \cap L^\infty(U,\mathbb{R}^m)$ which are tangential to M along x;

(ii) λ satisfies the inequality

$$(2.17)\quad \lambda \leq \mathcal{L}^2 \llcorner [(|x_u|^2 + |x_v|^2)(|H| - H_{\partial A})_+ \circ x] \quad on\ x^{-1}\partial A,$$

in particular $\lambda = 0$ and x is a weak H-surface in M (or constant) if $|H| \leq H_{\partial A}$ is valid along ∂A;

(iii) *If $|H(a)| < H_{\partial A}(a)$ holds at some point $a \in \partial A$ and the boundary trace $x|_{\partial U}$ does not meet a neighbourhood of a, then also the surface x omits a neighbourhood of this point.*

Here the **coincidence set** $x^{-1}\partial A$ is the set of $w \in U$ (defined up to $\mathcal{L}^2$ measure zero) such that $x(w) \in \partial A$, i.e. x at w touches the obstacle $M \backslash A$. Similarly, the meaning of (iii) is that $x^{-1}W$ has $\mathcal{L}^2$ measure zero for some neighbourhood W of a. Our sign convention for the boundary mean curvature $H_{\partial A}$ is such that $H_{\partial A} \geq 0$ in the case $M = \mathbb{R}^3$ and A a convex body.

Proof. We write $d(p) = \text{dist}\,(p, \partial A)$ for the distance of $p \in M$ to ∂A with respect to the inner metric on M and we extend ν to a bounded C^1 vector field on M, still denoted ν, which coincides with $\text{grad}_M d$ on a neighbourhood of ∂A. Assuming first that A is compact we see that $\zeta = \eta(\nu \circ x)$ is admissible in the variational inequality (2.15) if $0 \leq \eta \in C^1_{\text{cpt}}(U, \mathbb{R})$, and we obtain a nonnegative Radon measure λ on U such that

(2.18) $\delta \mathbf{E}_H(x, \eta(\nu \circ x)) = \int_U \eta d\lambda$

for all $\eta \in C^1_{\text{cpt}}(U, \mathbb{R})$.

We next choose $\vartheta \in C^\infty(\mathbb{R}, \mathbb{R})$ nonincreasing with $\text{spt}\vartheta \subset]-\infty, 1[$ and $\text{spt}(1-\vartheta) \subset]0, \infty[$, define $\vartheta_\varepsilon(t) = \vartheta(\varepsilon^{-1}t)$ for $\varepsilon > 0$, and consider $\zeta_\varepsilon = \eta(\vartheta_\varepsilon \circ d \circ x)(\nu \circ x)$ with $\eta \geq 0$ as above. Then $\zeta = \zeta_\varepsilon$ holds on the inverse image of a neighbourhood of ∂A under x, hence $\zeta - \zeta_\varepsilon$ and $\zeta_\varepsilon - \zeta$ are both admissible in the variational inequality and, consequently,

(2.19) $\delta \mathbf{E}_H(x; \zeta_\varepsilon) = \delta \mathbf{E}_H(x; \zeta) \geq 0.$

Noting that

$$(\vartheta_\varepsilon \circ d \circ x)_u = (\vartheta'_\varepsilon \circ d \circ x) x_u \cdot (\nu \circ x)$$

for ε small we compute

$$\begin{aligned} x_u \cdot (\zeta_\varepsilon)_u &= \eta_u (\vartheta_\varepsilon \circ d \circ x) x_u \cdot \nu \circ x + \eta(\vartheta'_\varepsilon \circ d \circ x)(x_u \cdot (\nu \circ x))^2 \\ &\quad + \eta(\vartheta_\varepsilon \circ d \circ x) x_u \cdot ((\nabla^M \nu) \circ x) x_u) \\ &\leq (\vartheta_\varepsilon \circ d \circ x)[\eta_u x_u \cdot (\nu \circ x) + \eta x_u \cdot ((\nabla^M \nu) \circ x) x_u]. \end{aligned}$$

Adding the analogous inequality for $x_v \cdot (\zeta_\varepsilon)_v$, letting $\varepsilon \searrow 0$, and using the identities (valid almost everywhere)

$$x_u \cdot (\nu \circ x) = x_v \cdot (\nu \circ x) = 0 \quad \text{on} \quad x^{-1}\partial A,$$

$$x_u \cdot ((\nabla^M \nu) \circ x) x_u + x_v \circ ((\nabla^M \nu) \circ x) x_v = -(|x_u|^2 + |x_v|^2) H_{\partial A} \circ x \quad \text{on} \quad x^{-1}\partial A$$

(this follows from the conformality of x and the definition $-2H_{\partial A} = \text{trace}(\nabla^M \nu)$ on ∂A) we arrive at the estimate

$$\delta \mathbf{E}_H(x; \zeta_\varepsilon) \leq \int_{x^{-1}\partial A} \eta(|x_u|^2 + |x_v|^2)(|H| - H_{\partial A}) \circ x du dv.$$

From this and (2.18), (2.19) we deduce assertion (ii).

To prove (i) we note that (ii) implies absolute continuity of λ together with $\lambda(U \setminus x^{-1}\partial A) = 0$, and (2.16) follows from (2.18) by approximation in the special case $\zeta = \eta(\nu \circ x)$ with $\eta \in W^{1,2}_0 \cap L^\infty(U, \mathbb{R})$. General vector fields $\zeta \in W^{1,2}_0 \cap L^\infty(U, \mathbb{R}^m)$ tangential to M along

x will be decomposed $\zeta = \zeta^\perp + \zeta^\top$ with $\zeta^\perp = \eta(\nu\circ x)$ and $\eta = \zeta\cdot(\nu\circ x) \in W_0^{1,2}\cap L^\infty(U,\mathbb{R})$. Then $\zeta^\top\cdot(\nu\circ x) = 0$ almost everywhere on the inverse image under x of some neighbourhood of ∂A, hence $\zeta^\top$ and $-\zeta^\top$ are both admissible vector fields in (2.15) so that $\delta\mathbf{E}_H(x;\zeta^\top) = 0$, and (2.16) now follows from $\delta\mathbf{E}_H(x;\zeta) = \delta\mathbf{E}_H(x;\zeta^\perp)$ and the special case of vector fields $\eta(\nu\circ x)$ treated first.

If A is not compact we can still apply the reasoning above with $\nu\circ x$ replaced by $(\psi\circ x)(\nu\circ x)$ where $\psi : M \to [0,1]$ is a C^1 function with compact support. We then obtain a measure λ_ψ depending on ψ for which the estimate in (ii) is valid. Letting ψ increase to the constant 1 in such a way that $|\mathrm{grad}_M\psi|$ tends to zero uniformly on M we find a limit measure λ of the λ_ψ for which assertions (i) and (ii) hold (cf. [Du]).

For the proof of (iii) we denote by $H_A = -\frac{1}{2}\mathrm{trace}\,\nabla^M\nu$ the mean curvature of the parallel surface to ∂A (on a neighbourhood of ∂A in M) and we consider $a\in\partial A$ such that $|H| < H_A$ holds on some ball $B = B_\varrho(a)$ in M and $x|_{\partial U}$ does not meet $B_{2\varrho}(a)$. We first assume that $x^{-1}(d^{-1}[0,\varepsilon]\cap B_{2\varrho}(a)\backslash B)$ has zero $\mathcal{L}^2$ measure for some $\varepsilon > 0$ and we choose $\tilde\psi \geq 0$ in $C^1(M,\mathbb{R})$ with $\tilde\psi = 1$ on B and $\mathrm{spt}\tilde\psi$ compact in the interior of $B_{2\varrho}(a)$. Then $\tilde\zeta_\varepsilon = (\varepsilon - d\circ x)_+(\tilde\psi\circ x)(\nu\circ x)$ is admissible in (2.15) and vanishes on $U\backslash x^{-1}B$. With similar calculations as above we obtain therefore

$$\begin{aligned}0 \leq \delta\mathbf{E}_H(x;\tilde\zeta_\varepsilon) = &-\int_{x^{-1}(B\cap d^{-1}[0,\varepsilon])}[(x_u\cdot(\nu\circ x))^2 + (x_v\cdot(\nu\circ x))^2]dudv\\ &+\int_{x^{-1}B}(\varepsilon - d\circ x)_+[x_u\cdot((\nabla^M\nu)\circ x)x_u + x_v\cdot((\nabla^M\nu\circ x)x_v) +\\ &\qquad + 2(H\circ x)(\nu\circ x)\cdot x_u\wedge_M x_v]dudv.\end{aligned}$$

We now use the conformality of x, the identity $(\nabla^M\nu)\nu = 0$ on $B\cap d^{-1}[0,\varepsilon]$ for $\varepsilon > 0$ sufficiently small, the selfadjointness of the shape operator $S = (\nabla^M\nu)\circ x$, and the general identity

$$1 - ((b\wedge c)\cdot n)^2 = (b\cdot n)^2 + (c\cdot n)^2$$

for orthonormal vectors b, c and unit vectors n in an oriented inner product space of dimension 3, to deduce

$$\begin{aligned}-(H_A\circ x)(|x_u|^2 + |x_v|^2) &= (\mathrm{trace}\,S)\tfrac{1}{2}(|x_u|^2 + |x_v|^2)\\ &= x_u\cdot Sx_u + x_v\cdot Sx_v + |x_u\wedge_M x_v|^{-1}(x_u\wedge_M x_v)\cdot S(x_u\wedge_M x_v)\end{aligned}$$

and

$$|x_u\wedge_M x_v|^{-1}|(x_u\wedge_M x_v)\cdot S(x_u\wedge_M x_v)| \leq \sup_B\|\nabla^M\nu\|[(x_u\cdot(\nu\circ x))^2 + (x_v\circ(\nu\cdot x))^2]$$

on $B\cap d^{-1}[0,\varepsilon]$. We conclude

$$\begin{aligned}0 \leq &-\int_{x^{-1}(B\cap d^{-1}[0,\varepsilon])}[1 - (\varepsilon - d\circ x)_+\sup_B\|\nabla^M\nu\|][(x_u\cdot(\nu\circ x))^2 + (x_v\cdot(\nu\circ x))^2]dudv\\ &+\int_{x^{-1}B}(\varepsilon - d\circ x)_+(|x_u|^2 + |x_v|^2)(|H| - H_A)\circ x\,dudv,\end{aligned}$$

from which we infer that $|x_u|^2 + |x_v|^2$ vanishes on $x^{-1}(B\cap d^{-1}[0,\varepsilon[)$ for $\varepsilon > 0$ sufficiently small. For every $\psi\in C^1(M,\mathbb{R})$ with compact support in $(\mathrm{int}\,B)\cap d^{-1}[0,\varepsilon[$ it follows that

$\psi\circ x$ is constant and, in view of the assumption $x(\partial U)\cap B=\emptyset$, i.e. $\psi\circ x\in W_0^{1,2}(U,\mathbb{R})$, that in fact $\psi\circ x=0$ on U. This means that $x^{-1}((\text{int } B)\cap d^{-1}[0,\varepsilon[)$ has $\mathcal{L}^2$ measure zero and x omits the neighbourhood $(\text{int } B)\cap d^{-1}[0,\varepsilon[$ of a as asserted in (iii).

To remove the assumption that $x^{-1}(d^{-1}[0,\varepsilon]\cap B_{2\varrho}(a)\backslash B_\varrho(a))$ has vanishing $\mathcal{L}^2$ measure for some $\varepsilon>0$ we consider the closure $\tilde{A}$ of a smooth domain $\tilde{A}\supset A$ such that $\partial\tilde{A}\cap\partial A\cap W=\{a\}$ for some neighbourhood W of a, but still $H_{\partial\tilde{A}}(a)=H_{\partial A}(a)>|H(a)|$. (This may be achieved by defining $\tilde{A}=\Phi_t A$ for some small $t>0$, where Φ_t is the flow of a smooth and compactly supported vector field on M which vanishes at a and is directed strictly to the exterior of A on a neighbourhood of a in ∂A.) We may then apply the previous reasoning with A,ν replaced by $\tilde{A},\tilde{\nu}$, provided ϱ is chosen small enough to ensure that ζ_ε, constructed now with $\tilde{A}$ instead of A and still with $\text{spt}\psi\subset B_{2\varrho}(a)$, is admissible in (2.15). ∎

2.5 Remarks. (1) The conformality assumption on x can be dropped if the boundary mean curvature $H_{\partial A}$ is replaced by the minimum of the principal curvatures of ∂A in all the statements of the previous theorem. This is seen with simple modifications of the proof above.

(2) If ∂A has bounded principal curvatures and global inner parallel surfaces $A\cap d^{-1}\{\varepsilon\}$ of class C^2 for $0\le\varepsilon<\varepsilon_0$, and if $|H|\le H_A$ holds on $A\cap d^{-1}[0,\varepsilon_0[$ (where $H_A(a)$ is the mean curvature of $A\cap d^{-1}\{\varepsilon\}$ at a as in the proof above), then solutions $x\in W^{1,2}(U,A)$ to the variational inequality (2.15) have their image contained in the interior parallel set $A\cap d^{-1}[\varepsilon_0,\infty[$ of A whenever their boundary trace $x|_{\partial U}$ has values in this parallel set. To see this one uses $\zeta_\varepsilon=(\varepsilon-d\circ x)_+(\nu\circ x)$ as in the proof of (iii) and one observes that the vanishing of $(x_u\cdot(\nu\circ x))^2+(x_v\cdot(\nu\circ x))^2$ on the set $x^{-1}(d^{-1}[0,\varepsilon[)$ implies constancy of $(\varepsilon-d\circ x)_+$ on U.

(3) Variants of the preceding theorem have long been known in the theory of parametric surfaces of prescribed mean curvature. However, we could not find in the literature the strong version above which is valid for all conformal $W^{1,2}(U,A)$ solutions x to the variational inequality (2.15). Hildebrandt [Hi4] used the variational inquality and a certain amount of smoothness of x which he established for energy minimizers in [Hi5]. Gulliver & Spruck [GS2] required continuity and energy minimality of x, too. Stronger hypotheses than above are also used in the treatments of the inclusion principle in [Gu1], [Gu5], [Hi3], [HK], [Ka]. In this connection we mention the inclusion principles of Dierkes [Di1], [Di2] which are of a different nature. The proof of Proposition 2.4 given above is an adaption of arguments from [Du] and [DS2]. ∎

3 The volume functional

The definition of volume used in the early work of Heinz [He1], Hildebrandt [Hi1] and others (see [DS3, Section 1]) for parametric surfaces of prescribed mean curvature in $\mathbb{R}^3$ depended on the choice of a vector field Z with the prescribed mean curvature as its divergence $H=\text{div } Z$. The integral of Z over a parametric surface x may then be interpreted, by the Gauss-Green theorem, as the volume enclosed by x and a fixed comparison surface y with the same boundary curve Γ as x. (At least this is true up to an irrelevant constant determined by y). The method works well also in the Riemannian case and has been used in [HK], [Gu1] in this context. In general, for a 3-dimensional manifold $M\subset\mathbb{R}^m$ as in

Section 2 and a bounded continuous vector field Z on M with $\mathrm{div}_M Z = H$ in the sense of distributions we may define

(3.1) $$\mathbf{V}_H(x) = \int_U (Z \circ x, x_u \wedge_M x_v)_M du dv \text{ for } x \in W^{1,2}(U, M)$$

or, using the 2-form ω on M which is dual to Z and satisfies $d\omega = H\Omega$ (where Ω is the volume form on M),

(3.2) $$\mathbf{V}_H(x) = \int_U x^{\#}\omega = \int_U \langle \omega \circ x, x_u \wedge x_v \rangle du dv.$$

There is a slight abuse of notation here, because $\mathbf{V}_H(x)$ depends on the choice of ω with $d\omega = H\Omega$, but if we restrict the considerations to surfaces with a given boundary curve Γ in the sense of (2.4), then different choices of ω will change $\mathbf{V}_H(x)$ by an ignorable constant depending on Γ only.

The homotopy formula (2.13) is valid with the definition of volume above for variations $X(s, u, v) = x_s(u, v)$ of type (2.14), for example. Since (2.13) can be written

$$\mathbf{V}_H(x_t) - \mathbf{V}_H(x) = \int_{[0,t] \times U} X^{\#}(H\Omega)$$

and we have $H\Omega = d\omega$, the homotopy formula is formally true by Stokes' theorem. It can be proved rigorously by first considering smooth bounded 2-forms ω on M and smooth variations $X : [0, t] \times \overline{U} \to M$, then with variations as in (2.14) for general $x \in W^{1,2}(U, M)$ by approximation with smooth variations, and finally for continuous ω with $d\omega = H\Omega$ in the distributional sense by approximating ω suitably with bounded smooth 2-forms. Thus, all the conclusions of Section 1 are valid with the definition (3.1) or (3.2) of the H-volume.

However, this method has certain drawbacks when one takes a global perspective. If M is compact without boundary, for example, then the condition $\int_M H\Omega = 0$ of vanishing mean value is necessary (and sufficient) for the existence of a 2-form ω with $d\omega = H\Omega$, and this is an undesirable restriction from our point of view. Furthermore, if M is noncompact there may still be obstructions to the existence of a *bounded* 2-form ω on M with $\mathrm{div}_M \omega = H\Omega$. The (boundedness) is needed to secure the existence of the integral (3.2) for general $x \in W^{1,2}(U, M)$.) For instance, if $M = \mathbb{R}^3$ and $|a|H(a) \to \infty$ as $|a| \to \infty$, then no bounded vector field Z with $\mathrm{div} Z = H$ exists on M as can be seen from the Gauss-Green theorem. For these reasons we adopt another, geometric measure theoretic, method for the definition of volume which was first used in [Ste1]. We now briefly develop this approach in the present Riemannian setting.

Given a parametric surface $x \in W^{1,2}(U, M)$ we define the **associated 2-current** J_x on M by integration of 2-forms over x,

(3.3) $$J_x(\beta) = \int_U x^{\#}\beta = \int_U \langle \beta \circ x, x_u \wedge x_v \rangle du dv \quad \text{for} \quad \beta \in \mathcal{D}^2(M),$$

where $\mathcal{D}^k(M)$ denotes the space of smooth k-forms on M with compact support. Then J_x is a current of finite mass

(3.4) $$\mathbf{M}(J_x) \le \mathbf{D}(x),$$

and using approximation by smooth mappings, in the sense of the $W^{1,2}$ norm as in [Ste1] or in the sense of a Lusin type theorem as in [EG, 6.6] one sees that J_x is an integer multiplicity rectifiable 2-current on M, i.e. J_x can be represented by integration (with respect to 2-dimensional Hausdorff measure) of 2-forms over a locally 2-rectifiable set with a measureable orientation and with a summable integer valued multiplicity function. (By definition, a k-current T on M is a continuous linear functional on $\mathcal{D}^k(M)$ and its mass $\mathbf{M}(T)$ is the supremum of values $T(\beta)$ on k-forms $\beta \in \mathcal{D}^k(M)$ with $|\beta| \leq 1$ pointwise on M. The standard references are [Fe], [Si]. Our terminology deviates from [Fe] in that our integer multiplicity rectifiable currents have finite mass but not necessarily compact support. In the language of [Fe] these would be called locally rectifiable currents of finite mass.)

If y is another surface in $W^{1,2}(U, M)$, then $(J_x - J_y)(\beta)$ is given by integration of $x^\# \beta - y^\# \beta$ over the set $G = \{w \in U : x(w) \neq y(w)\}$, because $Dx = Dy$ holds almost everywhere on $U \setminus G$. Hence, (3.4) can be sharpened to

$$(3.5) \quad \mathbf{M}(J_x - J_y) \leq \mathbf{D}_G(x) + \mathbf{D}_G(y) \quad \text{if } x = y \quad \text{on } U \setminus G,$$

where

$$(3.6) \quad \mathbf{D}_G(x) = \tfrac{1}{2} \int_G |Dx|^2 du dv$$

for measurable $G \subset U$.

The boundary ∂T of a k-current T on M is generally defined, if $k \geq 1$, by $\partial T(\alpha) = T(d\alpha)$ for $\alpha \in \mathcal{D}^{k-1}(M)$. If $x \in W^{1,2}(U, M)$ has a continuous rectifiable curve $x|_{\partial U}$ as boundary trace, then ∂J_x is given by integration of 1-forms over this curve (see [Ste1]). This implies, in particular, that $J_x - J_y$ is a closed current, i.e. $\partial(J_x - J_y) = 0$, if $x, y \in W^{1,2}(U, M)$ both satisfy the Plateau boundary condition (2.4) for the same oriented rectifiable Jordan curve Γ in M. It is not difficult to verify $\partial(J_x - J_y) = 0$ also in the case $x - y \in W_0^{1,2}(U, \mathbb{R}^3)$.

As a convenient notation we introduce

$$(3.7) \quad \mathcal{S}(\Gamma, A) = \{x \in W^{1,2}(U, A) : x|_{\partial U} \text{ satisfies (2.4) for } \Gamma\}$$

for the **class of admitted surfaces**. Here A is a closed subset of M defining the obstacle condition $x(U) \cap M \setminus A = \emptyset$ (if $A \neq M$) and Γ is a Jordan curve as above (which must be contained in A unless have $\mathcal{S}(\Gamma, A) = \emptyset$).

The idea for the geometric definition of the H-volume $\mathbf{V}_H(x, y)$ enclosed by $x, y \in \mathcal{S}(\Gamma, A)$ is to look for a 3-current Q on M with $\partial Q = J_x - J_y$ and to integrate H (or $H\Omega$, more precisely) over Q. Since M is of dimension 3, the integer multiplicity rectifiable 3-currents Q on M have a simple structure. Namely, they are represented by an integer valued summable multiplicity function i_Q on M such that

$$Q(\gamma) = \int_M i_Q \gamma \quad \text{for} \quad \gamma \in \mathcal{D}^3(M).$$

One may think of Q or i_Q as a set with integer multiplicities and finite absolute volume. The condition $\partial Q = J_x - J_y$ means that x and y parametrize the boundary of this set with multiplicities in the sense of Stokes' theorem

$$\int_M i_Q d\beta = \int_U x^\# \beta - \int_U y^\# \beta \quad \text{for} \quad \beta \in \mathcal{D}^2(M).$$

The finiteness of the mass of ∂Q is equivalent with i_Q being a BV function on M. The tentative definition of the H-volume is then $\mathbf{V}_H(x,y) = \int_M i_Q H\Omega$.

There are two obvious problems with this definition, the question of existence of Q and the question of uniqueness. To take care of the first problem one could try to fix a reference surface $y \in \mathcal{S}(\Gamma, A)$ and then work in the variational problem with the class of $x \in \mathcal{S}(\Gamma, A)$ such that $J_x - J_y$ is homologous to zero in A, i.e. the boundary of some integer multiplicity rectifiable 3-current on M with support in A. However, as we have pointed out in the introduction, weak convergence in $W^{1,2}(U, M)$ does not preserve such a homology condition. Therefore we are forced to require that $J_x - J_y$ is in fact homologically trivial in A for all surfaces $x, y \in \mathcal{S}(\Gamma, A)$ which we need to admit for competition in the later treatment of the variational problem. This amounts to the assumption that certain special 2-currents are boundaries in A.

3.1 Definition.

(i) A 2-current T on M with support in the closed subset A is called a **spherical 2-current** in A if it can be represented $T = f_\#[\![S^2]\!]$ with a map $f \in W^{1,2}(S^2, A)$ from the standard 2-sphere S^2 into A, i.e.

$$T(\beta) = \int_{S^2} f^\# \beta \quad \text{for} \quad \beta \in \mathcal{D}^2(M).$$

(ii) T is called **homologically trivial** in A if it is the boundary of an integer multiplicity rectifiable 3-current on M with support in A. If this is true for every spherical 2-current in A then we say that A is **homologically 2-aspherical in M.** ■

We note that homological triviality of $T = f_\#[\![S^2]\!]$ is equivalent to $T = \partial Q$ for some 3-current Q on M with $\mathbf{M}(Q) < \infty$ and $\operatorname{spt} Q \subset A$. By the constancy theorem [Fe, 4.1.7, 4.1.31] Q is unique up to a possible real multiple of $[\![M]\!]$, the 3-current defined by integration of 3-forms over M, and from the general theory of rectifiable currents [Fe, Chapter 4] it follows by localization in M that Q can be chosen rectifiable with integer multiplicity. Under mild assumptions on A it is possible to show that a spherical 2-current T in A can be approximated in mass by smooth maps from S^2 into A, and T is homologically trivial in A if this is true for the approximating smoothly represented spherical currents. This is made precise in

3.2 Lemma. *Suppose A is a uniform Lipschitz (resp. C^1) neighbourhood retract in $\mathbb{R}^m$ and $f \in W^{1,2}(S^2, A)$.*

(i) For every $\varepsilon > 0$ there exists $g \in W^{1,2}(S^2, A)$ such that $\|g - f\|_{W^{1,2}} < \varepsilon$, $g = f$ outside a set of measure less than ε in S^2, and g is Lipschitz continuous (resp. of class C^1).

(ii) If $0 < s \le \infty$, $0 < r < \infty$, $\mathbf{M}(f_\#[\![S^2]\!]) < s$, and $g_\#[\![S^2]\!]$ is the boundary of a 3-current of mass $\le r$ with support in A whenever $g\colon S^2 \to A$ is Lipschitz (resp. C^1) with $\mathbf{M}(g_\#[\![S^2]\!]) < s$, then also $f_\#[\![S^2]\!]$ is homologically trivial in A.

Proof. (i) From [EG, Theorem 6.6.3] we infer the existence of Lipschitz functions $g_\lambda: S^2 \to \mathbb{R}^m$ for $\lambda > 0$ such that $\|g_\lambda - f\|_{W^{1,2}} \to 0$ as $\lambda \to \infty$ and $g_\lambda = f$ outside $E_\lambda \subset S^2$ with $\lambda^2 |E_\lambda| \to 0$ as $\lambda \to \infty$. (We use stereographic projection from two antipodal points $p, -p$ in S^2 and a partition of unity subordinate to the covering $S^2 \setminus \{p\}, S^2 \setminus \{-p\}$ to pass from the sphere to the domain $\mathbb{R}^2$ treated in [EG]. By $|E_\lambda|$ we denote the 2-dimensional Hausdorff measure of E_λ.) Moreover, from step 4 in the proof of this approximation theorem presented in [EG] we see $\mathrm{Lip}(g_\lambda) \le C\lambda$ with a constant C depending on m only. Since for $|E_\lambda| < 4\pi$ no disc of radius $\pi(\frac{1}{4\pi}|E_\lambda|)^{1/2}$ can be contained in E_λ, we deduce that for each $w \in E_\lambda$ there exists $w' \in S^2 \setminus E_\lambda$ with $g_\lambda(w') = f(w')$ and

$$|g_\lambda(w') - g_\lambda(w)| \le C\lambda\pi(\frac{1}{4\pi}|E_\lambda|)^{1/2}.$$

On account of $\lim_{\lambda\to\infty} \lambda^2 |E_\lambda| = 0$ this implies, for λ sufficiently large, that $g_\lambda(S^2)$ is contained in a uniform neighbourhood $V_\varrho(A)$ where a Lipschitz retraction $\pi: V_\varrho(A) \to A$ is defined.

Setting $g = \pi \circ g_\lambda$ we now have $g \in \mathrm{Lip}(S^2, A)$, $g = f$ on $S^2 \setminus E_\lambda$, and

$$\|g - f\|_{W^{1,2},S^2} \le \|g\|_{W^{1,2},E_\lambda} + \|f\|_{W^{1,2},E_\lambda}.$$

Here the last summand has limit 0 as $\lambda \to \infty$, and

$$\|Dg\|^2_{L^2,E_\lambda} \le (\mathrm{Lip}\ g)^2 |E_\lambda| \le (\mathrm{Lip}\ \pi)^2 C^2 \lambda^2 |E_\lambda| \to 0,$$
$$\|g\|_{L^2,E_\lambda} \le \|\pi \circ g_\lambda - \pi \circ f\|_{L^2,E_\lambda} + \|f\|_{L^2,E_\lambda} \le (\mathrm{Lip}\pi)^2 \|g_\lambda - f\|_{L^2,E_\lambda} + \|f\|_{L^2,E_\lambda} \to 0.$$

Choosing λ sufficiently large the assertions $|E_\lambda| < \varepsilon$ and $\|g - f\|_{W^{1,2},S^2} < \varepsilon$ follow.

To treat the C^1 case we use [EG, Theorem 6.6.1] to approximate g_λ by C^1 mappings $\tilde{g}_\lambda: S^2 \to \mathbb{R}^m$ with $\mathrm{Lip}\ \tilde{g}_\lambda \le \tilde{C}\mathrm{Lip}\ g_\lambda$, $\tilde{g}_\lambda = g_\lambda$ on $S^2 \setminus \tilde{E}_\lambda$ and $|\tilde{E}_\lambda|$ small enough to ensure $\lambda^2 |\tilde{E}_\lambda| \to 0$ as $\lambda \to \infty$. ($\tilde{C}$ is a constant depending on m only.) Arguing as above we see

$$|\tilde{g}_\lambda(w) - g_\lambda(w)| \le (\mathrm{Lip}\ \tilde{g}_\lambda + \mathrm{Lip}\ g_\lambda)(\frac{1}{4\pi}|\tilde{E}_\lambda|)^{1/2} \le (\tilde{C}+1)C\lambda\pi(\frac{1}{4\pi}|\tilde{E}_\lambda|)^{1/2}$$

for $w \in S^2$, and we infer that $\tilde{g}_\lambda$ has image in $V_\varrho(A)$ for λ sufficiently large. Using a C^1 retraction $\pi: V_\varrho(A) \to A$ we now define $g = \pi \circ \tilde{g}_\lambda$ and readily verify the assertions made in (i), provided λ is chosen large enough.

(ii) This is an immediate consequence of (i) and the compactness theorem for integer multiplicity currents [Fe, 4.2.17] (in fact, we need only the easier two dimensional case here, i.e. the BV compactness theorem [EG, 5.2], [Zi, 5.3]) which is transferred from Euclidean spaces to Riemannian manifolds by standard localization procedures. ■

The following Lemma shows that $J_x - J_y$ is a 2-current of spherical type if $x, y \in \mathcal{S}(U, M)$ both satisfy the Plateau boundary condition for a given oriented rectifiable Jordan curve Γ in M. In the case $x - y \in W_0^{1,2}(U, \mathbb{R}^m)$ this property of $J_x - J_y$ is evident, for surfaces $x, y \in \mathcal{S}(\Gamma, M)$ with only weakly monotonic boundary traces, however, the proof requires a geometric construction which we present in part (i) of

3.3 Lemma.

(i) *Suppose $x \in \mathcal{S}(\Gamma, A)$ and $0 < \varepsilon < 1$. Then there exist $\tilde{x} \in \mathcal{S}(\Gamma, A)$ and $0 < \delta < \varepsilon$ such that $\mathbf{D}(\tilde{x}) < \varepsilon + \mathbf{D}(x)$, $\tilde{x}(w) = x\left(\frac{w}{1-\delta}\right)$ for $|w| \leq 1 - \delta$, $\tilde{x}(w) \in \Gamma$ for $1 - \delta \leq |w| \leq 1$, and $\tilde{x}|_{\partial U}$ is a biLipschitz parametrization of Γ; in particular $J_x = J_{\tilde{x}}$.*

(ii) *If $x, y \in W^{1,2}(U, A)$ with $x - y \in W_0^{1,2}(U, \mathbb{R}^m)$ or $x, y \in \mathcal{S}(\Gamma, A)$, then $J_x - J_y$ is a spherical 2-current in A.*

Proof. (i) We choose a biLipschitz parametrization $\gamma : S^1 \to \Gamma$. Then $x(e^{\mathrm{i}\vartheta}) = \gamma(e^{\mathrm{i}\varphi(\vartheta)})$ where $\varphi: [0, 2\pi] \to \mathbb{R}$ is continuous and nondecreasing with $\varphi(2\pi) - \varphi(0) = 2\pi$. We extend $\psi(\vartheta) = \varphi(\vartheta) - \vartheta$ from $[0, 2\pi]$ to $\mathbb{R}$ with period 2π. Since $e^{\mathrm{i}\varphi(\vartheta)} = \gamma^{-1} \circ x(e^{\mathrm{i}\vartheta})$ and γ^{-1} admits a Lipschitz extension $\mathbb{R}^m \to \mathbb{C}$, we see that ψ is of class $W_{2\pi}^{1/2,2}(\mathbb{R}, \mathbb{R})$. In fact, denoting by $\bar{x}: \overline{U} \to \mathbb{R}^m$ any continuous $W^{1,2}$ extension of $x|_{\partial U}$ we have an extension $(\vartheta, \varrho) \mapsto \gamma^{-1} \circ \bar{x}(e^{\mathrm{i}(\vartheta + \mathrm{i}\varrho)}) \in \mathbb{C}_{\neq 0}$ of $e^{\mathrm{i}\varphi}$ to a strip $\mathbb{R} \times [0, s]$ with $s > 0$ small, and this can be lifted via the exponential map to an extension of φ on this strip which is continuous and of class $W^{1,2}$ on $[0, 2\pi] \times [0, s]$.

We now consider the harmonic function $\eta : \mathbb{R} \times [0, 1] \to \mathbb{R}$ which is 2π-periodic in the first variable and has boundary values $\eta(\vartheta, 0) = \psi(\vartheta)$, $\eta(\vartheta, 1) = 0$. (This may be viewed as a harmonic function on $S^1 \times [0, 1]$.) Since the boundary values are of class $W_{2\pi}^{1/2,2}$, η has finite Dirichlet integral on $[0, 2\pi] \times [0, 1]$. From the maximum principle we infer $\frac{\partial}{\partial \vartheta}\eta(\vartheta, \sigma) \geq \sigma - 1$ for $0 < \sigma < 1$, because $\vartheta \mapsto \eta(\vartheta, 0) + \vartheta$ is nondecreasing and $\eta(\vartheta, 1) = 0$. (Consider difference quotients in the ϑ direction.)

We define

$$\tilde{x}(\varrho e^{\mathrm{i}\vartheta}) = \begin{cases} x(\frac{\varrho}{1-\delta} e^{\mathrm{i}\vartheta}), & \text{if } \ 0 \leq \varrho < 1 - \delta \\ \gamma \circ \exp[\mathrm{i}\vartheta + \mathrm{i}\eta(\vartheta, \varrho - 1 + \delta)], & \text{if } \ 1 - \delta \leq \varrho \leq 1. \end{cases}$$

Then $\tilde{x} \in W^{1,2}(U, \mathbb{R}^m)$, and $\tilde{x}|_{\partial U}$ is a biLipschitz parametrization of Γ since $\frac{\partial}{\partial \vartheta}(\vartheta + \eta(\vartheta, \delta)) \geq \delta > 0$ and, hence, $e^{\mathrm{i}\vartheta} \mapsto \exp[\mathrm{i}\vartheta + \mathrm{i}\eta(\vartheta, \delta)]$ is an orientation preserving diffeomorphism of S^1. The Dirichlet integral of $\tilde{x}$ exceeds $\mathbf{D}(x)$ by at most $(\mathrm{Lip}\gamma)^2$ times the energy of $(\vartheta, \sigma) \mapsto \vartheta + \eta(\vartheta, \sigma)$ on $[0, 2\pi] \times [0, \delta]$, and by choosing $\delta > 0$ sufficiently small this is less than ε. Finally, $J_{\tilde{x}} = J_x$ holds, because $\tilde{x}$ is a reparametrization of x on the disc $|w| < 1 - \delta$, while $\tilde{x}$ factors through γ and has zero area on the annulus $1 - \delta \leq \varrho \leq 1$.

(ii) In the case $x, y \in \mathcal{S}(\Gamma, A)$ we construct $\tilde{y}$ corresponding to y as in (i) and we extend the biLipschitz automorphism $(\tilde{y}|_{\partial U})^{-1} \circ (\tilde{x}|_{\partial U})$ of ∂U to a biLipschitz automorphism τ of $\overline{U}$. Then $J_x = J_{\tilde{x}}$, $J_y = J_{\tilde{y}} = J_{\tilde{y} \circ \tau}$, and $\tilde{y} \circ \tau - \tilde{x} \in W_0^{1,2}(U, \mathbb{R}^m)$. Composing $\tilde{x}$ with stereographic projection from the south pole and $\tilde{y} \circ \tau$ with sterographic projection from the north pole of S^2 we therefore obtain $f \in W^{1,2}(S^2, A)$ with

$$f_{\#}[\![S^2]\!] = J_{\tilde{x}} - J_{\tilde{y} \circ \tau} = J_x - J_y.$$

In the case $x - y \in W_0^{1,2}(U, \mathbb{R}^m)$ we apply the last construction directly to x and y. ∎

Suppose $J_x - J_y$ is homologically trivial in the closed set $A \subset M$, i.e. we have $J_x - J_y = \partial Q$ for some integer multiplicity rectifiable 3-current Q on M with support in A. By the constancy theorem [Fe, 4.1.7, 4.1.31] Q is unique up to an integer multiple of $[\![M]\!]$. It follows that Q is unique if A is a proper subset of M or if $A = M$ has infinite volume. On the other hand, if $A = M$ has finite volume $|M| < \infty$, in particular, if $A = M$ is closed manifold, then Q is not determined uniquely and we must impose an additional condition on Q to force uniqueness. A natural such condition is that the mass $\mathbf{M}(Q)$ is smaller than $\frac{1}{2}|M|$. (It would be sufficient to require $|\operatorname{spt} Q| < \frac{1}{2}|M|$.) With the uniqueness of Q being granted in this way we may now define the H-volume $\mathbf{V}_H(x, y)$ by integrating the bounded continuous function $H: M \to \mathbb{R}$ over Q.

3.4 Definition.

(i) We say that a spherical 2-current T in the closed set $A \subset M$ is **uniquely homologically trivial** in A if $T = \partial Q$ for some integer multiplicity rectifiable 3-current Q on M with support in A and either we are in **the non-closed case**, i.e. $A \neq M$ or $A = M$ with $|M| = \infty$, or we are **in the closed case** $A = M$ with $|M| < \infty$ and Q can be found with $\mathbf{M}(Q) < \frac{1}{2}|M|$.

(ii) Suppose x, y are in $W^{1,2}(U, A)$ with boundary condition $x, y \in \mathcal{S}(\Gamma, A)$ or $x - y \in W_0^{1,2}(U, \mathbb{R}^m)$. If $J_x - J_y$ is uniquely homologically trivial in A we define the **H-volume** enclosed by x and y in A

$$\mathbf{V}_H(x, y) = I_{x,y}(H\Omega) = \int_M i_{x,y} H\Omega,$$

where $I_{x,y}$ is the unique integer multiplicity rectifiable 3-current Q on M associated with the spherical 2-current $T = J_x - J_y$ as in (i), $i_{x,y}$ is the multiplicity function of $I_{x,y}$ and Ω the volume form of M. ■

The intuitive meaning of the H-volume $\mathbf{V}_H(x, y)$ is the integral of H over the set with multiplicities in A which is bounded in A by the parametric surfaces x and y.

3.5 Remarks. (1) The (not necessarily unique) homological triviality of $J_x - J_y$ in Definition 3.4 (ii) is granted, of course, when A is homologically 2-aspherical in the sense of Definition 3.1 (ii). For a connected uniform Lipschitz neighbourhood retract A this latter condition is equivalent to triviality of the Hurewicz homomorphism $\pi_2(A) \to H_2(A, \mathbb{Z})$, i.e. each continuous map $S^2 \to A$ induces the zero homomorphism $H_2(S^2, \mathbb{Z}) \to H_2(A, \mathbb{Z})$, and it is satisfied in particular if the homotopy group $\pi_2(A) = 0$ or the homolgy group $H_2(A, \mathbb{Z}) = 0$. These assertions follow from Lemma 3.2 above and from the description of the homology groups with integer coefficients in terms of integral currents [Fe, Section 4.4]. For example, if M is a 3-torus and A the complement of a smooth open ball in M, then $J_x - J_y$ is uniquely homologically trivial in A for all x, $y \in \mathcal{S}(\Gamma, A)$ (although $\pi_2(A) \neq 0$ and $H_2(A, \mathbb{Z}) \neq 0$).

(2) The H-volume $\mathbf{V}_H(x, y)$ is defined whenever $A \subset M$ is uniformly locally biLipschitz equivalent to the unit 3-ball, x, $y \in W^{1,2}(U, A)$ satisfy the boundary condition x, $y \in \mathcal{S}(\Gamma, A)$ or $x - y \in W_0^{1,2}(U, \mathbb{R}^m)$, and the mass $\mathbf{M}(J_x - J_y)$ is sufficiently small. This can be proved as in [DS2, Proposition 2.2]. ■

It remains to verify that the H-volume defined here has the properties used in Section 2 to derive the variational (in-)equality. For this, we prove

3.6 Lemma. *Suppose $x, y \in W^{1,2}(U, A)$ are as in Definition 3.4 (ii) so that $\mathbf{V}_H(x, y)$ is defined.*

(i) If $A \subset M$ admits a uniform Lipschitz neighbourhood retraction π in $\mathbb{R}^m$, $\tilde{x} \in W^{1,2}(U, A)$, $\tilde{x} - x \in W_0^{1,2}(U, \mathbb{R}^m)$ or $\tilde{x}, x \in \mathcal{S}(\Gamma, A)$, and $\|\tilde{x} - x\|_{L^\infty}$ is smaller than a positive constant (depending only on A and in the closed case additionally on an upper bound for $\mathbf{D}(x) + \mathbf{D}(\tilde{x})$ and a positive lower bound for $\frac{1}{2}|M| - \mathbf{M}(I_{x,y})$), then $\mathbf{V}_H(\tilde{x}, y)$, $\mathbf{V}_H(\tilde{x}, x)$ are also defined and satisfy

$$\mathbf{V}_H(\tilde{x}, y) - \mathbf{V}_H(x, y) = \mathbf{V}_H(\tilde{x}, x),$$

$$|\mathbf{V}_H(\tilde{x}, x)| \leq \sup_M |H| \, \|\tilde{x} - x\|_{L^\infty} (\mathrm{Lip}\pi)^3 \tfrac{1}{2}[\mathbf{D}_G(x) + \mathbf{D}_G(\tilde{x})],$$

where $G = \{w \in U : \tilde{x}(w) \neq x(w)\}$.

(ii) If Φ_t^Y is the flow of a compactly supported C^1 vector field Y on M with $\Phi_t^Y(A) \subset A$ for small $t > 0$, η is a nonnegative C^1 function with compact support in U, and $x_t(u, v) = X(t, u, v)$ with $X(s, u, v) = \Phi^Y(s\eta(u, v), x(u, v))$, then $\mathbf{V}_H(x_t, y)$ and $\mathbf{V}_H(x_t, x)$ are also defined for small $t \geq 0$ and

$$\mathbf{V}_H(x_t, y) - \mathbf{V}_H(x, y) = \mathbf{V}_H(x_t, x) = \int_U \int_0^t (H \circ X)\langle \Omega \circ X, X_s \wedge X_u \wedge X_v \rangle ds du dv.$$

Proof. (i) With the affine homotopy $X(s, u, v) = (1 - s)x(u, v) + s\tilde{x}(u, v)$ we define the 3-current Q on $\mathbb{R}^m$ by

$$(3.8) \quad Q(\gamma) = \int_U \int_0^1 \langle \gamma \circ X, X_s \wedge X_u \wedge X_v \rangle ds du dv$$

for $\gamma \in D^3(\mathbb{R}^m)$. From the homotopy formula [Fe, 4.1.9] and the boundary condition satisfied by x and $\tilde{x}$ we infer $\partial Q = J_{\tilde{x}} - J_x$. For the mass of Q we obtain from (3.8) the estimate

$$(3.9) \quad \mathbf{M}(Q) \leq \|\tilde{x} - x\|_{L^\infty} \tfrac{1}{2}[\mathbf{D}_G(x) + \mathbf{D}_G(\tilde{x})].$$

For $\|\tilde{x} - x\|_{L^\infty}$ sufficiently small $\pi_\# Q$ is then an integer multiplicity rectifiable 3-current in A with boundary

$$\partial \pi_\# Q = \pi_\# \partial Q = J_{\tilde{x}} - J_x$$

and with mass

$$(3.10) \quad \mathbf{M}(\pi_\# Q) \leq (\mathrm{Lip}\ \pi)^3 \mathbf{M}(Q).$$

In the non-closed case it follows immediately that

$$\pi_\# Q = I_{\tilde{x}, x}, \quad I_{\tilde{x}, y} = I_{\tilde{x}x} + I_{x,y}.$$

In the closed case the same equation is true if

$$\mathbf{M}(I_{x,y}) + \mathbf{M}(\pi_{\#}Q) < \frac{1}{2}|M|,$$

which can be deduced from $\mathbf{M}(I_{x,y}) < \frac{1}{2}|M|$ and (3.9), (3.10) if $\|\tilde{x} - x\|_{L^\infty}$ is sufficiently small.

For the H-volume we conclude

$$\mathbf{V}_H(\tilde{x}, y) - \mathbf{V}_H(x, y) = \int_M (i_{\tilde{x},y} H\Omega - i_{x,y} H\Omega) = \int_M i_{\tilde{x},x} H\Omega = \mathbf{V}_H(\tilde{x}, x) = \pi_{\#}Q(H\Omega).$$

Now the estimate asserted in (i) follows from (3.9), (3.10), and approximation of $H\Omega$ by smooth $\gamma \in \mathcal{D}^3(M)$ with $|\gamma| \leq |H|$ pointwise on M.

(ii) We now define, for small $t \geq 0$,

$$(3.11)\ Q_t(\gamma) = \int_U \int_0^t \langle \gamma \circ X, X_s \wedge X_u \wedge X_v \rangle ds du dv$$

for $\gamma \in \mathcal{D}^3(M)$ using the present homotopy X. Then Q_t is an integer multiplicity rectifiable 3-current in A with $\partial Q_t = J_{x_t} - J_x$. This follows from the homotopy formula [Fe, 4.1.9] if x is Lipschitz continuous on U, and it is readily proved with the present hypotheses on x by suitable approximation arguments (e.g. [EG, 6.6]). For the mass of Q_t we estimate

$$(3.12)\ \mathbf{M}(Q_t) \leq tC^3 \sup_U \eta \left[\mathbf{D}_G(x) + \tfrac{1}{3}t^2 \mathbf{D}_G(\eta)\right]$$

where $G = \operatorname{spt}[\eta(Y \circ x)]$ and C is a bound for $|D\Phi^Y|$ on $[0, t \sup_U \eta] \times \operatorname{spt} Y$. For $t \geq 0$ sufficiently small we deduce from (3.12)

$$\mathbf{M}(I_{x,y}) + \mathbf{M}(Q_t) < \tfrac{1}{2}|M|$$

in the closed case, and

$$I_{x_t,x} = Q_t, \quad I_{x_t,y} = I_{x_t,x} + I_{x,y}$$

follows as in (i). Appxoximating $H\Omega$ with $\gamma \in \mathcal{D}^3(M)$ we see from (3.11) that $\mathbf{V}_H(x_t, x) = Q_t(H\Omega)$ has the integral representation asserted in (ii). ■

With (ii) we have verified that the homotopy formula (2.13) is valid for the variations considered in (ii) and the volume definition $\mathbf{V}_H(x) = \mathbf{V}_H(x, y)$, where $y \in W^{1,2}(U, A)$ is a fixed reference surface and x, y satisfy the conditions of Definition 3.4. Thus, all the conclusions of Section 1 are again valid.

Continuity of H-volume with respect to convergence of surfaces in $W^{1,2}$ norm can be discussed as in [Ste1]. We will only need the continuity property expressed in part (i) of the preceding lemma. With regard to the variational approach based on minimization of the energy $\mathbf{E}_H(x) = \mathbf{D}(x) + 2\mathbf{V}_H(x, y)$ the continuity of H-volume with respect to weak $W^{1,2}$ convergence would be useful. However, the H-volume functional is not (semi-)continuous in this sense.

4 A general existence theorem

In this section we prove a general theorem about the existence of weak solutions to the Plateau problem $\mathcal{P}(H,\Gamma)$ in a Riemannian 3-manifold $M \subset \mathbb{R}^m$ as described in Section 2. We use the direct method of the calculus of variations to minimize the energy $\mathbf{E}_H(x) = \mathbf{D}(x) + 2\mathbf{V}_H(x,y)$ on suitable subclasses of $\mathcal{S}(\Gamma, A)$, the set of parametric surfaces $x \in W^{1,2}(U,M)$ which satisfy the Plateau boundary condition (2.4) for the oriented rectifiable closed Jordan curve Γ in M and have their image (essentially) contained in the closed subset A of M. $y \in \mathcal{S}(\Gamma, A)$ will denote a fixed reference surface.

The natural topology to use an $\mathcal{S}(\Gamma, A)$ is weak convergence in $W^{1,2}(U,\mathbb{R}^m)$. Dirichlet's integral $\mathbf{D}(\,\cdot\,)$ is lowersemicontinuous, however, the H-volume functional $\mathbf{V}_H(\,\cdot\,,y)$ is not (semi-)continuous for this type of convergence. The reason is that for a $W^{1,2}$ weakly convergent sequence $x_n \rightharpoonup x$ in $\mathcal{S}(\Gamma, A)$ a large part of volume and of area of x_n may be parametrized over a small subset of the unit disc U with measure approaching zero as $n \to \infty$. We may think of a "bubble" splitting off in the limit which carries away a certain amount of volume and area. In contrast with the geometric measure theoretic treatment of the Platen problem in [DS2], this bubbling phaenomenon can occur even for uniformly bounded weakly convergent sequences of parametric surfaces, and it may change the homology class in the limit. For the analytical treatment it is important to have a precise technical description of the possible bubbling. This is presented in the following lemma which states that we can replace the x_n by uniformly close parametric surfaces $\tilde{x}_n$ which coincide with the weak limit x on a large subset of the domain, while we have control of all the relevant quantities. The construction used in the proof may be useful also in other situations where one has to deal with bubbling phaenomena. In Euclidean space $\mathbb{R}^3$ another construction, using harmonic replacement on open subsets of U with small measure, has been applied in [Ste1] to prove similar statements as in the lemma below. However, it seems that this latter construction cannot easily be adapted to the case of parametric surfaces in a Riemannian 3-manifold, and our method here is more flexible.

4.1 Lemma. *Suppose $x_n \rightharpoonup x$ weakly in $W^{1,2}(U,\mathbb{R}^m)$ and $x_n|_{\partial U} \to x|_{\partial U}$ uniformly in $L^\infty(\partial U,\mathbb{R}^m)$. Then, for every $\varepsilon > 0$ there exist $R > 0$, a measurable set $G \subset U$, and mappings $\tilde{x}_n \in W^{1,2}(U,\mathbb{R}^m)$ such that the following assertions are true after passing to a subsequence (still denoted $x_n, \tilde{x}_n$):*

(i) $\tilde{x}_n = x$ *on* $U \backslash G$ *with* $\mathcal{L}^2(G) < \varepsilon$;

(ii) $\tilde{x}_n|_{\partial U} = x|_{\partial U}$;

(iii) $\tilde{x}_n(w) = x_n(w)$ *if* $|x_n(w)| \geq R$ *or* $|x_n(w) - x(w)| \geq 1$;

(iv) $\lim_{n\to\infty} \|\tilde{x}_n - x_n\|_{L^\infty(U,\mathbb{R}^m)} = 0$;

(v) $\tilde{x}_n \rightharpoonup x$ *weakly in* $W^{1,2}(U,\mathbb{R}^m)$ *as* $n \to \infty$;

(vi) $\limsup_{n\to\infty}[\mathbf{D}_G(\tilde{x}_n) + \mathbf{D}_G(x)] \leq \varepsilon + \liminf_{n\to\infty}[\mathbf{D}(x_n) - \mathbf{D}(x)]$.

(vii) if the x_n have values in a closed set $A \subset \mathbb{R}^m$ which admits neighbourhood retractions with Lipschitz constant arbitrarily close to 1 on neighbourhoods of compact subsets, then the $\tilde{x}_n$ can be chosen to have also values in A.

Proof. Using the theorems of Rellich and Egoroff we can find $R > 3, \frac{1}{3} \geq \delta_n \searrow 0$, and $G \subset U$ with $\mathcal{L}^2(G) < \varepsilon$ and $\mathbf{D}_G(x) < \varepsilon'$, where $\varepsilon' > 0$ will be determined later, such that $\|x|_{\partial U}\|_{L^\infty} \leq \frac{1}{3}R$, $\sup_{U\setminus G}|x| \leq \frac{1}{3}R$, $\sup_{U\setminus G}|x_n - x| \leq \delta_n$, and $\|x_n|_{\partial U} - x|_{\partial U}\|_{L^\infty} \leq \delta_n$ (after passing to a subsequence). We choose $\eta \in C^1(\mathbb{R})$ with $\eta = 1$ on $]-\infty, \frac{1}{3}R]$, $\eta = 0$ on $[\frac{2}{3}R, \infty[$, $0 \leq -\eta' \leq 4R^{-1}$ on $\mathbb{R}$, and we set $\vartheta_n(t) = 1$ for $t \leq \delta_n$, $\vartheta_n(t) = (t^{-1} - 1)/(\delta_n^{-1} - 1)$ for $\delta_n \leq t \leq 1$, $\vartheta_n(t) = 0$ for $t \geq 1$.

Letting

$$\tilde{x}_n = x_n + (\eta \circ |x|)(\vartheta_n \circ |x_n - x|)(x - x_n) \tag{4.1}$$

and noting that $\eta \circ |x|$ and $\vartheta_n \circ |x_n - x|$ have values in $[0,1]$ and boundary traces 1 on ∂U we deduce immediately (i) and (ii). From the properties of ϑ_n and the observation that $|x_n(w)| \geq R$ implies $|x(w)| \geq \frac{2}{3}R$ or $|x_n(w) - x(w)| \geq \frac{1}{3}R > 1$, i.e. $\eta(|x(w)|) = 0$ or $\vartheta_n(|x_n(w) - x(w)|) = 0$, we infer (iii). Since $0 \leq \eta \leq 1$ and $\sup_{t\geq 0}\vartheta_n(t)t = \delta_n \to 0$ as $n \to 0$ we also see (iv). To prove (vi) we differentiate (4.1) and calculate (with the interpretation $\frac{x}{|x|}\cdot Dx = 0$ where $x = 0$)

$$\begin{aligned} D\tilde{x}_n = {} & Dx_n + (\eta \circ |x|)D[(\vartheta_n \circ |x_n - x|)(x - x_n)] \\ & + (\eta' \circ |x|)(\frac{x}{|x|}\cdot Dx)(\vartheta_n \circ |x_n - x|)(x - x_n), \end{aligned} \tag{4.2}$$

with

$$D[\vartheta_n \circ |x_n - x|)(x - x_n)] = D(x - x_n) \quad \text{where} \quad |x - x_n| \leq \delta_n$$

and

$$\begin{aligned} &= (\vartheta_n' \circ |x_n - x|)\Big[\frac{x_n - x}{|x_n - x|}\cdot D(x_n - x)\Big] \otimes (x - x_n) + (\vartheta_n \circ |x_n - x|)D(x - x_n) \\ &= -|x_n - x|(\vartheta_n' \circ |x_n - x|)PD(x_n - x) - (\vartheta_n \circ |x_n - x|)D(x_n - x) \\ &= \frac{\delta_n}{1 - \delta_n}PD(x_n - x) - (\vartheta_n \circ |x_n - x|)P^\perp D(x_n - x) \quad \text{where} \quad |x - x_n| > \delta_n. \end{aligned}$$

Here P denotes the field of rank 1 orthogonal projections

$$P : \mathbb{R}^m \ni \xi \mapsto |x_n - x|^{-2}((x_n - x)\cdot\xi)(x_n - x)$$

on $\mathbb{R}^m$, $P^\perp = \mathrm{Id} - P$, and for ϑ_n' we have used

$$-t\vartheta_n'(t) = \vartheta_n(t) + \frac{\delta_n}{1 - \delta_n} \quad \text{for} \quad t > \delta_n.$$

Inserting into (4.2) we end up with

$$\begin{aligned} D\tilde{x}_n = {} & (1 - \eta \circ |x|)Dx_n + (\eta \circ |x|)Dx \\ & + (\eta' \circ |x|)(\frac{x}{|x|}\cdot Dx)(x - x_n) \quad \text{where} \quad |x - x_n| \leq \delta_n, \end{aligned} \tag{4.3}$$

and

$$
(4.4)\qquad \begin{aligned} D\tilde{x}_n = {} & [1-(\eta\circ|x|)(\vartheta_n\circ|x_n-x|)]P^{\perp}Dx_n + \Big[1+(\eta\circ|x|)\frac{\delta_n}{1-\delta_n}\Big]PDx_n \\ & +(\eta\circ|x|)(\vartheta_n\circ|x_n-x|)P^{\perp}Dx-(\eta\circ|x|)\frac{\delta_n}{1-\delta_n}PDx \\ & +(\eta'\circ|x|)\Big(\frac{x}{|x|}\cdot Dx\Big)(\vartheta_n\circ|x_n-x|)(x-x_n) \quad \text{where} \quad |x-x_n|>\delta_n. \end{aligned}
$$

We can now estimate $|D\tilde{x}_n|$ on G. At (almost all) points $w\in G$ with $|x(w)-x_n(w)|\le\delta_n$ we have, by (4.3) and the properties of η and ϑ_n,

$$
|D\tilde{x}_n|\le|Dx_n|+|Dx|+\frac{4}{R}\delta_n|Dx|\le|Dx_n|+2|Dx|,
$$

while in the case $|x(w)-x_n(w)|>\delta_n$ we use (4.4) and $\vartheta_n(t)t\le\delta_n$ to see, at w,

$$
\begin{aligned} |D\tilde{x}_n| \le {} & \Big[|P^{\perp}Dx_n|^2+(1-\delta_n)^{-2}|PDx_n|^2\Big]^{1/2}+|P^{\perp}Dx| \\ & +\frac{\delta_n}{1-\delta_n}|PDx|+\frac{4}{R}\delta_n|Dx|. \end{aligned}
$$

In any case, we have

$$
|D\tilde{x}_n|\le\frac{1}{1-\delta_n}|Dx_n|+2|Dx| \quad \text{on} \quad G,
$$

and with Young's inequality we deduce, for given $\lambda>0$

$$
(4.5)\qquad \mathbf{D}_G(\tilde{x}_n)\le(1+\lambda)\mathbf{D}_G(x_n)+\frac{4(1+\lambda)}{1+\lambda-(1-\delta_n)^{-2}}\mathbf{D}_G(x),
$$

provided n is large enough to secure $(1-\delta_n)^{-2}<1+\lambda$. From (4.5) we conclude

$$
(4.6)\qquad \begin{aligned} & \limsup_{n\to\infty}[\mathbf{D}_G(\tilde{x}_n)+\mathbf{D}_G(x)] \\ & \le(1+\lambda)\limsup_{n\to\infty}[\mathbf{D}(x_n)-\mathbf{D}(x)]+(2+\lambda+4\frac{1+\lambda}{\lambda})\mathbf{D}_G(x), \end{aligned}
$$

because $\delta_n\to 0$ as $n\to\infty$ and

$$
\limsup_{n\to\infty}\mathbf{D}_{U\setminus G}(x_n)\ge\mathbf{D}_{U\setminus G}(x)
$$

on account of the weak convergence $x_n\rightharpoonup x$ in $W^{1,2}(U,\mathbb{R}^m)$.

We now fix $\lambda>0$ such that $\lambda\sup_n\mathbf{D}(x_n)\le\frac{1}{2}\varepsilon$, and then determine ε' with $\mathbf{D}_G(x)<\varepsilon'$ such that $[2+\lambda+4\lambda^{-1}(1+\lambda)]\varepsilon'\le\frac{1}{2}\varepsilon$. Then assertion (vi) follows from (4.6) after passing to a subsequence in order to replace lim sup by lim inf on the right-hand side. Since $\sup_n\mathbf{D}_G(\tilde{x}_n)<\infty$ by (vi) and $\mathbf{D}_{U\setminus G}(\tilde{x}_n)=\mathbf{D}_{U\setminus G}(x)$ by (i) we deduce (v) from (iv) and the weak convergence $x_n\rightharpoonup x$. Finally, to prove (vii) we apply the proceding construction (with $\frac{1}{2}\varepsilon$ instead of ε) and we replace the resulting $\tilde{x}_n$ by $\pi\circ\tilde{x}_n$, where $\pi:B\to A$ is a Lipschitz retraction with $B \subset \mathbb{R}^m$ containing a neighbourhood of the compact set $\{a \in A : |a| \le R\}$ and with Lipschitz constant sufficiently close to one. ∎

One may interprete the quantity $\liminf_{n\to\infty}[\mathbf{D}(x_n) - \mathbf{D}(x)]$ as the Dirichlet integral of the bubble that splits off in the weak $W^{1,2}$ limit $x_n \rightharpoonup x$. To prove lower semicontinuity of the energy $\mathbf{E}_H(x_n) = \mathbf{D}(x_n) + 2\mathbf{V}_H(x_n, y)$ as $x_n \rightharpoonup x$ in $\mathcal{S}(\Gamma, A)$, we must estimate the H-volume jump $\limsup_{n\to\infty} 2|\mathbf{V}_H(x_n, y) - \mathbf{V}_H(x, y)|$ by this quantity. This will be done by passing from x_n to $\tilde{x}_n$ constructed in the preceding lemma and assuming a suitable isoperimetric inequality for the H-volume which is formulated in the definition below. This isoperimetric condition, with constant $c < 1$, will also imply a bound for the $W^{1,2}$ norm of energy minimizing sequences, which is the remaining ingredient we need for the direct method of the calculus of variations. We recall the notions of spherical 2-current from Definition 3.1 and of unique homological triviality from Definition 3.4 (i).

4.2 Definition. Suppose $0 < s \le \infty$ and every spherical 2-current T with support in $A \subset M$ and $\mathbf{M}(T) \le s$ is uniquely homologically trivial in A, i.e. there exists an integer multiplicity rectifiable 3-current Q on M with support in A and $\partial Q = T$, and additionally with $\mathbf{M}(Q) < \frac{1}{2}|M|$ in the closed case.

(i) If $0 \le c < \infty$ and the inequality

$$(4.7)\quad 2|\langle Q, H\Omega\rangle| = 2\left|\int_A i_Q H\Omega\right| \le c\mathbf{M}(T)$$

holds for all such T and Q (where i_Q is the multiplicity function of Q and Ω the volume form on M) then we say that a **spherical isoperimetric condition** of type c, s is valid for H on A.

(ii) If $0 \le \tilde{c} < \infty$ and the inequality

$$(4.8)\quad \mathbf{M}(Q) = \int_A |i_Q|\Omega \le \tilde{c}\mathbf{M}(T)$$

is satisfied for all T and Q as above, then we say that a **linear spherical isoperimetric inequality** of type $\tilde{c}$, s holds on A. ∎

The difference between this definition and the unrestricted isoperimetric condition or inequalities used in [DS2] is that, on the one side, we need only consider the special spherical 2-currents here but, on the other side, we have to require unique homological triviality for these. In [DS2] this requirement was not necessary, because there we could work in a given homology class. Moreover, by the decomposition theorem for integer multiplicity rectifiable currents in the top dimension one could formulate the unrestricted isoperimetric condition simply in the form

$$2\left|\int_F H\Omega\right| \le c\mathbf{P}(F) \quad \text{for} \quad \mathbf{P}(F) \le s$$

for all sets $F \subset A$ with finite perimeter $\mathbf{P}(F) \le s$, and the unrestricted linear isoperimetric inequality similarly

$$|F| := \int_F \Omega \le \tilde{c}\mathbf{P}(F)$$

for the same sets F. Of course, if the unique homological triviality assumption made in Definition 4.2 is satisfied, then such unrestricted inequalities imply the corresponding inequalities for spherical currents. Therefore the results of [DS2] apply here with the additional hypothesis of unique homological triviality for spherical 2-currents of mass $\leq s$. Apart from the uniqueness requirement (in the closed case) this additional hypothesis is satisfied when A is homologically 2-aspherical, in particular when A is a uniform Lipschitz neighbourhood retract with $H_2(A, \mathbb{Z}) = 0$ (cf. Remark 3.5 (1)).

On the other hand, it is clear that there are situations where a *spherical* isoperimetric condition or linear isoperimetric inequality is less restrictive than an unrestricted condition of the same type. For example, if $A = M = S^1 \times S^1 \times \mathbb{R}$, then a linear spherical isoperimetric condition of type $c(s), s$ holds for each $0 < s < \infty$ with the same optimal constant $c(s)$ as in the case of the Euclidean space $\mathbb{R}^3$, because continuous maps $f : S^2 \to M$ can be lifted to the universal cover $\mathbb{R}^3$ of M. However, clearly no unrestricted linear isoperimetric inequality of type $\tilde{c}, s$ is valid on M a finite constant $\tilde{c}$ if $s \geq 2|S^1 \times S^1| = 8\pi^2$. A similar observation holds for isoperimetric conditions, e.g. with $H \equiv 1$ on M.

4.3 Remark. By Lemma 3.3 (ii) and Definition 3.4 a spherical isoperimetric condition of type c, s for H on A implies that the H-volume $\mathbf{V}_H(x, y)$ is defined and satisfies

$$(4.9) \quad 2|\mathbf{V}_H(x, y)| \leq c\mathbf{M}(J_x - J_y),$$

whenever x, y are parametric surfaces in $W^{1,2}(U, A)$ with $\mathbf{M}(J_x - J_y) \leq s$ and with boundary condition $x, y \in \mathcal{S}(\Gamma, A)$ or $x, y \in W_0^{1,2}(U, \mathbb{R}^m)$. It is this form in which the spherical isoperimetric condition will be used in the sequel. ∎

We are now prepared for the proof of the main theorem. A result of this type was first proved, for surfaces with constant prescribed mean curvature H in Euclidean 3-space, by Wente [We]. An improved version, valid for variable H and surfaces in $\mathbb{R}^3$, was later given by the second author [Ste1]. We recall the definition of the class $\mathcal{S}(\Gamma, A)$ from (3.7), the closed case and the non-closed case distinguished in Definition 3.4 (i), and the notion of weak solution to the Plateau problem $\mathcal{P}(H, \Gamma)$ in M from Section 2.

4.4 Theorem. *Suppose A is closed in the 3-manifold $M \subset \mathbb{R}^m$, Lipschitz neighbourhood retractions onto A exist in $\mathbb{R}^m$ with Lipschitz constant arbitrarily close to 1, the function $H : A \to \mathbb{R}$ is bounded and continuous, a spherical isoperimetric condition of type c, s is valid for H on A, a linear spherical isoperimetric inequality of type $\tilde{c}, s$ holds in the closed case with $\tilde{c}s < \frac{1}{2}|M|$, the oriented rectifiable closed Jordan curve Γ is contained in A, and the reference surface $y \in \mathcal{S}(\Gamma, A)$ satisfies $(1 + \sigma)\mathbf{D}(y) \leq s$ for some $1 < \sigma \leq \infty$. Let $\mathcal{S}(\Gamma, A; \sigma)$ be the class of parametric surfaces $\tilde{x} \in \mathcal{S}(\Gamma, A)$ with $\mathbf{D}(\tilde{x}) \leq \sigma\mathbf{D}(y)$. Then the following assertions hold:*

(i) If $\sigma < \infty$ and $c \leq 1$ or if $\sigma = \infty$ and $c < 1$, then the variational problem

$$(4.10) \quad \mathbf{E}_H(\tilde{x}) = \mathbf{D}(\tilde{x}) + 2\mathbf{V}_H(\tilde{x}, y) \to \min \quad \text{on} \quad \mathcal{S}(\Gamma, A; \sigma)$$

has a solution.

(ii) If

$$(4.11)\ c \le \frac{\sigma - 1}{\sigma + 1} \quad \textit{resp.} \quad c < 1 \quad \textit{in the case} \quad \sigma = \infty,$$

then the variational problem in (i) posesses a solution x with $\mathbf{D}(x) < \sigma\mathbf{D}(y)$; if strict inequality holds in (4.11) or y itself is not a solution to (4.10), then $\mathbf{D}(x) < \sigma\mathbf{D}(y)$ is true for each solution x of (4.10).

(iii) If A is the closure of a C^2 domain in M and

$$(4.12)\ |H| \le H_{\partial A} \quad \textit{pointwise on} \quad \partial A,$$

where $H_{\partial A}$ denotes the (inward) boundary mean curvature of A (no condition in the case $A = M$), then each minimizer x of (4.10)*with $\mathbf{D}(x) < \sigma\mathbf{D}(y)$ is a weak solution to the Plateau problem $\mathcal{P}(H, \Gamma)$ in A; moreover, if $|H(a)| < H_{\partial A}(a)$ holds at some point $a \in (\partial A)\backslash\Gamma$, then x does not meet a neighbourhood of this point.*

Proof. (i) Recalling (3.5) we first observe

$$(4.13)\ \mathbf{M}(J_{\tilde{x}} - J_y) \le \mathbf{D}(\tilde{x}) + \mathbf{D}(y) \le (\sigma + 1)\mathbf{D}(y) \le s$$

for $\tilde{x} \in \mathcal{S}(\Gamma, A; \sigma)$ and we infer from Definition 3.4 and the spherical isoperimetric condition that $\mathbf{V}_H(\tilde{x}, y)$ is defined for $\tilde{x} \in \mathcal{S}(\Gamma, A; \sigma)$ with

$$(4.14)\ |\mathbf{D}(\tilde{x}) - \mathbf{E}_H(\tilde{x})| = 2|\mathbf{V}_H(\tilde{x}, y)| \le c[\mathbf{D}(\tilde{x}) + \mathbf{D}(y)],$$

by (4.9) and (4.13). Choosing a minimizing sequence x_n for (4.10) we next note $\sup_n \mathbf{D}(x_n) < \infty$, because we have assumed $c < 1$ in the case $\sigma = \infty$ and (4.14) with $c < 1$ clearly implies a bound for $\mathbf{D}(x_n)$.

Since the energy functional $\mathbf{E}_H$ is invariant with respect to reparametrization of surfaces by conformal automorphisms of the unit disc U we may assume a three-point-condition, i.e. $x_n(w_i) = a_i$ holds for three fixed points $w_0, w_1, w_2 \subset \partial U$ and $a_0, a_1, a_2 \in \Gamma$ numbered compatible with the orientation. As is well known from the theory of parametric minimal surfaces (see [DHKW], [Jo3], [Ni1], [Ni2] or [Str]), it then follows from a Lemma of Courant and Lebesgue and the Jordan curve property of Γ that the sequence of boundary traces $x_n|_{\partial U}$ is equicontinuous. Passing to a subsequence and taking Rellich's theorem into account we may therefore assume that the x_n converge $W^{1,2}$ weakly and almost everywhere on U to a surface $x \in \mathcal{S}(\Gamma, A; \sigma)$ such that $x_n|_{\partial U} \to x|_{\partial U}$ holds uniformly on ∂U.

We now apply Lemma 4.1 with a given $\varepsilon > 0$ to obtain, after passage to another subsequence, the surfaces $\tilde{x}_n \in \mathcal{S}(\Gamma, A)$. From statements (i), (ii), (iv) and (vi) of this lemma and from part (i) of Lemma 3.6 we see

$$(4.15)\ \mathbf{V}_H(\tilde{x}_n, y) - \mathbf{V}_H(x_n, y) = \mathbf{V}_H(\tilde{x}_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

(It is clear from the proof of Lemma 3.6 that we do not need a *uniform* Lipschitz neighbourhood retraction onto A here, because we have $\tilde{x}_n(w) = x_n(w)$ for $|x_n(w)| \ge R$, by Lemma 4.1 (iii).)

Choosing $2\varepsilon < \mathbf{D}(x)$ we infer from (3.5) and Lemma 4.1 (vi) that

(4.16) $\mathbf{M}(J_{\tilde{x}_n} - J_x) \leq \mathbf{D}_G(\tilde{x}_n) + \mathbf{D}_G(x) \leq 2\varepsilon + \mathbf{D}(x_n) - \mathbf{D}(x) < \sigma\mathbf{D}(y) \leq s$

holds for large n. Therefore, the spherical isoperimetric condition with constant $c \leq 1$ implies for sufficiently large n, by Remark 4.3 and (4.16),

(4.17) $2|\mathbf{V}_H(\tilde{x}_n, x)| \leq 2\varepsilon + \mathbf{D}(x_n) - \mathbf{D}(x)$.

We claim

(4.18) $\mathbf{V}_H(\tilde{x}_n, y) = \mathbf{V}_H(\tilde{x}_n, x) + \mathbf{V}_H(x, y)$.

To prove this we note that, on account of (4.14), (4.15) and (4.17), we already know that the H-volumes in (4.18) are defined, i.e. we have the integer multiplicity rectifiable 3-currents $I_{\tilde{x}_n,y}, I_{\tilde{x}_n,x}, I_{x,y}$ with support in A which are uniquely determined by their boundaries $J_{\tilde{x}_n} - J_y$, $J_{\tilde{x}_n} - J_x$, $J_x - J_y$ and, in the closed case, by the additional condition of mass $< \frac{1}{2}|M|$. (4.18) follows if we verify

$$I_{\tilde{x}_n,y} = I_{\tilde{x}_n,x} + I_{x,y}.$$

This is clear in the non-closed case, because the currents on both sides have the same boundary, and it can be seen in the closed case from the following chain of inequalities, valid for $\varepsilon > 0$ fixed suffciently small and for all sufficiently large n:

$$\begin{aligned}\mathbf{M}(I_{\tilde{x}_n,x} + I_{x,y}) &\leq \mathbf{M}(I_{\tilde{x}_n,x}) + \mathbf{M}(I_{x,y}) \leq \tilde{c}\mathbf{M}(J_{\tilde{x}_n} - J_x) + \tilde{c}\mathbf{M}(J_x - J_y)\\ &\leq \tilde{c}[2\varepsilon + \mathbf{D}(x_n) - \mathbf{D}(x)] + \tilde{c}[\mathbf{D}(x) + \mathbf{D}(y)] \leq \tilde{c}[2\varepsilon + (1+\sigma)\mathbf{D}(y)]\\ &\leq \tilde{c}(2\varepsilon + s) < \tfrac{1}{2}|M|,\end{aligned}$$

where we have used (4.13), (4.16), and the linear spherical isoperimetric inequality with $\tilde{c}s < \frac{1}{2}|M|$ which is assumed in the closed case.

Now, from (4.15), (4.18), (4.17) we deduce

$$\begin{aligned}\mathbf{E}_H(x_n) &= \mathbf{D}(x_n) + 2\mathbf{V}_H(x_n, y)\\ &= \mathbf{E}_H(x) - \mathbf{D}(x) + \mathbf{D}(x_n) + 2\mathbf{V}_H(\tilde{x}_n, x) - 2\mathbf{V}_H(\tilde{x}_n, x_n)\\ &\geq \mathbf{E}_H(x) - 3\varepsilon\end{aligned}$$

for sufficiently large n, and we have proved that x minimizes $\mathbf{E}_H$ on $\mathcal{S}(\Gamma, A; \sigma)$.

(ii) For solutions x to (4.10) we infer from $\mathbf{E}_H(x) \leq \mathbf{E}_H(y)$, (4.14), the definition of $\mathcal{S}(\Gamma, A; \sigma)$, and the assumption (4.11), that the following chain of inequalities holds:

$$\begin{aligned}\mathbf{D}(x) &= \mathbf{E}_H(x) - 2\mathbf{V}_H(x,y) \leq \mathbf{E}_H(y) - 2\mathbf{V}_H(x,y)\\ &= \mathbf{D}(y) - 2\mathbf{V}_H(x,y) \leq \mathbf{D}(y) + c[\mathbf{D}(x) + \mathbf{D}(y)]\\ &\leq \mathbf{D}(y)[1 + c(\sigma+1)] \leq \sigma\mathbf{D}(y).\end{aligned}$$

Strict inequality $\mathbf{D}(x) < \sigma\mathbf{D}(y)$ follows if $\sigma = \infty$, or $1 + c(\sigma+1) < \sigma$, or y is not a solution to (4.10) (i.e. $\mathbf{E}_H(x) < \mathbf{E}_H(y)$), or strict inequality holds in (4.14) (with $\tilde{x}$ replaced by x there). If, on the other hand, y is a solution to (4.10), then $\mathbf{D}(x) < \sigma\mathbf{D}(y)$ is true for $x = y$ because $\sigma > 1$.

(iii) This follows from Section 2, in particular Propositions 2.3 and 2.4, since we have verified in Lemma 3.6 (ii) that the assumptions made in Section 2 about the H-volume functional are satisfied. ∎

4.5 Remarks. (1) The same proof works for the Dirichlet problem, i.e. $\mathcal{S}(\Gamma, A; \sigma)$ is replaced by $\mathcal{S}(y, A; \sigma)$, the set of $\tilde{x} \in W^{1,2}(U, A)$ with $\tilde{x} - y \in W_0^{1,2}(U, A)$ and $\mathbf{D}(\tilde{x}) \leq \sigma\mathbf{D}(y)$, where $y \in W^{1,2}(U, A)$ is given. To obtain weak solutions $x \in W^{1,2}(U, A)$ for the H-surface equation (2.2), (2.10) with Dirichlet boundary condition $x|_{\partial U} = y|_{\partial U}$ we have to replace, by Remark 2.5 (1), the mean boundary curvature $H_{\partial A}$ in part (iii) of Theorem 4.4 by the minimum of the principal curvatures of ∂A.

(2) In the case $A \neq M$ it is not really necessary to assume that the integer multplicity rectifiable 3-currents $I_{\tilde{x},y}$ etc. occuring in the proof have their support in A as required in the definition of the spherical isoperimetric condition. If we modify Definition 4.2 by allowing that the integer multiplicity rectifiable 3-current Q with $\partial Q = T$, where T is a given spherical 2-current in A, may have support $\operatorname{spt} Q$ in some closed subset $\tilde{A}$ of M with $\tilde{A} \supset A$ (i.e. we require unique homological triviality of T in $\tilde{A}$), then the proof of Theorem 4.4 remains valid, provided H is bounded and continuous on $\tilde{A}$. Of course, in the case $\tilde{A} = M$ with $|M| < \infty$ we then must impose the restriction $\mathbf{M}(Q) < \frac{1}{2}|M|$ on the mass of Q, and for the theorem we need the hypothesis $\tilde{c}s < \frac{1}{2}|M|$ for the linear isoperimetric constant $\tilde{c}$ of M.

(3) If we work with the definition

$$(4.19)\quad \mathbf{V}_H(x, y) = \mathbf{V}_H(x) - \mathbf{V}_H(y) = \int_U x^{\#}\omega - \int_U y^{\#}\omega$$

of H-volume, where ω is a continuous bounded 2-form on M with $d\omega = H\Omega$ in the distributional sense on a neighbourhood of A, then Theorem 4.4 is valid with the isoperimetric condition defined by (4.9). No assumption of homological triviality for the spherical currents $J_x - J_y$ is needed here, neither do we have to assume $\tilde{c}s < \frac{1}{2}|M|$ in the closed case. To adapt the proof given above, we need only observe that with a homotopy formula as in (2.13) and in the proof of Lemma 3.6 (i) we can show (4.15) also for the H-volume defined by (4.19). All the other steps of the proof remain valid. (However, as we have already pointed out, the assumption $H\Omega = d\omega$ with a bounded 2-form ω imposes restrictions on the prescribed mean curvature H which may be unwarranted.)

(4) A natural choice for the reference surface y is a minimizer of Dirichlet's integral in $\mathcal{S}(\Gamma, A)$ (which exists by the argument in the second paragraph of the proof of Theorem 4.4 above.) Under some uniformity condition on A in the noncompact case (see Section 6) y is then continuous on $\overline{U}$. Thus, the assumption $\mathcal{S}(\Gamma, A) \neq \emptyset$ implies in particular that Γ is contractible in A, and the converse is also true, of course. One may then lift Γ to a Jordan curve $\widetilde{\Gamma}$ by the universal covering $p\colon \widetilde{M} \to M$ and apply the theorem to $\widetilde{M}$ in order to obtain sometimes a better existence result for the Plateau problem $\mathcal{P}(H, \Gamma)$ by projecting the solutions to $\mathcal{P}(H \circ p, \widetilde{\Gamma})$ in $\widetilde{M}$ onto M (see [Gu1, Section 6]). ■

5 Geometric criteria sufficient for existence

In this section we combine the results of [DS2] on isoperimetric inequalities and conditions in Riemannian manifolds with Theorem 4.4 to derive concrete geometric conditions on the Jordan curve Γ and the prescribed mean curvature H which are sufficient for the existence of a solution to the Plateau problem $\mathcal{P}(H, \Gamma)$ in a given Riemannian 3-manifold M (satisfying the general assumptions of Section 2) or in the closure A of a smooth subdomain A of M.

The first treatment of this problem by Gulliver [Gul] and Hildebrandt & Kaul [HK] was based on the simple version of Theorem 4.4 described in Remark 4.5 (3), i.e. they worked with a 2-form ω on M satisfying $d\omega = H\Omega$ on A (in fact, they used the vector field Z dual to ω which satisfies $\mathrm{div}_M Z = H$ on A). The isoperimetric condition

$$2|\mathbf{V}_H(x,y)| = 2\left|\int_U x^\# \omega - \int_U y^\# \omega\right| \leq c\mathbf{M}(J_x - J_y)$$

for $x, y \in \mathcal{S}(\Gamma, A)$ (see (3.7)) is then clearly satisfied with constant $c = 2\sup_A |\omega|$. Therefore, to apply Theorem 4.4 with the choice $s = \sigma = \infty$, one needs the condition $\sup_A |\omega| < \frac{1}{2}$ (or $\sup_A |Z| < \frac{1}{2}$). With this assumption the proof of lower semicontinuity of the energy functional $\mathbf{E}_H = \mathbf{D}(x) + \mathbf{V}_H(x,y)$ is also much simpler than in Theorem 4.4 (cf. [DS3]).

A natural method to produce a 2-form ω with $d\omega = H\Omega$ on a geodesically star shaped domain in M is radial integration of $H\Omega$ as in the usual proof of Poincare's Lemma. Using this approach and Riemannian comparison techniques to deduce $\sup_A |\omega| < \frac{1}{2}$ from suitable geometric assumptions on A and H Gulliver and Hildebrandt & Kaul were able to show that in the case of a closed normal geodesic ball A of radius $R > 0$ on M the Plateau problem $\mathcal{P}(H, \Gamma)$ can be solved in A for all rectifiable closed Jordan curves $\Gamma \subset A$, provided

$$(5.1) \quad \sup_A |\omega| \leq \kappa \cot(\kappa R)\,,$$

where κ is real or purely imaginary with $|\kappa| R < \frac{\pi}{2}$ in the real case and the sectional curvature of M is bounded above by $\mathrm{Sec}(M) \leq \kappa^2$ on A. The geometric interpretation of the constant $\kappa \cot(\kappa R)$ appearing in (5.1) is the mean curvature of a sphere of radius R in the simply connected 3-manifold N_κ with constant sectional curvature κ^2. In the case $\kappa = 0$ one has to interprete $\kappa \cot(\kappa R) = R^{-1}$, of course, so that the result of Gulliver and Hildebrandt & Kaul appear as the natural analogue of Hildebrandts theorem [Hil] on the solvability of $\mathcal{P}(H, \Gamma)$ in a ball of radius R in Euclidean space $\mathbb{R}^3$ for prescribed mean curvature H satisfying $\sup |H| \leq R^{-1}$.

It follows from Gullivers work [Gul] that the constant $\kappa \cot(\kappa R)$ in (5.1) may be replaced by a computable larger constant, provided $|H|$ does not exceed on ∂A the mean boundary curvature $H_{\partial A}$ of A. Furthermore, Hildebrandt & Kaul [HK] treated not only balls but general geodesically star shaped domains in M. By definition, such a domain is the diffeomorphic image under the exponantial map $\exp_M$ of M at some point p (the star point) of a domain in the tangent space T_pM which is star shaped with respect to the origin of T_pM. It seems, however, that the following improvement of the results of Gulliver and Hildebrandt & Kaul, which we deduce from Theorem 4.4, cannot be easily obtained with their methods.

5.1 Theorem. *Suppose D is a geodesically star shaped domain in M with star point p, the sectional curvature of M is bounded above by $\mathrm{Sec}(M) \leq \kappa^2 \in \mathbb{R}$ on D, A is the closure of a C^2 domain in D, and geodesic arcs connecting p to points in A have length $\leq R$ where $2|\kappa| R < \pi$ in the case $\kappa^2 > 0$. If the (continuous) prescribed mean curvature $H\colon A \to \mathbb{R}$ satisfies the two conditions*

$$(5.2) \quad \sup_A |H| < \frac{\sin^2(\kappa R)}{R - \frac{1}{2\kappa}\sin(2\kappa R)}\,,$$

(5.3) $|H| \leq H_{\partial A}$ *pointwise on* ∂A,

where $H_{\partial A}$ *is the (inward) mean boundary curvature of* A, *then the Plateau problem* $\mathcal{P}(H, \Gamma)$ *has a weak solution in* A *for every rectifiable closed Jordan curve* Γ *which is contractible in* A.

We also have here and in the following theorems the strong inclusion statement of Theorem 4.4 (iii), i.e. if $|H| < H_{\partial A}$ holds at a point of $A \setminus \Gamma$ then each weak solution of $\mathcal{P}(H, \Gamma)$ in A omits a neighbourhood of this point.

Proof. We have the linear isoperimetric inequality

(5.4) $\mathbf{M}(Q) \leq c(\kappa, R)\mathbf{M}(\partial Q)$

for 3-dimensional integer multiplicity rectifiable currents Q on M with support in D where the constant $c(\kappa, R)$ is the optimal one when $M = N_\kappa$ is simply connected with constant sectional curvature κ^2 (see Section 2, in particular Remark 2.8 (2), in [DS2]). From the isoperimetric property of balls in N_κ ([Sch], [DeG], [BZ, 10.2]) one deduces that $c(\kappa, R)$ is the ratio of the volume of a ball of radius R in N_κ and its boundary area. One computes readily (cf. [DS2, Example 2.5]) that

(5.5) $$2c(\kappa, R) = 2\left(\frac{\sin \kappa R}{\kappa}\right)^{-2} \int_0^r \left(\frac{\sin \kappa r}{\kappa}\right)^2 dr = \frac{R - \frac{1}{2\kappa}\sin(2\kappa R)}{\sin^2 \kappa R}$$

is the reciprocal of the constant appearing in (5.2).

On account of

(5.6) $$|\langle Q, H\Omega\rangle| = \left|\int_A i_Q H\Omega\right| \leq \sup_D |H|\mathbf{M}(Q)$$

for $\operatorname{spt} Q \subset D$ we see that (5.4) implies an isoperimetric condition of type $c < 1, \infty$ for H on D if the hypthesis (5.2) is satisfied for D instead of A. Since we may clearly assume $\sup_D |H| = \sup_A |H|$ the assertion now follows from Theorem 4.4 with $s = \sigma = \infty$ and from Remark 4.5 (2). ∎

In the case of a normal geodesic ball A of radius R condition (5.3) is automatically satisfied if (5.1) holds on ∂A. Indeed, by the comparison theorem for Jacobians [Gü], [BZ, 3.3], [GHL, 3.101] the mean curvature $H_{\partial A}$ is not smaller on ∂A than the mean curvature of spheres of radius R in N_κ which is $\kappa \cot(\kappa R)$ (cf. the proof of Proposition 2.6 in [DS2]). In the interior of A, however, we need only the less restrictive inequality (5.2) for $|H|$. For example, in the case $\kappa = 0$ the constant appearing in (5.2), i.e. the limit as $\kappa \to 0$, equals $\frac{3}{2}R^{-1}$ in contrast with the constant R^{-1} in (5.1). It has already been observed by Gulliver & Spruck [GS2] that the proof of Hildebrandts existence theorem [Hil] for a Euclidean ball of radius R remains valid if instead of $\sup |H| \leq R^{-1}$ on the ball we have $\sup |H| < \frac{3}{2}R^{-1}$ in the interior and $\sup |H| \leq R^{-1}$ on the boundary. Thus, Theorem 5.1 may be considered as a Riemannan version of this stronger form of Hildebrandt's theorem.

In another direction we can generalize to the Riemannian context the result of the second author [Ste2] that the inequality

(5.7) $$\int_{\mathbb{R}^3} |H|^3 d\mathcal{L}^3 < \frac{9\pi}{2}$$

is sufficient for the existence of a solution to the Plateau problem $\mathcal{P}(H,\Gamma)$ in $\mathbb{R}^3$ for any given rectifiable boundary Γ. This is based on the nonlinear isoperimetric inequality (see [Kl] and [DS2, Section 2])

$$(5.8)\quad \mathbf{M}(Q) \le \left(\int_M |i_Q|^{3/2}\Omega\right)^{2/3} \le \frac{1}{\sqrt{36\pi}}\mathbf{M}(\partial Q)$$

for integer rectifiable 3-currents Q in a simply connected 3-manifold M of nonpositive sectional curvature, where the constant is the optimal isoperimetric constant in Euclidean space $\mathbb{R}^3$.

Using Hölder's inequality in (5.6) and applying (5.8) we find that H satisfies the required isoperimetric condition of type c, ∞ with $c < 1$ if the analogue of (5.7) is valid for integration on M with respect to its Riemannian volume form Ω. In the case $\mathrm{Sec}(M) \le \kappa^2 < 0$ we can prove more, making use also of the linear isoperimetric inequality (5.4). We then see from the proof of Theorem 3.10 (ii) in [DS2] that it is sufficient to integrate in (5.7) over the subset of M where $|H|$ is larger or equal to the constant appearing in (5.2). We thus obtain the following theorem which in the case $\mathrm{Sec}(M) \le \kappa^2 \le 0$ simultaneously sharpens the result of [Gu1], [HK] and extends [Ste2] to the Riemmanian situation:

5.2 Theorem. *Suppose the assumptions of Theorem 5.1 are satisfied for M, D, R, and A with* $\mathrm{Sec}(M) \le \kappa^2 \le 0$ *on D. If $H: A \to \mathbb{R}$ satisfies (5.3) on ∂A and*

$$(5.9)\quad \int_{\{a\in A: |H(a)|\ge h(\kappa,R)\}} |H|^3\Omega < \frac{9\pi}{2}$$

where

$$h(\kappa, R) = \frac{\sinh^2|\kappa R|}{R - \frac{1}{2|\kappa|}\sinh|2\kappa R|},$$

then the Plateau problem $\mathcal{P}(H,\Gamma)$ has a weak solution in A for every rectifible closed Jordan curve Γ which is contractible in A. ∎

Note that $h(\kappa, R)$ decreases to $|\kappa|$ as $R \to \infty$. Hence the domain of integration in (5.9) may be replaced by $\{a \in A : |H| > |\kappa|\}$ to obtain a condition which is sufficient for the solvability of $\mathcal{P}(H,\Gamma)$ for all rectifiable boundary curves Γ in a simply connected 3-manifold M of nonpositive sectional curvature $\mathrm{Sec}(M) \le \kappa^2 \le 0$. Hypothesis (5.9) is clearly an improvement over (5.2) since it is implied by (5.2) but allows arbitrarily large values of the prescribed mean curvature modulus $|H|$ (on sets of small volume). On account of

$$\left|\int i_Q H\Omega\right| \le \max\left\{\int |i_Q|H_+\Omega, \int |i_Q|H_-\Omega\right\},$$

where $H_+ = \max\{H, 0\}$ and $H_- = \max\{-H, 0\}$ are the positive and negative parts of H, it is actually sufficient to require (5.9) for H_+ and for H_- instead of $|H| = H_+ + H_-$.

So far we have discussed the consequences of Theorem 4.4 for the Plateau problem $\mathcal{P}(H,\Gamma)$ in (subdomains of) normal geodesic balls in M. The next theorem is not restricted to this special situation. It is again derived from the case $s = \sigma = \infty$ in Theorem 4.4.

5.3 Theorem. *Suppose A is the closure of a proper C^2 subdomain in M, A is homologically 2-aspherical, and the prescribed mean curvature H is related to the parallel mean curvature function H_A of A by the inequality*

(5.10) $|H| \leq cH_{\partial A}$ *pointwise on A*

with a constant $0 \leq c < 1$. Then the Plateau problem $\mathcal{P}(H, \Gamma)$ has a weak solution in A for every rectifiable Jordan curve Γ which is contractible in A.

Here the **parallel mean curvature function** $H_A(a)$ is the (inward) mean curvature of the local parallel surface to ∂A through a at each point $a \in A$ where this is defined (i.e. a has unique nearest point b in ∂A and it is not a focal point of b) and $H_A(a) = \infty$ otherwise (on a subset of measure zero in A). The **proof** of Theorem 5.3 is based on the fact that the gradient of the distance function to ∂A is a unit vector field $2Z$ which satisfies $\operatorname{div}_M Z \geq H_A$ on A in the distributional sense (see the proof of Proposition 2.9 in [DS2]). With the Gauss & Green Theorem and (5.10) one deduces

$$2\int_F |H|\Omega \leq c\mathbf{P}(F)$$

for sets $F \subset A$ of finite perimeter $\mathbf{P}(F)$, and an unrestricted isoperimetric condition of type c, ∞ follows for H on A by the decomposition theorem for integer multiplicity rectifiable currents of top dimension. (For details we refer to [DS2], in particular Remark 2.10 (1) and Proposition 3.4 (v).) Theorem 5.3 above then follows from an application of Theorem 4.4 with $s = \sigma = \infty$, since inequality (4.12) is also implied by (5.10) and $c < 1$.

Variants of the preceding theorem which allow $c = 1$ or even values of c slightly larger than 1 in (5.10) (insisting in $|H| \leq H_{\partial A}$ on the boundary, of course) can be proved using modifications of the vector field Z above. We refer the reader to [DS3, Section 2] to see what can be done in this direction. In Euclidean space $\mathbb{R}^3$ Theorem 5.3 is due to Gulliver & Spruck [GS2]. They used an existence theorem of Serrin [Se] for the nonparametric mean curvature equation

$$\operatorname{div}\frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 2H \quad \text{on } A$$

in order to find a vectorfield Z with $\operatorname{div} Z = H$ on A and with $\sup_A |Z| < \frac{1}{2}$. Working with the 2-form ω dual to Z one can then argue as in the beginning of this section to obtain an isoperimetric condition of type $c < 1, \infty$ for H on A. This method could also be used in the Riemannian situation, but our approach here is more flexible since we need only an inequality $|H| \leq \operatorname{div}_M Z$ on A.

It is clear that other isoperimetric inequalities and conditions proved in [DS2] can also be used in combination with Theorem 4.4 to extend classical existence theorems for the Plateau Problem in $\mathbb{R}^3$, e.g. from [GS1], [Hi2], [Hi3], [Ste1], [Ste2], to the Riemannian case. We leave this to the reader with one exception. Namely, we want to reach our initial goal and prove a Riemannian analogue of Wente's theorem [We] on the solvabiltity of the Plateau problem in $\mathbb{R}^3$ for all Jordan curves Γ spanning sufficiently small minimal area. This is based on the linear spherical isoperimetric inequality (see Definition 4.2)

(5.11) $\mathbf{M}(Q) \leq c_A(s)\mathbf{M}(\partial Q)$

for integer multiplicity rectifiable 3-currents Q on M with support in the closed subset A and with a spherical 2-current ∂Q of mass $\mathbf{M}(\partial Q) \leq s$ as boundary. In the closed case $A = M$ with $|M| < \infty$ we also understand $\mathbf{M}(Q) < \frac{1}{2}|M|$ for the currents allowed in (5.11).

For the next theorem we introduce, for a given rectifiable Jordan curve Γ with $\mathcal{S}(\Gamma, A) \neq \emptyset$ (see (3.7)), the **least spanning area** of Γ in A

$$a_\Gamma = \inf\{\mathbf{D}(x) : x \in \mathcal{S}(\Gamma, A)\} .$$

By the (weak) solution of the Plateau problem in A with zero prescribed mean curvature (the argument in the second paragraph of the proof of Theorem 4.4) there exists a surface $y_\Gamma \in \mathcal{S}(\Gamma, A)$ of least area $\mathbf{D}(y_\Gamma) = a_\Gamma$.

5.4 Theorem. *Suppose $0 < s \leq \infty$, A is the closure of a C^2 domain in M, each spherical 2-current of mass $\leq s$ in A is uniquely homologically trivial in A, and a linear isoperimetric inequality (5.11) with finite constant $c_A(s)$ holds on A where $sc_A(s) < \frac{1}{2}|M|$ in the closed case. Suppose further that the rectifiable Jordan curve Γ is contractible in A with least spanning area $a_\Gamma < \frac{1}{2}s$, and the prescribed mean curvature H satisfies (5.3) on ∂A (no condition if $A = M$) and*

$$(5.12) \quad \sup_A |H| < \frac{1}{2c_A(s)} \quad \textit{in the case } s = \infty,$$

$$(5.13) \quad \sup_A |H| < \frac{s - 2a_\Gamma}{2sc_A(s)} \quad \textit{in the case } s < \infty.$$

Then the Plateau problem $\mathcal{P}(H, \Gamma)$ has a weak solution in A.

Proof. We choose $y_\Gamma \in \mathcal{S}(\Gamma, A)$ with $\mathbf{D}(y_\Gamma) = a_\Gamma$ as reference surface. The unique homological triviality assumption on A and the linear spherical isoperimetric inequality (5.11) imply, by (5.6), a spherical isoperimetric condition of type c, s for H on A where

$$c = 2c_A(s) \sup_A |H| .$$

We can therefore apply Theorem 4.4 with $s = \sigma = \infty$ or with $1 < \sigma = sa_\Gamma^{-1} - 1 < \infty$ noting that inequality (4.11) in Theorem 4.4 reduces to (5.12) or (5.13). ■

Of course, instead of (5.12) or (5.13) we can use other conditions on H that imply an isoperimetric condition of type c, s for H on A with $c < 1$ or $c < s^{-1}(s - 2a_\Gamma)$ respectively.

It follows from Theorem 5.4 that the Plateau problem is solvable whenever a_Γ and $\sup_A |H|$ are sufficiently small. Indeed, in [DS2, Section 2] it is proved that a linear (unrestricted) isoperimetric inequality (5.11) is valid with a finite constant $c_A(s)$ for all sufficiently small $s > 0$ if A is **homogeneously regular** (in the sense of Morrey). The latter condition means that A is uniformly locally biLipschitz equivalent to the unit ball in $\mathbb{R}^3$. This is automatically guaranteed if A is the closure of a Lipschitz domain on M and A is compact. In the noncompact case the condition requires that we have uniform Lipschitz coordinate systems for A at its boundary points and also a certain control for the Riemannian metric on A. Since the homological triviality hypothesis of Theorem 5.4 is also satisfied for sufficiently small $s > 0$, by [DS2, Section 2], we obtain the

5.5 Corrollary. *If A is compact or homogeneously regular and either $A = M$ or $A \neq M$ with positive inward mean boundary curvature bounded away from zero, then there exist constants $\alpha(A) > 0$, $\beta(A) > 0$ such that the Plateau problem $\mathcal{P}(H, \Gamma)$ is weakly solvable in A whenever*

$$a_\Gamma < \alpha(A) \quad \textit{and} \quad \sup_A |H| < \beta(A).$$

■

In order to make the idea precise that we should be able to solve the Plateau problem for large values of the prescribed mean curvature if the minimal spanning area of the boundary curve is sufficiently small, we make use of the **nonlinear spherical isoperimetric inequality**

(5.14) $\mathbf{M}(Q) \leq \gamma_A \mathbf{M}(\partial Q)^{3/2}$ if $\mathbf{M}(\partial Q) \leq s_A$,

which is valid with constants $0 < s_A \leq \infty$ and $0 \leq \gamma_A < \infty$ for all integer multiplicity rectifiable 3-currents on M with $\operatorname{spt} Q \subset A$, with spherical boundary ∂Q, and with $\mathbf{M}(Q) < \frac{1}{2}|M|$ in the closed case, provided A is homogeneously regular (see Proposition 2.2, Corollary 2.3, and Remark 2.4 in [DS2]). For a simply connected 3-manifold $A = M$ of nonpositive curvature we have $s_A = \infty$, and $\gamma_A = (36\pi)^{-1/2}$ is the optimal isoperimetric constant of Eclidean space $\mathbb{R}^3$, by [Kl].

5.6 Corollary. *If M is compact or homogeneously regular then there exists $\alpha(M) > 0$ such that the Plateau problem $\mathcal{P}(H, \Gamma)$ is weakly solvable in M whenever $a_\Gamma \leq \alpha(M)$ and*

(5.15) $$\sup_M |H| < \frac{1}{\gamma_M \sqrt{54 a_\Gamma}}$$

where $\gamma_M > 0$ is the isoperimetric constant from the nonlinear spherical isoperimetric inequality (5.14) (with $A = M$).

Proof. (5.14) implies the nonlinear isoperimetric inequality

$$\mathbf{M}(Q) \leq \gamma_M \sqrt{s} \mathbf{M}(\partial Q) \quad \text{for } Q \text{ with } \mathbf{M}(\partial Q) \leq s \leq s_A$$

(and with $\mathbf{M}(Q) < \frac{1}{2}|M|$ in the compact case). Therefore we can apply Theorem 5.4 with $c_A(s)$ replaced by $\gamma_M \sqrt{s}$, provided $s > 0$ is sufficiently small. For small a_Γ we then choose $s = 6a_\Gamma$, and (5.13) reduces to the inequality (5.15). ■

We conclude with an **example:** Consider the standard sphere S^3. Then the hypothesis $s c_M(s) < \frac{1}{2}|M|$ in Theorem 5.4 is equivalent to $s < 4\pi$, and the condition $a_\Gamma < \frac{1}{2}|M|$ means that the least spanning area of Γ should be smaller then the area of a great half sphere in S^3. By Corollary 5.5 we can then solve $\mathcal{P}(H, \Gamma)$ in S^3 for $\sup_{S^3} |H|$ sufficiently small. On the other hand, Gulliver [Gu3] has proved that $\mathcal{P}(H, \Gamma)$ has no solution if Γ is a great circle in S^3 and $H \neq 0$ is constant. Thus, Corollary 5.5 is sharp with respect to the conditions imposed on Γ to secure solvability of $\mathcal{P}(H, \Gamma)$ for sufficiently small values of $\sup_M |H|$.

However, Corollary 5.6 is most likey not optimal with respect to the bound required for $\sup_M |H|$. If one optimizes (5.13) using the explicitely known form of $c_M(s)$ for $M = S^3$, one finds (see [DS2, Example 4.4]) that the condition $\sup_{S^3} |H| < \cot r$ is sufficient for existence where r is the solution of $r \cot r = 1 + (2\pi)^{-1/2} a_\Gamma$ with $(2\pi)^{-1/2} a_\Gamma^{1/2} < r < \frac{\pi}{2}$. For small a_Γ this is much more restrictive than necessary. In the Euclidean case $M = \mathbb{R}^3$ (and for simply connected M of nonpositive curvature) (5.15) reduces to the condition

$$(5.16)\quad \sup_M |H| \leq \sqrt{\frac{2}{3}}\sqrt{\frac{\pi}{a_\Gamma}}.$$

Even in this case the optimal constant is not known. Considering plane circles $\Gamma \subset \mathbb{R}^3$ one is lead to the conjecture that the constant $\sqrt{2/3}$ in (5.16) (which is an improvement [Ste2] of Wente's original constant [We]) can be replaced by 1. This would be best possible by the non-existence theorem of Heinz [He3]. (It has been observed by Struwe [Str, III.3] that $\sqrt{2/3}$ can be replaced by a larger constant depending on Γ if only constant H are admitted. Such statements can also be derived from an analysis of the proofs for Theorem 4.4 and Corollary 5.6 above).

6 Concluding remarks on regularity

Here we want to indicate briefly how one can prove regularity of the solutions $x \in W^{1,2}(U, M)$ to the Plateau problem which we have produced in Theorem 4.4 and Section 5.

With regard to **analytic regularity**, i.e. the smoothness of the function x, it is sufficient to establish continuity on the closed disc $\overline{U}$. Then one can localize in the target manifold M and introduce coordinates to rewrite the variational equation (2.10) valid for x as an elliptic system of partial differential equations (2.5) satisfied by x in the weak sense. Now the arguments given in [Gu1], [HK], [Hi5] apply to give interior $C^{k+2,\beta}$ regularity for x if the precribed mean curvature H is of class $C^{k,\beta}$. The simplest method is [Grü], making strong use of the (weak) conformality of x. Boundary regularity was established for minimal surfaces in a Riemannian manifold M by Heinz & Hildebrandt [HH2] using earlier regularity theorems of Heinz [He2], [He4]. Their reasoning can also be applied to H-surfaces in M, because the extra terms appearing in (2.5) when $H \not\equiv 0$ are of the same quadratic nature with respect to x_u, x_v as the terms which come from the Levi-Cevita connection on M and are already present in the case $H \equiv 0$ (see also [DHKW, Chap. 7]).

To prove continuity of x on $\overline{U}$ one can make use of the H-energy minimizing character of our solutions to employ an old device of Morrey which is based on harmonic replacement and comparison of energies. In order to produce comparison surfaces which lie in the manifold M, or in the closed subset A in which we have solved the energy minimization problem by Theorem 4.4 (i), we need to locally transform M, or A, to a convex set in $\mathbb{R}^3$, because harmonic maps into $\mathbb{R}^3$ take their values in the convex hull of their boundary trace. A natural assumption is therefore that A is uniformly locally biLipschitz equivalent to a convex set in $\mathbb{R}^3$, a property called **quasiregularity** of A in [HK], [Hi5]. In the case $A = M$ this condition means that M is homogeneously regular in the sense of Morrey. For completeness we now outline the continuity proof for x based on quasiregularity of A and the ideas of Morrey. For more details we refer to [HK].

Given a disc $U_r(w)$ in U such that the boundary trace of our minimizer $x \in \mathcal{S}(\Gamma, A)$ on $\partial U_r(w)$ is an absolutely continuous curve of sufficiently small length, one can choose a biLipschitz map Φ from a convex set in $\mathbb{R}^3$ onto a neighbourhood of this trace in A such that the biLipschitz constant Λ of Φ is bounded by a constant depending on A only. Defining $h \in W^{1,2}(U_r(w), \mathbb{R}^3)$ as the harmonic function with the boundary values of $\Phi^{-1} \circ x$ on $\partial U_r(w)$ we may then use $\tilde{x} = \Phi \circ h$ on $U_r(w)$, $\tilde{x} = x$ on $U \setminus U_r(w)$, as comparison surface. The energy minimality of x, the identity

$$\mathbf{E}_H(\tilde{x}) - \mathbf{E}_H(x) = \mathbf{D}_{U_r(w)}(\tilde{x}) - \mathbf{D}_{U_r(w)}(x) + 2\mathbf{V}_H(\tilde{x}, x)$$

(valid for small $r > 0$, cf. (4.18)), and the isoperimetric condition

$$2\,|\mathbf{V}_H(x, \tilde{x})| \leq c\left[\mathbf{D}_{U_r(w)}(\tilde{x}) + \mathbf{D}_{U_r(w)}(x)\right]$$

(valid by (4.9) and (3.5)) then imply

$$\int_{U_r(w)} |Dx|^2\,dudv \leq \frac{1+c}{1-c}\Lambda^2 \int_{U_r(w)} |Dh|^2\,dudv\,,$$

provided we have required $0 \leq c < 1$ for the constant in the isoperimetric condition excluding the case $c = 1$ in Theorem 4.4 (i). Since the Dirichlet integral of a harmonic function on the unit disc does not exceed the Dirichlet integral of its boundary trace (as can be seen from a Fourier expansion, e.g.) one infers an inequality

$$\int_{U_r(w)} |Dx|^2\,dudv \leq \text{const}\, r \int_{\partial U_r(w)} |Dx|^2\,ds\,.$$

As this inequality is also satisfied (trivially) in the case where x on $U_r(w)$ has a boundary trace of large length one can integrate to obtain a growth condition

$$\int_{U_r(w)} |Dx|^2\,dudv \leq \text{const}\left(\frac{r}{R}\right)^{2\alpha} \int_{U_R(w)} |Dx|^2\,dudv$$

for $0 < r \leq R \leq \min\{1 - |w|, R_0\}$, where $R_0 > 0$ is sufficiently small and $0 < \alpha < 1$ a suitable exponent. By Morrey's well known Dirichlet growth theorem this condition implies Hölder continuity of x with exponent α on U and, if $x\big|_{\partial U}$ is continuous, also continuity on $\overline{U}$ (see [HK] for a proof of the latter assertion).

With regard to **geometric regularity**, i.e. the immersed character of our parametric solutions $x\colon U \to A \subset M$ of the Plateau problem with prescribed mean curvature, it is known from an asymptotic expansion given by Hildebrandt & Heinz [HH1] (see also [DHKW, Chap. 8]) that branch points, where x fails to have maximal rank, are isolated in $\overline{U}$. As a consequence, the boundary mapping $x\big|_{\partial U}$ must map the circle ∂U homeomorphically onto the Jordan curve Γ although for the admissible surfaces in the class $\mathcal{S}(\Gamma, A)$ we have only assumed that their boundary traces are weakly monotonic parametrizations of Γ.

True interior branch points, i.e. branch points accompanied by lines of transversal self-intersection of the surface emanating from the point, can be ruled out by an argument of Osserman [Os] for energy minimizing minimal surfaces in $\mathbb{R}^3$ which has been extended to surfaces with Lipschitz prescribed mean curvature in Riemannian manifolds by Gulliver [Gu2]. The missing arguments in Ossermanns work with regard to false branch points, where the parametric surface is locally a branched covering of an embedded smooth 2-submanifold of M, have been worked out by Alt [Al] in $\mathbb{R}^3$ and by Gulliver [Gu2] also in a Riemannian manifold M (see also [GOR], [Gu4], [Gu5]) to exclude such points in the interior. Gulliver worked with the volume definition explained at the beginning of Section 5, where the vector field Z with $\operatorname{div}_M Z = H$ was used (or the dual 2-form ω with $d\omega = H\Omega$), and he required $\sup |Z| < \frac{1}{2}$. An inspection of Gulliver's proof reveals, however, that it is sufficient to have the isoperimetric condition (4.9) for H with constant $c < 1$ (see [SW]).

The situation is less satisfactory with respect to geometric boundary regularity. With the exception of the article [GL] by Gulliver & Lesly adressing energy minimizing solutions to the Plateau problem with given analytic boundary curve and with constant mean curvature in an analytic Riemannian 3-manifold, there is no work on the exclusion of boundary branch points. Thus, in contrast with the geometric measure theory setting where complete boundary regularity has been established [DS1], [DS2], the question of geometric boundary regularity of energy minimizing solutions to the 2-dimensional parametric Plateau problem is largely open.

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