EURPOEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN/ISR-TH/73-55

THE WIDTH OF NON-LINEAR RESONANCES EXCITED BY BEAMS CROSSING AT SMALL ANGLES IN LOW-β SECTIONS

by

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Geneva - December 1973

1. INTRODUCTION

The linear tune shift of one beam due to crossing another beam at a small angle in a low-β section has been calculated [1]. It was found that the contribution of long-range forces occurring when the beams are physically separated but still interact electromagnetically can be very important. It was implied that the same holds for the widths of non-linear resonances believed to be responsible for the incoherent beam-beam limit. An approximate calculation by Voss has shown that this is not necessarily true. In the following we give an accurate derivation of the width of non-linear resonances due to the crossing of a beam at a small angle in a low-β section.

2. REVIEW OF THE WIDTH OF NON-LINEAR RESONANCES

For convenience a short summary of the Schoch-Guignard theory of non-linear resonances [3] is given below. A resonance can occur when a relation of the type

$$n_x Q_x + n_z Q_z = p \tag{1}$$

holds, where n_x, n_z and p are integers and Q_x and Q_z are the horizontal and vertical betatron wave numbers, respectively. The order of the resonance is

$$N = |n_x| + |n_z| \tag{2}$$

The width of a resonance is defined as

$$\Delta e = \Delta(n_x Q_x + n_z Q_z - p) \tag{3}$$

and is given by

$$\Delta e = 2n\ k\ r^{n-2}\ \left(A + \frac{N-n}{n}\ r^2\right)^{\frac{1}{2}(N-n)}\ \left(n + \frac{N-n}{\frac{n\ A}{r^2(N-n)} + 1}\right) \tag{4}$$

where n, r and A have indices x or z, and $n \neq 0$. Hence, when n_x or n_z vanishes (for uncoupled resonances) the choice of the index is fixed, otherwise it is immaterial. Furthermore, we have

$$\begin{aligned} A_x &= r_z^2 - \frac{n_z}{n_x}\ r_x^2 \qquad (n_x \neq 0) \\ A_z &= r_x^2 - \frac{n_x}{n_z}\ r_z^2 \qquad (n_z \neq 0) \end{aligned} \tag{5}$$

r is a normalized amplitude

$$r^2 = R\ E \tag{6}$$

with the machine radius R and the emittance E. The factor k is given by the Fourier component of the non-linear perturbation.

$$k \simeq \left| \int_{-\pi}^{+\pi} \frac{\left(\frac{\beta_x}{R}\right)^{\frac{1}{2}|n_x|} \left(\frac{\beta_z}{R}\right)^{\frac{1}{2}|n_z|}}{2^{N-1}\ \pi |n_x|!\ |n_z|!}\ \frac{R^2}{B\ \rho}\ \frac{\partial^N\ V}{\partial x^{|n_x|}\ \partial z^{|n_z|}} \exp\left\{ i[n_x\ \mu_x + n_z\ \mu_z - (n_x\ Q_x + n_z\ Q_z - p)\theta] \right\} d\ \theta \right| \tag{7}$$

Here, β_x and β_z are the amplitude functions, $B\rho$ is the magnetic rigidity of the particles, and V is a magnetic force potential, i.e. its spatial derivatives are the cross products $\vec{v} \times \vec{B}\ /\ |v|$. Introducing V here is a generalization with respect to [3] since it does not imply that V obeys

Laplace's equation. The phase advances μ_x and μ_z are also functions of θ, the angle around the machine. Their origin is arbitrary.

For our purposes we specialize the above formulae for the case where the horizontal and vertical emittances and amplitude functions are equal, $E_x = E_z = E$ and $\beta_x = \beta_z = \beta_x$, and hence also $\mu_x = \mu_z = \mu_x$. We restrict ourselves to sum resonances with $n_x \geqslant 0$ and $n_z \geqslant 0$. For a while we impose the stricter condition $n_x > 0$, and use (4) with the index x. We then find that

$$A_x + \frac{N - n_x}{n_x} r_x^2 = r_z^2$$

$$n_x + \frac{N - n_x}{\frac{n_x A_x}{r_x^2 (N - n_x)} + 1} = \frac{n_x^2 r_z^2 + n_z^2 r_x^2}{n_x r_z^2}$$

Finally,

$$r_x^2 = r_z^2 = R E$$

Inserting all this into (4) yields

$$\Delta e = \frac{n_x^2 + n_z^2}{2^{N-1} \pi n_x! n_z!} \frac{E^{\frac{1}{2}N - 1}}{B \rho} \left| \int_{-\ell/2}^{+\ell/2} \beta_x^{\frac{1}{2}N} \frac{\partial^N V}{\partial_x^{n_x} \partial_z^{n_z}} \right.$$

$$\left. \exp \left\{ i[N\mu_x - (n_x Q_x + n_z Q_z - p) \frac{s}{R}] \right\} ds \right| \tag{8}$$

Here we have also changed the integration variable from angle θ to length s, and have introduced the length ℓ of the crossing region in which the two beams interact electromagnetically. In what follows we shall assume that the tunes Q_x and Q_z are chosen to be on resonance, i.e. that (1) is satisfied. We shall therefore drop the corresponding term in (8). It may also be seen that in (8) we may take $n_x \geq 0$ and $n_z \geq 0$. At least for machines with $Q_x \simeq Q_z$ the restriction to sum resonances is of no practical importance, since there are fewer difference resonances than sum resonances in the vicinity of the $Q_x = Q_z$ diagonal.

In the case where the force is due to the crossing with another particle beam, we have to add electric and magnetic forces and obtain

$$\frac{\partial^N V}{\partial x^{n_x} \partial z^{n_z}} = \begin{cases} \dfrac{1 + \beta^2}{\beta c} \dfrac{\partial^{N-1} E_x}{\partial x^{n_x - 1} \partial z^{n_z}} & n_x \geq 1 \\[2ex] \dfrac{1 + \beta^2}{\beta c} \dfrac{\partial^{N-1} E_z}{\partial x^{n_x} \partial z^{n_z - 1}} & n_z \geq 1 \end{cases} \tag{9}$$

Here, E_x and E_z are the electric field components and β is the usual relativistic factor.

3. EVALUATION OF THE ELECTRIC FIELD ALONG THE PARTICLE TRAJECTORY

The following is merely a repetition of the field calculation in [1]. We use the co-ordinate system shown in Fig. 1, i.e. we assume that the two beams cross vertically at a small angle α. For a round beam we may use Gauss's theorem and find for the field perpendicular to the beam

$$E_r(r) = \frac{4\pi}{r} \int_0^r \rho(r')\, r'\, dr' \tag{10}$$

where ρ is the charge density in the beam. The components in the beam co-ordinate system are

$$E_x' = E_r \cos\phi$$
$$E_z' = E_r \sin\phi \tag{11}$$

To obtain the field components in the particle co-ordinate system we use the transformations

$$x' = x$$
$$z' = z\cos\alpha + s\sin\alpha \tag{12}$$
$$s' = s\cos\alpha - z\sin\alpha$$

and

$$x = x'$$
$$z = z'\cos\alpha - s'\sin\alpha \tag{13}$$
$$s = s'\cos\alpha + z'\sin\alpha$$

We find

$$E_x = E_x' \qquad\qquad = E_r(r)\cos\phi$$
$$E_z = E_z'\cos\alpha = E_r(r)\sin\phi\cos\alpha \tag{14}$$

We now specialize to a Gaussian beam

$$\rho(r) = \frac{\lambda\ e}{2\pi\ \sigma^2}\exp(-\ r^2/2\sigma^2) \tag{15}$$

where σ is the rms half-width and λ is the number of particles per unit length of the beam. Combining (10) and (15) we obtain

$$E_r(r) = \frac{2\lambda e}{r} [1 - \exp(- r^2/2\sigma^2)] \tag{16}$$

and hence

$$E_x = \frac{2\lambda e \cos\phi}{r} [1 - \exp(- r^2/2\sigma^2)]$$

$$E_z = \frac{2\lambda e \sin\phi \cos\alpha}{r} [1 - \exp(- r^2/2\sigma^2)] \tag{17}$$

In evaluating (17) one recalls that

$$\sin\phi = z'/r$$

$$\cos\phi = x'/r \tag{18}$$

$$r^2 = x'^2 + z'^2$$

One must also remember that, because of the β variation

$$\beta_x(s') = \beta_o + s'^2/\beta_o \tag{19}$$

σ is a function of s':

$$\sigma(s') = \sigma_o[\beta_x(s')/\beta_o]^{\frac{1}{2}} \tag{20}$$

β_o and σ_o are the amplitude function and the rms beam radius at the crossing point, respectively. Because of (19), the phase advance, counted from the crossing point, is given by

$$\mu_x = \tan^{-1} (s/\beta_o) \tag{21}$$

4. CALCULATION OF THE WIDTH OF A GIVEN RESONANCE

For given values of n_x and n_z the width of a resonance is given by (8) which we rewrite as follows:

$$\Delta e = \frac{2(n_x^2 + n_z^2)(1 + \beta^2)\, r_o\, \lambda\, E^{\frac{1}{2}N - 1}}{2^N\, \pi\, n_x! n_z!\, \beta^2\, \gamma}$$

$$\left| \int_{-\ell/2}^{+\ell/2} \beta_x^{\frac{1}{2}N} \; \frac{1}{2\lambda\; e} \; \begin{matrix} \dfrac{\partial^{N-1} E_x}{\partial x^{n_x - 1}\, \partial z^{n_z}} \\ \text{or} \\ \dfrac{\partial^{N-1} E_z}{\partial x^{n_x}\, \partial z^{n_z - 1}} \end{matrix} \; \exp(iN\mu_x)\, ds \right| \tag{22}$$

In evaluating (22) one must remember that β_x is a function of s, given by (19), and that σ is a function of s', given by (20). In order to obtain the derivative of the electric field one uses (17), differentiates it the appropriate number of times, and evaluates the result on the trajectory of the test particle, x = z = 0. If we decide to evaluate (22) for a particle with the rms betatron amplitude we may put

$$E = \sigma_o^2\; \beta_o^{-1} \tag{23}$$

and obtain

$$\Delta e = \frac{2(n_x^2 + n_z^2)(1 + \beta^2)\, r_o\, \lambda\, \sigma_o^{N-2}\, \beta_o}{2^N\, \pi\, n_x! n_z!\, \beta^2\, \gamma}$$

$$\left| \int_{-\ell/2}^{+\ell/2} \left(\frac{\beta_x}{\beta_o}\right)^{\frac{1}{2}N} \; \frac{1}{2\lambda\; e} \; \begin{matrix} \dfrac{\partial^{N-1} E_x}{\partial x^{n_x - 1}\, \partial x^{n_z}} \\ \text{or} \\ \dfrac{\partial^{N-1} E_z}{\partial x^{n_x}\, \partial z^{n_z - 1}} \end{matrix} \; \exp(iN\mu_x)\, ds \right| \tag{24}$$

It may be seen from (17) that E_x, and hence its even derivatives with respect to x, are all odd functions of x and vanish for x = 0. It follows from (24) that only resonances with even values of n_x are excited. Similarly, E_x is an even function of z'. This means that the dependence of E_x on s and z is such that terms with even powers of s and odd powers of z, and vice versa, do not appear in a power series expansion of E_x and of its even derivatives with respect to z. For z = 0, E_x and all its even derivatives with respect to z are even functions of s, whereas the odd derivatives are odd functions of s. Since β_x is an even function of s, the integrals (24) vanish when n_z is odd. We conclude that only resonances of even order with both n_x and n_z even are excited by beam crossings.

5. WIDTHS OF THE RESONANCES OF ORDER 2, 4 AND 6

The evaluation of (24), and in particular the differentiation of E_x and E_z is quite lengthy, and was partly done using the algebraic manipulation program SCHOONSHIP [4]. The widths of these resonances can be written in the form

$$\Delta e(nx\ ,\ nz) = k_{n_x n_z}\ \Delta Q_x\ I_{n_x n_z}\ \left(\frac{\ell}{2\beta_o}\ ,\ \frac{\alpha\beta_o}{\sigma_o}\right) \tag{25}$$

where $k_{n_x n_z}$ is a numerical factor, ΔQ_x is the linear tune shift given by [1]

$$\Delta Q_x = \frac{\lambda\ r_o\ \ell\ \beta_o(1 + \beta^2)}{4\pi\ \beta^2\gamma\ \sigma_o{}^2}\ I_x\left(\frac{\ell}{2\beta_o}\ ,\ \frac{\alpha\ \beta_o}{\sigma_o}\right) \tag{26}$$

with

$$I_x = \frac{1}{\xi\ \eta^2}\int_{-\xi}^{+\xi} (1 + x^{-2})\left\{1 - \exp[-\tfrac{1}{2}\eta^2 x^2/(1 + x^2)]\right\} \tag{27}$$

and $I_{n_x n_z}(\xi,\eta)$ is a correction factor which allows for the variation of β_x and for finite crossing angles α. It is normalized such that it tends towards one for $\xi,\eta << 1$. The values of $k_{n_x n_z}$ and the expressions for $I_{n_x n_z}(\xi,\eta)$ are listed in Table I. This means that for collinear crossings with a small phase advance the tune shift is indeed a measure of the widths of the non-linear resonances excited by the other beam.

If the region for which $|\xi| < \xi_o$ is excluded from all the integrals in Table I, $(1 + x^2)$ may be replaced by x^2 in which case all the integrals become closed expressions. These long-range contributions to the resonance width are listed in Table II, for two limiting cases, nearly head-on collisions with $\eta << 1$ and $\xi >> \xi_o$, and well separated beams with $\eta >> 1$ and $\xi >> \xi_o$.

6. DISCUSSION

Figs. 2 to 8 show the correction factors $I_{n_x n_z}$. There is a striking difference between the curves for $\eta \lesssim 1$ and $\eta >> 1.0$. The first shows an oscillatory behaviour. This is due to a cancellation effect because of the phase factor in (22). Since this factor is varying more rapidly for 6th order than for 4th order resonances, the period of the oscillation in ξ is also shorter for the 6th order resonances. When the beams are well separated at the end of the intersection region, $\eta >> 1$, there is no such behaviour. Instead the curves come down rather smoothly with increasing ξ. This can be explained by observing that the long-range contribution to the resonance width tends to be smaller than the contribution to the tune shift, hence their ratio comes down with increasing ξ. The behaviour of different resonances of the same order is quite different, for $\eta >> 1$. This is explained by the observation that the forces, in the vertical plane of crossing, cancel out to a large extent when the beams cross at an angle, whereas the forces, perpendicular to the plane of crossing, do not cancel out. Hence the resonances come down in strength when n_z goes up for a given order of resonance.

The long-range contribution to the resonance widths shown in Table II are in excellent agreement with the accurate results given in Figs. 2 to 8 for the case of nearly head-on collisions, $\eta \ll 1$ and $\xi \gg 1$. In the case of well separated beams, the widths of the non-linear resonances are reduced by a factor of η^2 each time the order of the resonances is increased by two. This agrees with Voss's calculation [2]. However, the widths of the resonances as calculated from the formulae in Table II are much smaller than those shown in Figs. 2 to 8. This means, of course, that most of the resonance excitation occurs close to the crossing point and not where the beams are well separated. Hence, the observation that the long-range contribution to the resonance width comes down rapidly with increasing η, is of little practical importance because that contribution is only a small fraction of the total effect.

As an example, we consider the optimised proton machines described in [5] with $\xi = 5$ and $\eta = 17.8$. When one assumes that the widths of the resonances are just proportional to the tune shift ΔQ_x, one implies that $I_{n_x n_z} = 1$. If, in addition, one observes that because of symmetry the resonances with $n_z \neq 0$ should not be excited, one expects $I_{n_x n_z} = 0$ for these resonances. Comparing this to the computed results in Table III shows that the I_{N0} values are overestimated by a factor of about two, and that the other $I_{n_x n_z}$ are underestimated.

REFERENCES

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2) G.A. Voss; CRISP 72-49 (1972)

3) A. Schoch; CERN 57-23 (1958)
G. Guignard; CERN 70-24 (1970)

4) M. Veltman; CERN Library Program R 201

5) E. Keil; Nuclear Instruments and Methods 113, 333 (1973)

TABLE I

Widths of resonances of order 2, 4, 6

$$\Delta e\ (n_x, n_z) = K_{n_x,n_z} \cdot \Delta Q_x \cdot I_{n_x,n_z}\ (\xi,\eta), \quad \xi = \frac{\ell}{2\beta_o}\ , \ \eta = \frac{\alpha\beta_o}{\sigma_o}$$

$N = 2,\ n_x = 0,\ n_z = 2,\ K_{02} = 4$

$$I_{02}(\xi,\eta) = \frac{1}{\xi\eta^2 D} \int_{-\xi}^{\xi} \frac{1+x^2}{x^2} \left\{1 - \exp\left(\frac{-x^2\eta^2}{2(1+x^2)}\right)\left[1 + \frac{\eta^2x^2}{1+x^2}\right]\right\} \cdot \cos(2\tan^{-1}x)\ dx$$

$N = 2,\ n_x = 2,\ n_z = 0,\ K_{20} = 4$

$$I_{20}(\xi,\eta) = \frac{1}{\xi\eta^2 D} \int_{-\xi}^{\xi} \frac{1+x^2}{x^2} \left\{1 - \exp\left(\frac{-x^2\eta^2}{2(1+x^2)}\right)\right\} \cdot \cos(2\tan^{-1}x)\ dx$$

$N = 4,\ n_x = 0,\ n_z = 4,\ K_{04} = \frac{1}{2}$

$$I_{04}(\xi,\eta) = \frac{2}{3\eta^4\xi D} \int_{-\xi}^{\xi} \left(\frac{1+x^2}{x^4}\right) \left\{- 6(1+x^2) + \exp\left(\frac{-x^2\eta^2}{2(1+x^2)}\right) \cdot \left[6(1+x^2) + 3x^2\eta^2 + \frac{x^6\eta^6}{(1+x^2)^2}\right]\right\} \cdot \cos(2\tan^{-1}x)\ dx$$

$N = 4,\ n_x = 2,\ n_z = 2,\quad K_{22} = \frac{1}{2}$

$$I_{22}(\xi,\eta) = \frac{2}{\eta^4 \xi D} \int_{-\xi}^{\xi} \left(\frac{1 + x^2}{x^4}\right) \left\{ 6(1 + x^2) + \exp\left(\frac{-\, x^2\eta^2}{2(1 + x^2)}\right) \left[-\, 6(1 + x^2) - 3x^2\eta^2 - \frac{x^4\eta^4}{(1 + x^2)} \right] \right\} \cdot \cos(4\tan^{-1}x)\, dx$$

$N = 4,\ n_x = 4,\ n_z = 0,\quad K_{40} = \frac{1}{2}$

$$I_{40}(\xi,\eta) = \frac{2}{3\eta^4 \xi D} \int_{-\xi}^{\xi} \left(\frac{1 + x^2}{x^4}\right) \left\{ -\, 6(1 + x^2) + \exp\left(\frac{-\, x^2\eta^2}{2(1 + x^2)}\right) \left[6(1 + x^2) + 3x^2\eta^2 \right] \right\} \cdot \cos(4\tan^{-1}x)\, dx$$

$N = 6,\ n_x = 0,\ n_z = 6,\quad K_{06} = 1/32$

$$I_{06}(\xi,\eta) = \frac{3}{\eta^6 \xi D} \int_{-\xi}^{\xi} \left(\frac{1 + x^2}{x^6}\right) \left\{ -\, 8(1 + x^2)^2 + \exp\left(\frac{-\, x^2\eta^2}{2(1 + x^2)}\right) \left[8(1 + x^2)^2 + 4x^2\eta^2(1 + x^2) + x^4\eta^4 + \frac{x^6\eta^6}{3(1 + x^2)} - \frac{x^8\eta^8}{3(1 + x^2)^2} + \frac{x^{10}\eta^{10}}{15(1 + x^2)^3} \right] \right\} \cdot \cos(6\tan^{-1}x)\, dx$$

$N = 6,\ n_x = 2,\ n_z = 4,\ \ K_{24} = 5/3 \cdot 32$

$$I_{24}(\xi,\eta) = \frac{15}{\eta^6 \xi D} \int_{-\xi}^{\xi} \left(\frac{1 + x^2}{x^6}\right) \left\{ 8(1 + x^2)^2 + \exp\left(\frac{- x^2\eta^2}{2(1 + x^2)}\right) \left[- 8(1 + x^2)^2 \right.\right.$$

$$\left.\left. - 4x^2\eta^2(1 + x^2) - x^4\eta^4 - \frac{2x^6\eta^6}{15(1 + x^2)} - \frac{x^8\eta^8}{15(1 + x^2)^2} \right] \right\} \cdot \cos(6\tan^{-1}x)\ dx$$

$N = 6,\ n_x = 4,\ n_z = 2,\ \ K_{42} = 5/3 \cdot 32$

$$I_{42}(\xi,\eta) = \frac{15}{\eta^6 \xi D} \int_{-\xi}^{\xi} \left(\frac{1 + x^2}{x^6}\right) \left\{ - 8(1 + x^2)^2 + \exp\left(\frac{- x^2\eta^2}{2(1 + x^2)}\right) \left[8(1 + x^2)^2 \right.\right.$$

$$\left.\left. + 4x^2\eta^2(1 + x^2) + x^4\eta^4 + \frac{x^6\eta^6}{5(1 + x^2)} \right] \right\} \cdot \cos(6\tan^{-1}x)\ dx$$

$N = 6,\ n_x = 6,\ n_z = 0,\ \ K_{60} = 1/32$

$$I_{60}(\xi,\eta) = \frac{3}{\eta^6 \xi D} \int_{-\xi}^{\xi} \left(\frac{1 + x^2}{x^6}\right) \left\{ 8(1 + x^2)^2 + \exp\left(\frac{- x^2\eta^2}{2(1 + x^2)}\right) \left[- 8(1 + x^2)^2 \right.\right.$$

$$\left.\left. - 4x^2\eta^2(1 + x^2) - x^4\eta^4 \right] \right\} \cdot \cos(6\tan^{-1}x)\ dx$$

where $D = \frac{1}{\xi\eta^2} \int_{-\xi}^{\xi} \frac{1 + x^2}{x^2} \left\{ 1 - \exp\left(\frac{- x^2\ \eta^2}{2(1 + x^2)}\right) \right\} dx$

TABLE II

Long-range contribution to the resonance widths

For nearly head-on collisions, $\eta \ll 1$ and $\xi \gg 1$

$$I_{04}(\xi,\eta) = I_{22}(\xi,\eta) = I_{40}(\xi,\eta) = \left| 1 + \frac{4}{1+\xi^2} - \frac{4}{\xi}\tan^{-1}\xi \right|$$

$$I_{06}(\xi,\eta) = I_{24}(\xi,\eta) = I_{42}(\xi,\eta) = I_{60}(\xi,\eta) = \left| 1 + \frac{12}{1+\xi^2} - \frac{8}{(1+\xi^2)^2} = \frac{6}{\xi}\tan^{-1}\xi \right|$$

For well separated beams, $\eta \gg 1$ and $\xi \gg 1$

$$I_{04}(\xi,\eta) = \frac{4}{\eta^2}\left| 1 + \frac{4}{1+\xi^2} - \frac{4}{\xi}\tan^{-1}\xi \right|$$

$$I_{22}(\xi,\eta) = \frac{12}{\eta^2}\left| 1 + \frac{4}{1+\xi^2} - \frac{4}{\xi}\tan^{-1}\xi \right|$$

$$I_{40}(\xi,\eta) = \frac{4}{\eta^2}\left| 1 + \frac{4}{1+\xi^2} - \frac{4}{\xi}\tan^{-1}\xi \right|$$

$$I_{06}(\xi,\eta) = \frac{24}{\eta^4}\left| 1 - \frac{8}{(1+\xi^2)^2} + \frac{12}{1+\xi^2} - \frac{6}{\xi}\tan^{-1}\xi \right|$$

$$I_{24}(\xi,\eta) = \frac{120}{\eta^4}\left| 1 - \frac{8}{(1+\xi^2)^2} + \frac{12}{1+\xi^2} - \frac{6}{\xi}\tan^{-1}\xi \right|$$

$$I_{42}(\xi,\eta) = \frac{120}{\eta^4}\left| 1 - \frac{8}{(1+\xi^2)^2} + \frac{12}{1+\xi^2} - \frac{6}{\xi}\tan^{-1}\xi \right|$$

$$I_{60}(\xi,\eta) = \frac{24}{\eta^4}\left| 1 - \frac{8}{(1+\xi^2)^2} + \frac{12}{1+\xi^2} - \frac{6}{\xi}\tan^{-1}\xi \right|$$

TABLE III

Correction factors $I_{n_x n_z}$ for $\xi = 5$, $\eta = 17.8$

n_x	n_z	$I_{n_x n_z}$
4	0	0.51981
2	2	0.05873
0	4	0.00318
6	0	0.45421
4	2	0.07975
2	4	0.00409
0	6	0.00010

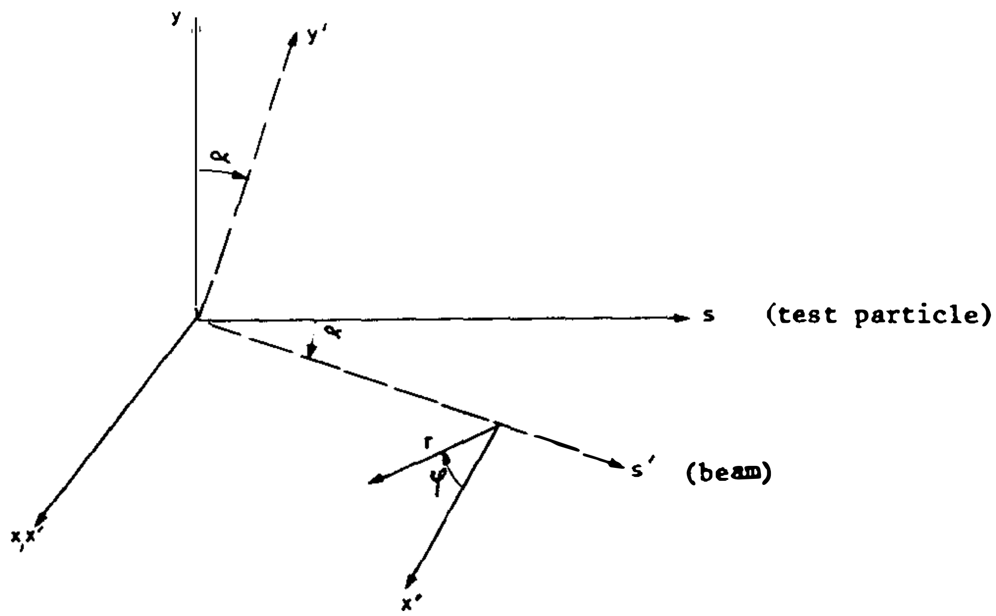
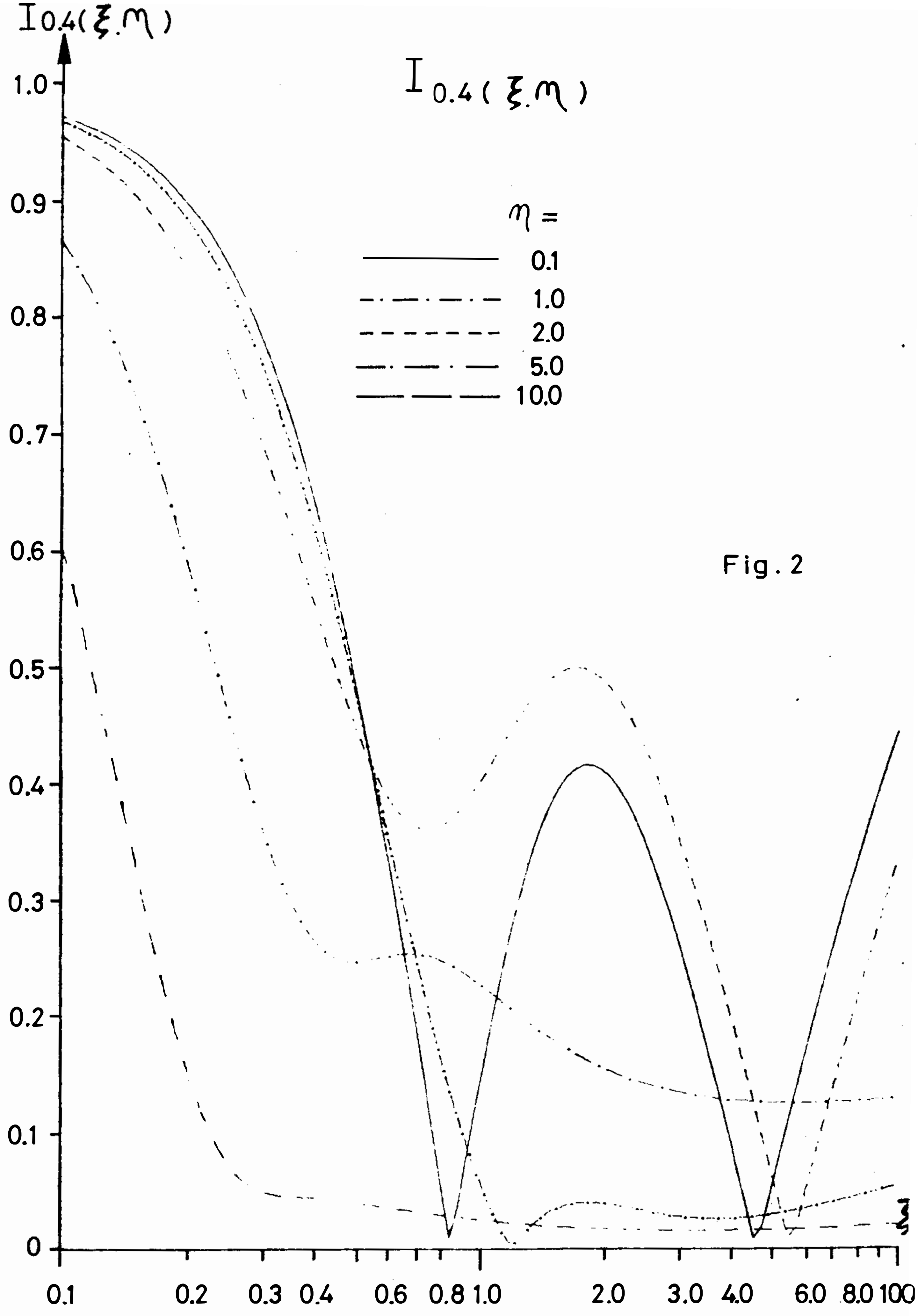
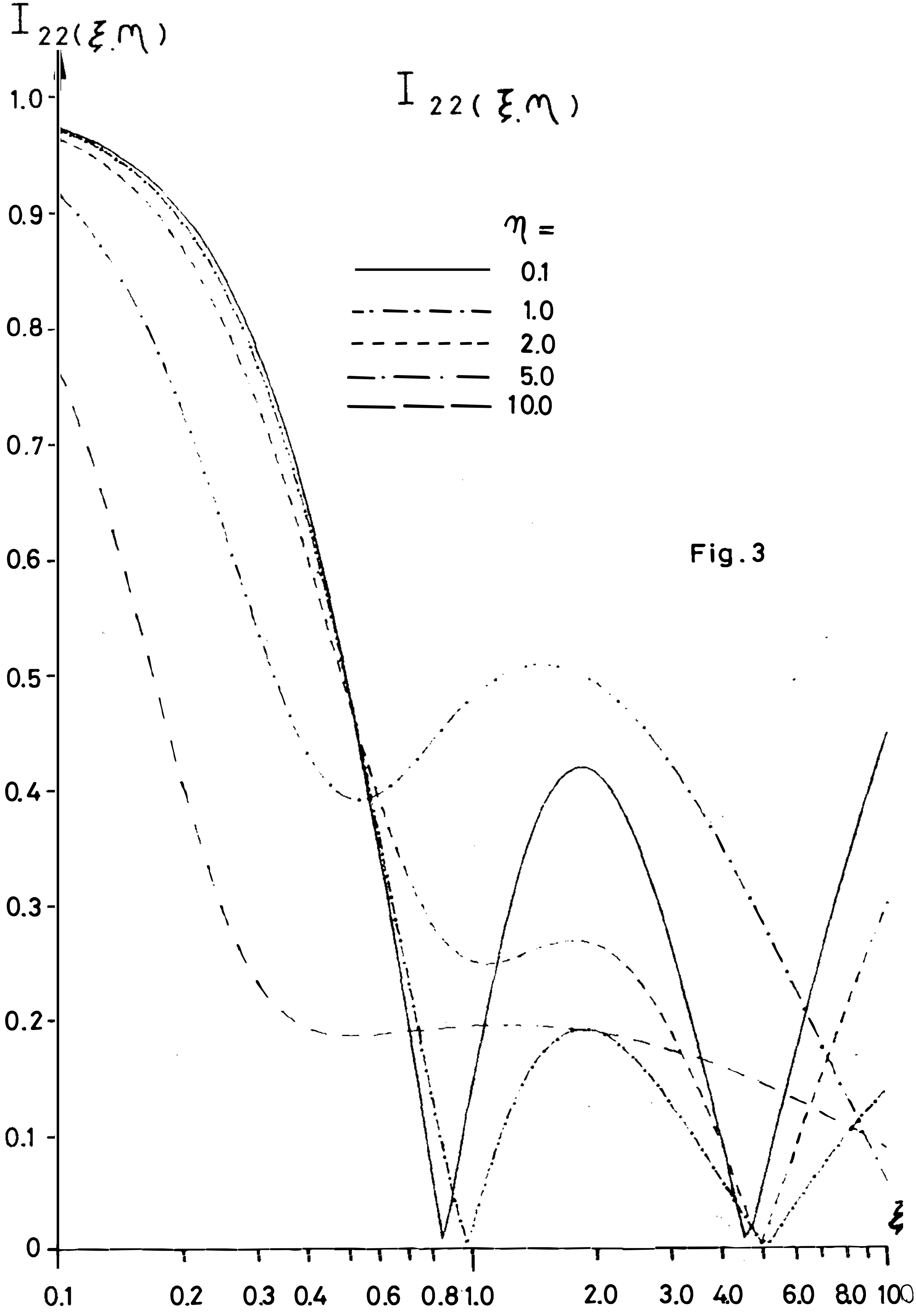


Fig. 1 - Co-ordinate systems. The angle between the beams is α;
$r-\phi$ is a cylindrical co-ordinate system about the beam direction



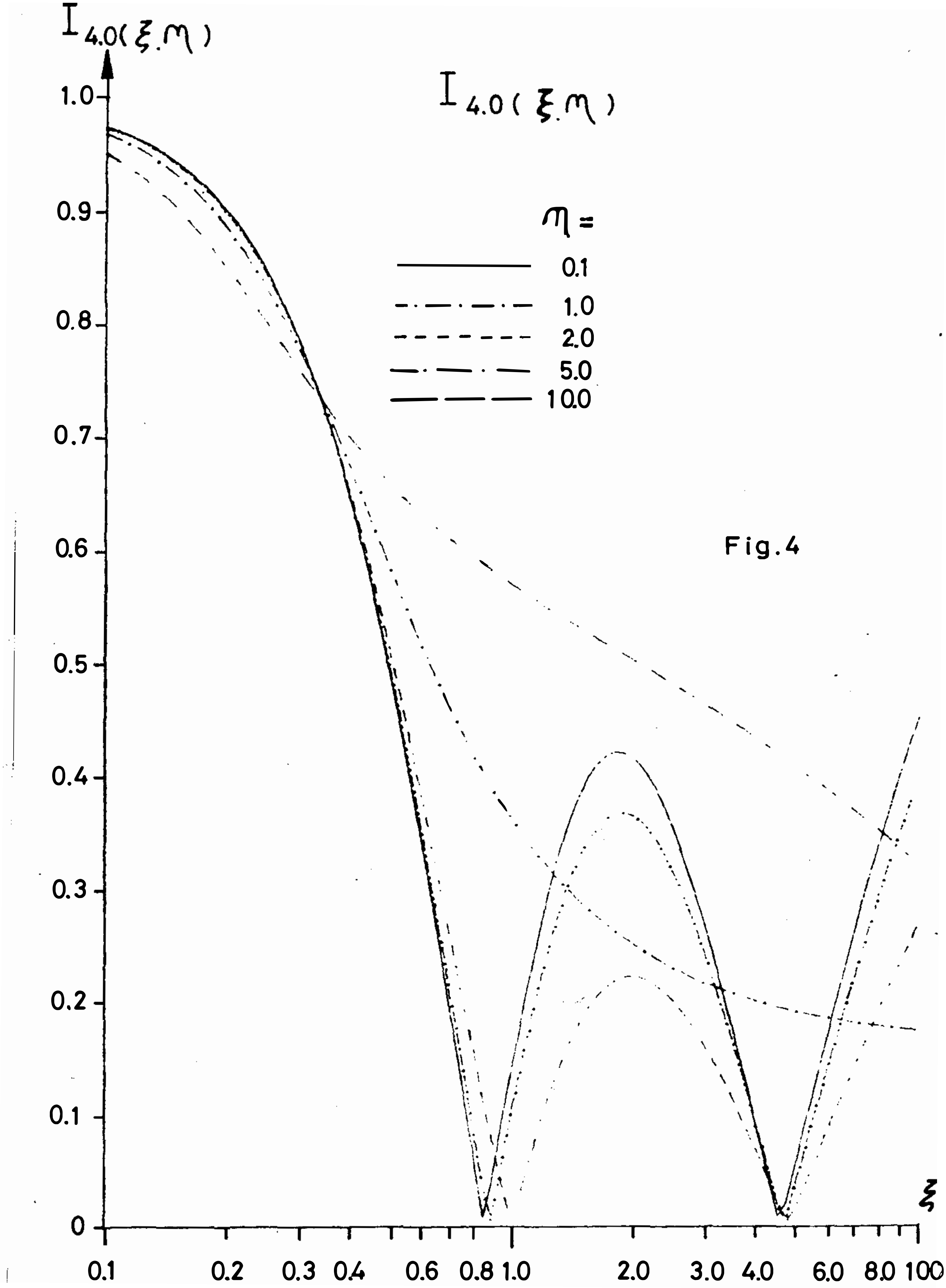
$I_{0.4}(\xi.\eta)$
$I_{0.4}(\xi.\eta)$
η =
0.1
1.0
2.0
5.0
10.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2.0
3.0
4.0
6.0
8.0
10.0
ξ

Fig. 2

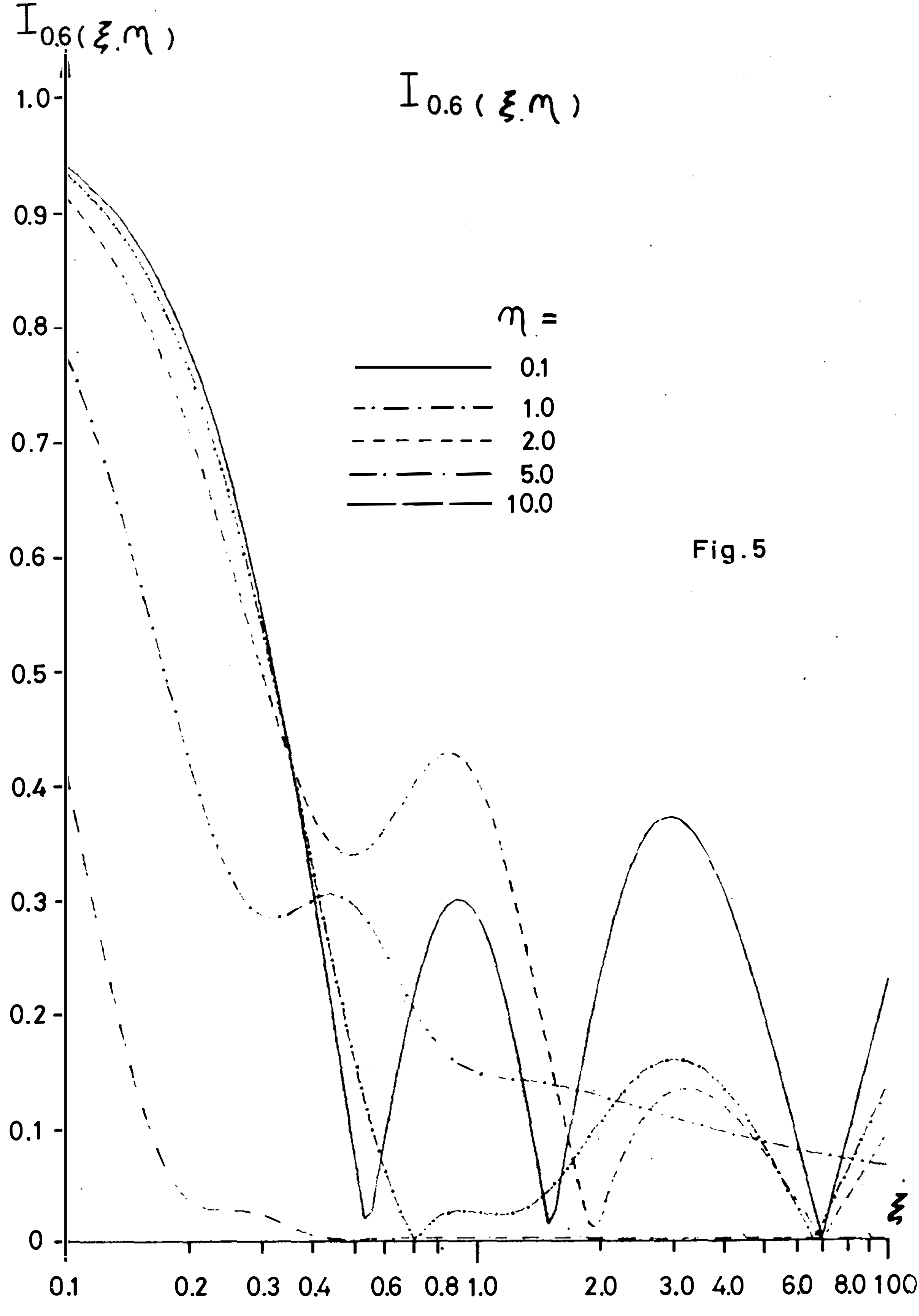


I 22(ξ.η)
I 22(ξ.η)
η =
0.1
1.0
2.0
5.0
10.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2.0
3.0
4.0
6.0
8.0
10.0
ξ

Fig. 3

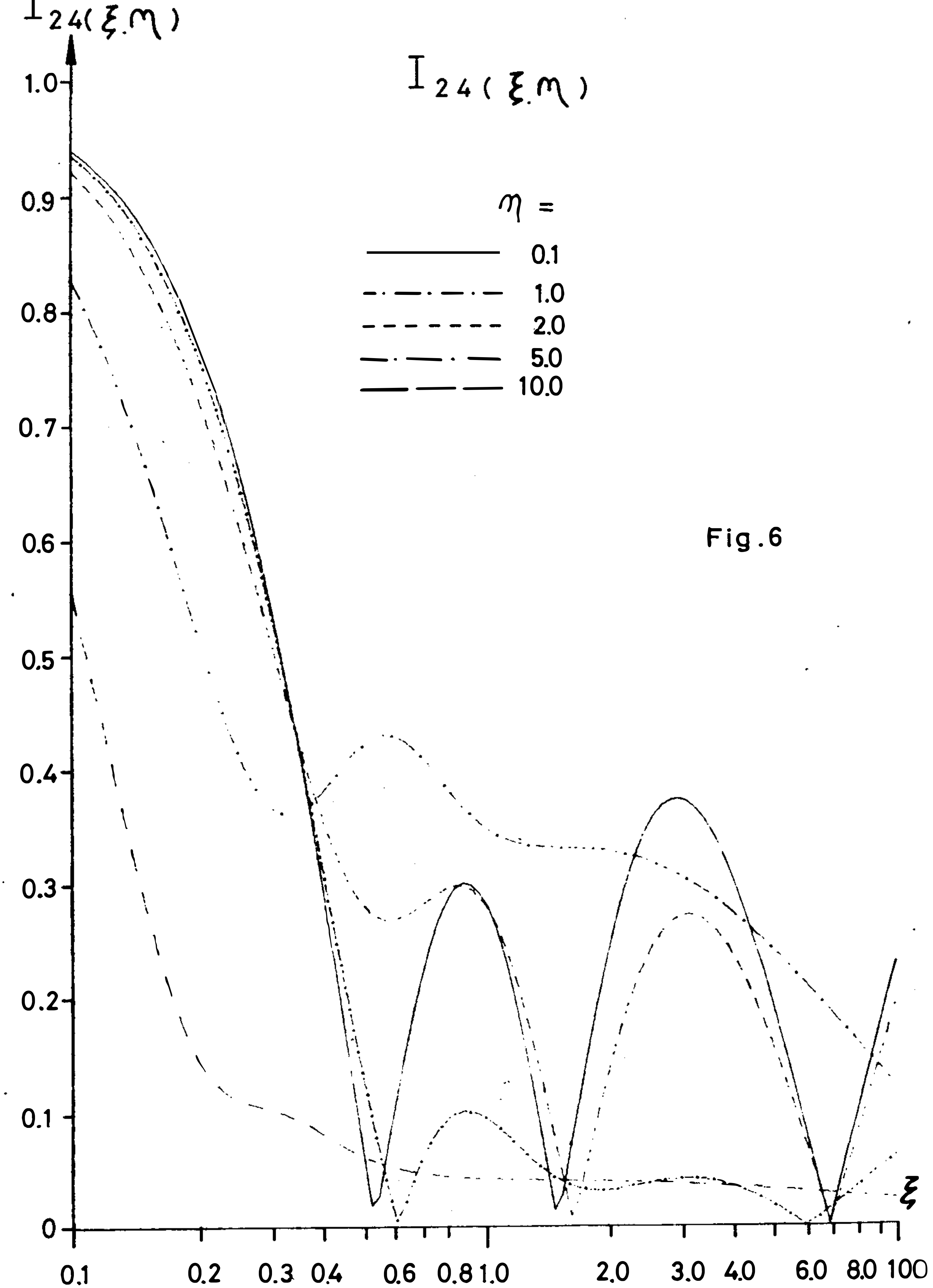


I 4.0(ξ.η)
I 4.0(ξ.η)
η =
0.1
1.0
2.0
5.0
10.0
Fig.4
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ξ
0.1 0.2 0.3 0.4 0.6 0.8 1.0 2.0 3.0 4.0 6.0 8.0 100



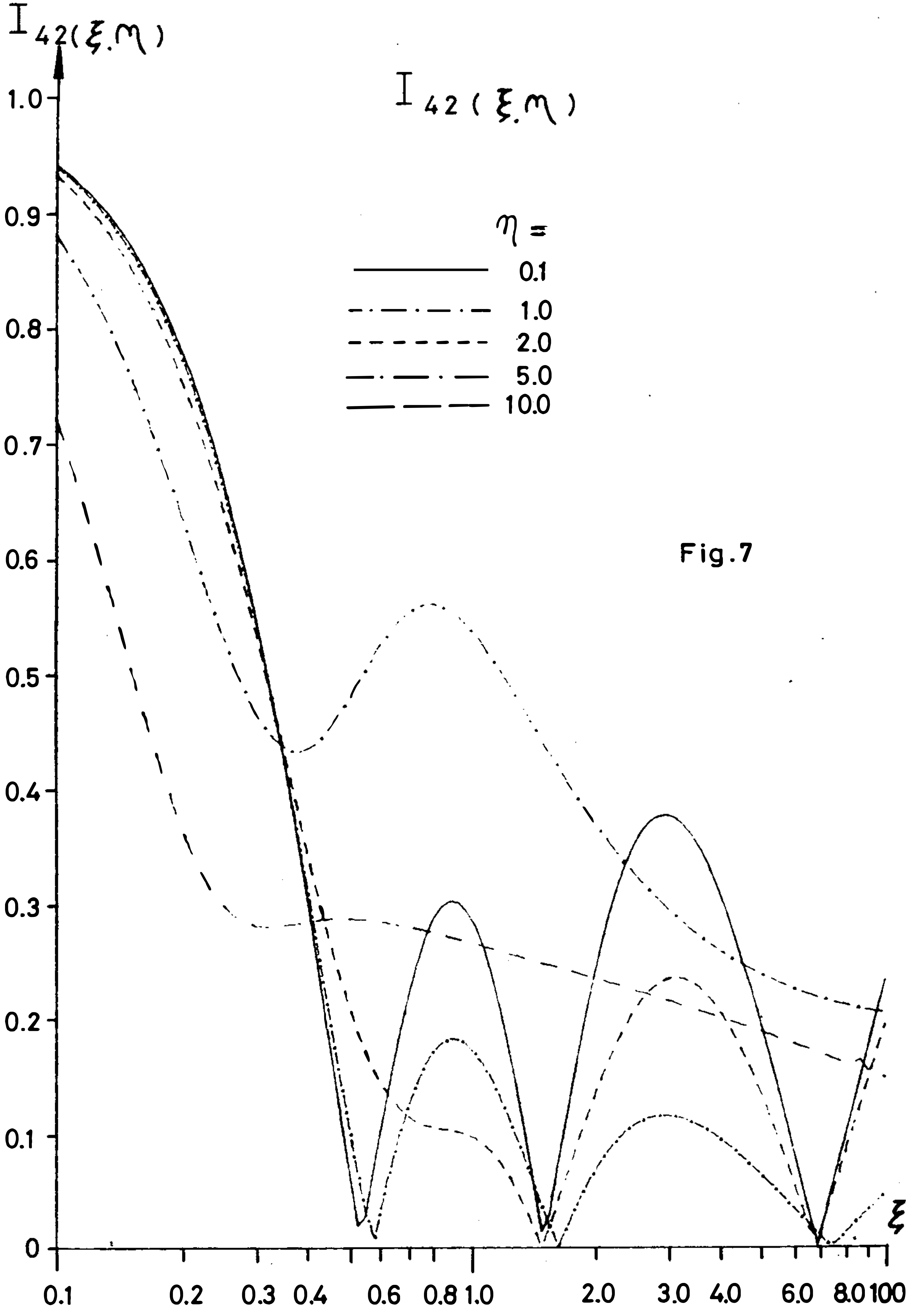
$I_{0.6}(\xi.\eta)$
$I_{0.6}(\xi.\eta)$
η =
0.1
1.0
2.0
5.0
10.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ξ
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2.0
3.0
4.0
6.0
8.0
100

Fig.5

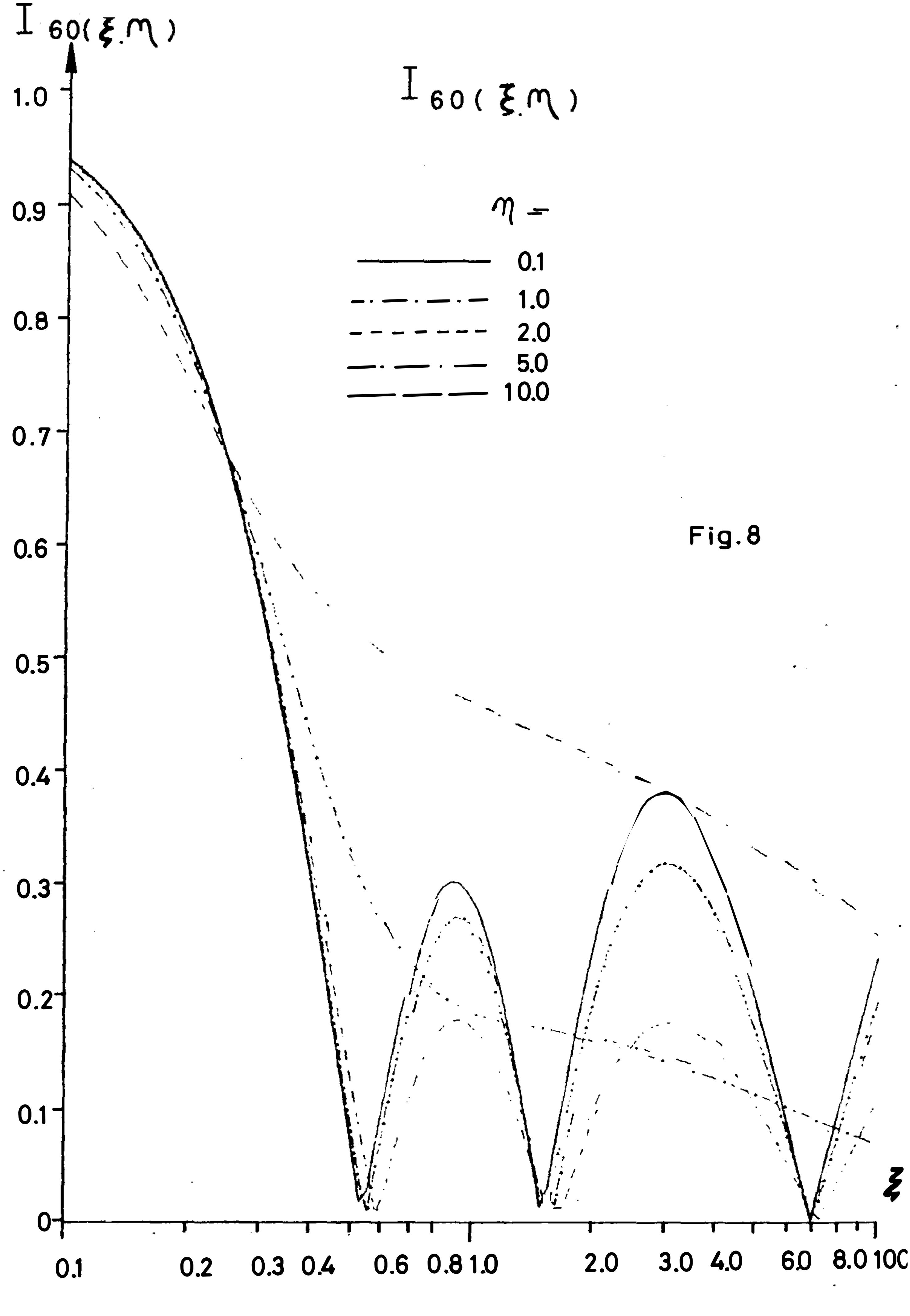


I24(ξ.η)
I24(ξ.η)
η =
0.1
1.0
2.0
5.0
10.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ξ
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2.0
3.0
4.0
6.0
8.0
100

Fig.6



$I_{42}(\xi, \eta)$
$I_{42}(\xi, \eta)$
$\eta =$
0.1
1.0
2.0
5.0
10.0
Fig.7
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ξ
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2.0
3.0
4.0
6.0
8.0
100



I 60(ξ.η)
I 60(ξ.η)
η =
0.1
1.0
2.0
5.0
10.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ξ
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2.0
3.0
4.0
6.0
8.0
100

Fig.8