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153 10 AG. PARASKEVI ATTIKIS POB 60228 TEL. 6513111-9 6518911-9 TLX 216199 ATOM

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PARAFERMIONS AND GENERALIZED PARAFERMIONS IN PHYSICAL SYSTEMS

Dennis BONATSOS

Institute of Nuclear Physics, N.C.S.R. "Demokritos"
GR-15310 Aghia Paraskevi, Attiki, Greece

C. DASKALOYANNIS, K. KANAKOGLOU, G. PETSOS

Department of Physics, Aristotle University of Thessaloniki
GR-54006 Thessaloniki, Greece

ABSTRACT

A brief review of applications of parafermionic oscillators and parafermionic algebras in molecular, nuclear, and superintegrable systems is given first. Subsequently two new applications are described in more detailed: i) The mapping of spinors with $j = p/2$ onto polynomial algebras, which are proved to be generalized parafermionic algebras of order p. ii) An extension of the Jaynes–Cummings model of quantum optics for the case of a $(p+1)$-level atom interacting with an one-mode electromagnetic field.

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Dennis BONATSOS
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ABSTRACT

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1. Introduction

In addition to fermions (particles which have the characteristic property that only one of them can occupy each quantum state) and bosons (particles of which infinitely many identical to each other can occupy the same quantum state), parafermions of order p (with p being a positive integer) have been introduced [1,2], having the characteristic property that at most p identical particles of this kind can be found in the same quantum state. Ordinary fermions clearly correspond to parafermions with $p = 1$, since only one fermion can occupy each quantum state according to the Pauli principle. While fermions obey the Fermi–Dirac statistics and bosons obey the Bose–Einstein statistics [3], parafermions are assumed to obey an intermediate kind of statistics, called parastatistics [3,4,5]. For the description of systems of parafermions, parafermionic algebras [4] and generalized deformed parafermionic oscillators [6] have been introduced, the latter being related to the notion of the generalized deformed oscillator [7,8]. The properties of parafermions and parabosons, as well as the parastatistics and the field theories associated with them, have been the subject of many recent investigations [9,10,11,12,13,14].The relations between parafermionic algebras and other algebras (finite W algebras $\tilde{W}_0$ [15] and $W_3^{(2)}$ [16,17], $su_\Phi(2)$ algebras [18,19,20,21]) have been studied in [22,23].

Furthermore, parafermionic oscillators and parafermionic algebras find applications in the description of several physical systems. Known examples include:

i) The generalized deformed oscillators used for the description of the Morse potential [24,25], which have the form of parafermionic oscillators with the order p of the parafermions being related to the inverse of the anharmonicity constant. A similar generalized deformed oscillator has also been introduced [26] for the modified Pöschl-Teller potential. These oscillators have been used for the description of the vibrational spectra of diatomic [27] and polyatomic [25] molecules.

ii) The generalized deformed oscillators used for the exact description of fermion pairs of zero angular momentum in a single-j nuclear shell [28]. Again in this case the oscillators are parafermionic ones, the order p of the parafermions being related to the size (degeneracy) of the single-j shell.

iii) The generalized deformed parafermionic oscillators corresponding to [22] the isotropic harmonic oscillator in a 2-dimensional curved space with constant curvature [29,30], to the Kepler problem in a 2-dimensional curved space with constant curvature [29,30], to the Fokas-Lagerstrom potential [30,31], to the Smorodinsky-Winternitz potential [30,32], to the Holt potential [30,33].

In the present work we are going to deal with two new applications of parafermionic techniques to physical systems:

i) The description of spinors with $j = p/2$ by polynomial algebras which are generalized parafermionic algebras of order p. This will be carried out in detail in sections 2-5.

ii) The extension of the Jaynes-Cummings model [34,35], which describes the interaction between a two-level atom (described by a fermion) and a one-mode electromagnetic field (described by a boson), to the case of an atom with $p+1$ levels (described by a parafermion of order p) coupled with the electromagnetic field (described by a boson). This will be briefly described in sec. 6.

2. Spinors and polynomial algebras

The study of systems of many spins is of interest in many branches of physics. This study is in many cases facilitated through boson mapping procedures (see [36] for a comprehensive review). Some well-known examples are the Holstein-Primakoff mapping of the spinor algebra onto the harmonic oscillator algebra [37] and the Schwinger mapping of Lie algebras (or of q-deformed algebras) onto the usual (or onto the q-deformed) oscillator algebras [38,39,40].

Parafermions and parabosons have also been involved in mapping studies. A mapping of the spinor algebra onto a parafermionic algebra has been discussed in [1,2,4]. Mappings of so(2n), sp(2n,R), and other Lie algebras onto parafermionic and parabosonic algebras have been studied in [4,41], while parabosonic mappings of osp(m,n) superalgebras have been given in [42,43].

Recently [44] the algebras of the operators of a single spinor with fixed spin value j have been mapped onto polynomial algebras, which constitute a quite recent subject of investigations in physics [6,22,45,46]. In polynomial algebras the commutator of two generators does not result in a linear combination of generators, as in the case of the usual Lie algebras, but rather into a combination of polynomials of the generators. The mappings of ref. [44] connect the class of spinor algebras to the class of polynomial algebras.

In the present study we show that the polynomial algebras of ref. [44], which are connected to the single spinor algebras, are indeed examples of either parafermionic algebras [1,2,4] or generalized parafermionic algebras [6,22]. As a result, a mapping of the single spinor algebras with $j = p/2$ onto parafermionic algebras of order p (or generalized parafermionic algebras of order p) is established.

The consequences of these findings are twofold:

i) A connection between parafermionic algebras and polynomial algebras is established, yielding new results of mathematical nature. For example, it is proved that the number operator for a single parafermion can be written as a combination of monomials of the ladder operators.

ii) The single parafermionic algebra is consistently imbeded into a many parafermion algebra [4]. Then a system of spinors can be viewed as a system of parafermions, obeying intermediate statistics. The practical consequences of this assumption on the study of the statistics of systems of many spinors, with spin greater than 1/2, remain to be seen. The situation is analogous to the case of a system of many q-deformed oscillators (or quons) obeying fractional statistics [47,48].

3. Parafermionic algebras

Let us start by defining the algebra $\mathcal{A}_n^{[p]}$, corresponding to n parafermions of order p. This algebra is generated by n parafermionic generators $b_i, b_i^\dagger$, where $i = 1, 2, \ldots, n$, satisfying the *trilinear* commutation relations:

$$\left[M_{k\ell}, b_m^\dagger\right] = \delta_{\ell m} b_k^\dagger, \quad \left[M_{\ell k}, b_m\right] = -\delta_{\ell m} b_k, \tag{1}$$

where $M_{k\ell}$ is an operator defined by:

$$M_{k\ell} = \frac{1}{2}\left(\left[b_k^\dagger, b_\ell\right] + p\delta_{k\ell}\right). \tag{2}$$

From this definition it is clear that eq. (1) is a *trilinear* relation, i.e. a relation relating three of the operators $b_i^\dagger, b_i$. Finally the definition of the parafermionic algebra is completed by the relation:

$$[b_i, [b_j, b_k]] = \left[b_i^\dagger, \left[b_j^\dagger, b_k^\dagger\right]\right] = 0. \tag{3}$$

Each parafermion separately is characterized by the ladder operators $b_i^\dagger$ and b_i and the number operator M_{ii}. The basic assumption is that the parafermionic creation and annihilation operators are nilpotent ones:

$$(b)^{p+1} = \left(b^\dagger\right)^{p+1} = 0. \tag{4}$$

In ref. [8] it is proved that the single parafermionic algebra is a generalized oscillator algebra [7], satisfying the following relations (for simplicity we omit the parafermion indices):

$$\left[M, b^\dagger\right] = b^\dagger, \quad [M, b] = -b, \tag{5}$$

$$b^\dagger b = [M] = M\,(p+1-M)\,, \quad bb^\dagger = [M+1] = (M+1)\,(p-M)\,, \tag{6}$$

$$M\,(M-1)\,(M-2)\ldots(M-p) = 0. \tag{7}$$

The definition (2) – or equivalently eq. (6) – imply the commutation relation:

$$\left[b^\dagger, b\right] = 2(M - p/2). \tag{8}$$

The above relation combined with (5) suggests the use of the parafermions as spinors of spin $p/2$

$$S_+ \leftrightarrow b^\dagger, \quad S_- \leftrightarrow b, \quad S_o \leftrightarrow (M - p/2). \tag{9}$$

In most publications no special commutation relations are considered for a collection of spinors. Usually the operators corresponding to different half-integer (integer) spinors are assumed to anticommute (commute). In contrast, in the present paper the use of the trilinear commutation relations is suggested for the many spinor problem. Another interesting point is that the following relation is usually not taken into consideration:

$$\prod_{k=-p/2}^{p/2} (S_o - k) = 0.$$

It is worth noticing that in the case of parafermions the commutation relation (8) is some how trivial because it is inherent in the definition of the number operator (2). This relation switches the trilinear commutation relations to ordinary commutation relations, where two operators are involved. In contrast, in the case of parabosons this construction is not trivial, because anticommutation relations are involved in the definition of the number operator[14].

4. Spinors and parafermionic algebras

We start now examining in detail the connection between spinors with $j = p/2$ and parafermions of order p.

The $p = 1$ parafermions coincide with the ordinary fermions, i.e. the usual spin 1/2 spinors [49].

For spinors with $j = 1$ Chaichian and Demichev [44] use the following mapping

$$S_+ \leftrightarrow \sqrt{2}a^\dagger, \qquad S_- \leftrightarrow \sqrt{2}a, \tag{10}$$

where

$$a^3 = a^{\dagger 3} = 0, \tag{11}$$

$$aa^\dagger + a^{\dagger 2}a^2 = 1. \tag{12}$$

Using the above two relations we can define the number operator N

$$N = 1 - \left[a, a^\dagger\right] = a^\dagger a + a^{\dagger 2}a^2. \tag{13}$$

This number operator satisfies the linear commutation relations:

$$\left[N, a^\dagger\right] = a^\dagger, \quad [N, a] = -a.$$

The self-contained commutation relations for the $p = 2$ parafermions are given in ref. [4] (eqs (5.13) to (5.20))

$$b^3 = b^{\dagger 3} = 0, \tag{14}$$

$$bb^\dagger b = 2b, \tag{15}$$

$$b^\dagger b^2 + b^2 b^\dagger = 2b. \tag{16}$$

The set of relations (14)-(16) imply the following definition of the number operator M:

$$M = \frac{1}{2}\left(\left[b^\dagger, b\right] + 2\right). \tag{17}$$

The set of relations (11)–(12) imply relations (14)–(16) after taking into consideration the correspondence:

$$a = \frac{1}{\sqrt{2}} b, \qquad a^\dagger = \frac{1}{\sqrt{2}} b^\dagger. \tag{18}$$

For example one can easily see the following:

i) Eq. (14) occurs trivially from eq. (11).

ii) Eq. (15) is obtained by multiplying eq. (12) by a on the right and using eq. (11).

iii) Eq. (16) is obtained by multiplying eq. (12) by a on the left and using eq. (11) and (12).

In ref. [8] the parafermionic algebra (14)–(17) was shown to be equivalent to the deformed oscillator algebra [7], which is defined by relations (4)–(7), for $p = 2$. This deformed oscillator algebra satisfies in addition the relations (11) to (13). Therefore the Chaichian - Demichev polynomial algebra (11)–(13), the $p = 2$ parafermionic algebra (14)–(17) and the deformed oscillator algebra (4)–(7) are equivalent.

Relations (12) and (13) indicate that $aa^\dagger$ and N can be expressed as a linear combination of monomials $\left(a^\dagger\right)^k a^k$. This is the reason the algebra described by eqs (12)–(13) is called in [44] a "polynomial" algebra.

What we have just seen is that the polynomial algebra (11)–(13) is in fact the $p = 2$ parafermionic algebra (14)–(17). The new result which arises from this discussion is that the parafermionic algebra can be written as a polynomial algebra through the r.h.s of eq. (13). It seems that this fact has been ignored, while the "dual" relation, giving $b^\dagger b$ or $bb^\dagger$ as polynomial functions of the number operator,

$$b^\dagger b = M(3 - M), \qquad bb^\dagger = (M + 1)(2 - M),$$

is known [6,8,23].

For spinors with $j = 3/2$ Chaichian and Demichev [14] use the following mapping

$$S_+ \leftrightarrow \sqrt{3}a^+, \quad S_- \leftrightarrow \sqrt{3}a. \tag{19}$$

where

$$a^4 = a^{\dagger 4} = 0, \tag{20}$$

$$aa^\dagger = 1 + \frac{1}{3}a^\dagger a - \frac{1}{3}a^{\dagger 2}a^2 - \frac{2}{3}a^{\dagger 3}a^3, \tag{21}$$

$$\left[a, a^\dagger\right] = 1 - \frac{2}{3}N, \tag{22}$$

$$N = a^\dagger a + \frac{1}{2}a^{\dagger 2}a^2 + a^{\dagger 3}a^3. \tag{23}$$

These relations are the analogues of eqs. (11)-(13) for the $j = 3/2$ case.

The complicated self-consistent commutation relations for the $p = 3$ parafermionic algebra are given in Appendix B of ref. [4]. After long but straightforward calculations the $p = 3$ parafermionic relations are deduced from the above eqs (21)-(23) by taking into account the correspondence:

$$a = \frac{1}{\sqrt{3}}b, \quad a^\dagger = \frac{1}{\sqrt{3}}b^\dagger. \tag{24}$$

Therefore the polynomial algebra (21)-(23) is in fact the $p = 3$ parafermionic algebra. The new result which again arises from this discussion is that the parafermionic algebra can be written as a polynomial algebra through eq. (23), while the "dual" relation

$$b^\dagger b = M(4 - M), \quad bb^\dagger = (M + 1)(3 - M),$$

is again already known [8].

Stimulated by the above results we can show the following proposition:

Proposition 1 *The $j = p/2$ spinor algebra $\{S_\pm, S_o\}$ is mapped onto the p-parafermionic algebra $\{b^+, b, M\}$ which is a polynomial algebra given by the relations:*

$$\begin{aligned}
\left[M, b^\dagger\right] &= b^\dagger, \\
\left[M, b\right] &= -b, \\
b^{p+1} &= \left(b^\dagger\right)^{p+1} = 0, \\
b^\dagger b &= M\left(p + 1 - M\right), \\
bb^\dagger &= \left(M + 1\right)\left(p - M\right), \\
M &= \tfrac{1}{2}\left(\left[b^\dagger, b\right] + p\right),
\end{aligned} \tag{25}$$

where the number operator M is given by the following polynomial relation

$$M = \sum_{k=1}^{p} \frac{c_k}{p^k}\, b^{\dagger k} b^k, \tag{26}$$

and the coefficients $c_1, c_2, \ldots, c_p$ can be determined from the solution of the system of equations:

$$\left.\begin{array}{l} \rho(1) = 1 \\ \rho(2) = 2 \\ \ldots \\ \rho(p) = p \end{array}\right\} \tag{27}$$

where

$$\rho(z) = \sum_{k=1}^{p} \frac{c_k}{p^k} \frac{\Gamma(z)}{\Gamma(z-k)}.$$

This is true because we can see that

$$b^{\dagger k} b^k = \prod_{\ell=0}^{k-1} (M - \ell) \equiv \frac{\Gamma(M)}{\Gamma(M-k)}.$$

To the best of our knowledge the fact that the number operator of a parafermionic algebra can be written as a combination of monomials, i.e. eq. (26), was not previously known in the context of parafermionic algebras. It has been derived in another context by Chaichian and Demichev [44], without reference to parafermions. The analytic calculation of the coefficients c_k seems to be a complicated task. In Table 1 the coeffients up to $p = 5$ are explicitly given.

Table 1: Coefficients appearing in eq. (26).

p	c_1	c_2	c_3	c_4	c_5
1	1	--	--	--	--
2	1	1	--	--	--
3	1	1/2	1	--	--
4	1	1/3	1/3	7/9	--
5	1	1/4	1/6	19/96	23/48

In ref. [44] an alternative to the parafermionic algebra was also used for constructing a mapping of the spinor algebra. This alternative is characterized by the relations:

$$\begin{array}{c} (\alpha^{\dagger})^{p+1} = (\alpha)^{p+1} = 0, \\ \alpha\alpha^{\dagger} + \alpha^{\dagger p}\alpha^{p} = 1. \end{array} \tag{28}$$

This algebra is a generalized parafermionic algebra [6,8] corresponding to the structure function:

$$\alpha^{+}\alpha = \Phi(N) = N(p+1-N)\left(f_1 + f_2 N + \ldots f_p N^{p-1}\right), \quad \alpha\alpha^{+} = \Phi(N+1). \tag{29}$$

In this case eq. (28) imposes the constraint:

$$\sigma(N) \equiv \Phi(N+1) + \prod_{k=0}^{p-1} \Phi(N-k) = 1. \tag{30}$$

After solving the system:

$$\left.\begin{array}{r} \sigma(0) = 1 \\ \sigma(1) = 1 \\ \sigma(2) = 1 \\ \dots \\ \sigma(p-1) = 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \Phi(1) = 1 \\ \Phi(2) = 1 \\ \Phi(3) = 1 \\ \dots \\ \Phi(p) = 1 \end{array}\right. \tag{31}$$

we can find the values of the parameters $f_1, f_2, \dots, f_p$. Their values up to $p = 6$ are reported in Table 2.

Table 2: Coefficients appearing in eq. (29).

p	f_1	f_2	f_3	f_4	f_5
1	1	−−	−−	−−	−−
2	1/2	−−	−−	−−	−−
3	7/12	−1/3	1/12	−−	−−
4	5/12	−5/24	1/24	−−	−−
5	157/360	−11/30	29/180	−1/30	1/360
6	7/20	−49/180	77/720	−7/360	1/720

Thus we have shown the following proposition:

Proposition 2 *The polynomial algebra (28) is a generalized deformed parafermionic algebra, characterized by the structure function (29), where the coefficients are chosen so that eq. (30) is satisfied.*

5. Discussion

In summary, it has been shown that the single spinor algebra with $j = p/2$, which is a polynomial algebra as shown in ref. [44], is the single parafermionic algebra of order p. Furthermore, the parafermionic algebras can be considered as polynomial algebras, their diagonal number operator M_{ii} being able to be written as a combination of monomials of the ladder operators. The general problem of finding an expression of the number operator M_{ij} as a combination of monomials of the ladder operators is still open. A similar problem exists in quonic algebras [50,51,52,53].

The realization that the single spinor algebra is a single parafermionic algebra can have nontrivial consequences in the study of the many indistinguishable spinor problem. Usually a many spinor system is considered as a many fermion or many boson model, depending on the individual spin being half integer or integer. In this case the system obeys the Fermi-Dirac statistics or the Bose-Einstein statistics, respectively. In contrast, a many parafermion system obeys more complicated fractional-like statistics [3]. Consequently the acceptance that the one spinor algebra is described by a polynomial algebra, which as we have proved is in fact the parafermionic algebra, opens the question whether the many spinor system can be considered as a many parafermion model rather than a many fermion (or boson) model and to what extend its associated statistics is the Fermi-Dirac (or Bose-Einstein) or the more complicated fractional statistics [3].

6. Parafermionic extension of the Jaynes–Cummings model

The Jaynes–Cummings model (JCM) [34,35] is a simple model describing the interaction of a two-level atom with a single-mode bosonic field. The success of this model is due to its simplicity and solubility. The two-level atom is simulated by an ordinary fermion. It has two possible states, with only one fermion being able to occupy each state. Multiphoton generalizations of the JCM have also been developed, in which the fermionic operators describing the two-level atom are coupled to combinations of bosonic operators representing the multiphoton system. It has been shown recently [54] that all these variants of the two-level JCM possess an underlying symmetry structure resembling a generalized Schwinger realization of su(2). Furthermore, a generalization dealing with a fermion interacting with a paraboson has been introduced [55].

In the above mentioned generalizations of the JCM the part of the Hamiltonian describing the photon field is generalized, while the part of the Hamiltonian describing the two-level atom remains intact. In this section we study a generalization of the JCM in a different direction: we conserve the single-mode bosonic field in the Hamiltonian, but we generalize the part of the Hamiltonian describing the atom. Instead of a fermion coupled to the photon field, we use a parafermion. In this way we obtain a parafermionic generalization of the JCM, dealing with a parafermion of order p interacting with a one-mode field. This model can be thought of as simulating a $(p+1)$-level atom interacting with a one-mode electromagnetic field. The underlying symmetry algebra turns out to be a generalized deformed su(2) algebra corresponding to a Schwinger realization in terms of a parafermion and a boson.

The original Jaynes–Cummings model (JCM) [34] refers to the Hamiltonian of a system having a bosonic sector (electromagnetic field) and a fermionic sector (two-level atom), as well as a term corresponding to the coupling of the bosonic and fermionic sectors:

$$H = H_{\text{bos}} + H_{\text{fer}} + H_{\text{fer-bos}} = \omega \left(N + \frac{1}{2} \right) + \Omega M + g \left(b^\dagger a + a^\dagger b \right), \qquad (32)$$

where $a, a^\dagger, N$ and $b, b^\dagger, M$ are the bosonic and fermionic annihilation, creation, and number operators, satisfying the algebraic relations:

$$
\begin{array}{ll}
\underline{\text{Bosonic sector}} & \underline{\text{Fermionic sector}} \\
\left[N, a^\dagger\right] = a^\dagger & \left[M, b^\dagger\right] = b^\dagger \\
\left[N, a\right] = -a & \left[M, b\right] = -b \\
a^\dagger a = N & b^\dagger b = M \\
a a^\dagger = N + 1 & b b^\dagger = 1 - M \\
 & (b)^2 = \left(b^\dagger\right)^2 = 0 \\
 & M(M-1) = 0
\end{array}
\qquad (33)
$$

The bosonic sector of the algebra corresponds to the electromagnetic field, while the fermionic one simulates the two-level atomic system. Since the fermion corresponds to the nilpotent operator b, which satisfies the relation $b^2 = 0$, there can be no more than two fermions in each site.

There are several variations of the two-level JC model (see [35] and references therein). All two-level models conserve the fermionic sector of the model, but the bosonic sector is replaced either by complicated combinations of the bosonic operators or by parabosonic or q-deformed operators. The common feature of all these models is their solubility, due to the nilpotency of the fermionic operators $b, b^\dagger$, which has as a consequence that all these models can be treated by manipulations of 2 by 2 matrices. This property allows the construction of "generalized" JC models, i.e. abstract models which can be solved for any form of the "deformed" bosonic sector [56].

The fermionic operators are the most elementary examples of deformed oscillator algebras [49] satisfying the nilpotency condition (4). From this point of view the JC Hamiltonian (34) is in fact the interaction Hamiltonian of the most elementary nilpotent field with one usual boson.

An extension of the JC model can be constructed through the replacement of the fermionic sector of the Hamiltonian (32) by a parafermionic one. The relevant Hamiltonian reads

$$H = H_{\text{bos}} + H_{\text{pf}} + H_{\text{bos-pf}} = \omega\left(N + \frac{1}{2}\right) + E\left(M\right) + g\left(b^\dagger a + a^\dagger b\right). \qquad (34)$$

The above Hamiltonian has a parafermionic sector linearly coupled to a bosonic sector. The parafermionic sector is described by the "parafermionic" algebra of eqs (5) – (7). This algebra corresponds [6] to the ordinary parafermionic algebra used in parastatistics. From the physical point of view it describes a particle (parafermion) having $p+1$ possible states. The energy of each state is given by the function $E\left(M\right)$, which is a polynomial of degree p

$$E\left(M\right) = a_o + a_1 M + a_2 M^2 + \ldots + a_p M^p. \qquad (35)$$

The Fock space has $p+1$ elements $|0\rangle, |1\rangle, \ldots, |p\rangle$, on which the operators of the algebra act in the following way

$$\begin{aligned} b^\dagger \,|M_{\rm pf}\rangle &= \sqrt{[M_{\rm pf}+1]}\,|M_{\rm pf}+1\rangle\,, \\ b\,|M_{\rm pf}\rangle &= \sqrt{[M_{\rm pf}]}\,|M_{\rm pf}-1\rangle\,, \\ M\,|M_{\rm pf}\rangle &= M_{\rm pf}\,|M_{\rm pf}\rangle\,, \end{aligned} \tag{36}$$

where by definition

$$[x] = x\,(p+1-x)\,. \tag{37}$$

Each state $|k\rangle$ (with $k = 0, 1, \ldots, p$) is an eigenstate of the energy operator (35), corresponding to the eigenvalue $E(k)$. The operator $b^\dagger$ causes the transition from a state $|k\rangle$ to the following excited one $|k+1\rangle$, while the operator b forces the system to go to the previous energy state $|k-1\rangle$. We assume a descending ordering of energy eigenstates:

$$E(0) > E(1) > \ldots > E(p). \tag{38}$$

Starting from here one can prove the following:

i) One can construct a deformed u(2) algebra, generated by the operators

$$\begin{aligned} J_+ &= a^\dagger b, & J_- &= a b^\dagger, \\ J_0 &= \tfrac{1}{2}\,(N-M)\,, & L &= \tfrac{1}{2}\,(N+M)\,. \end{aligned} \tag{39}$$

It is clear that this algebra possesses a Schwinger realization in terms of one parafermion and one boson. One can see that this is the symmetry algebra of the extended JCM [57].

ii) The representation theory of the above-mentioned deformed su(2) algebra can be constructed [57].

iii) Mean values as well as time averaged mean values of physically interesting quantities (number of emitted photons, "angular momentum" projection J_0) can then be calculated in a straightforward way [57].

The proposed formalism permits the numerical study of the revivals and collapses in a multi-level laser system. This is the subject of an ongoing project.

A straightforward extension of this study will be the use of multimode electromagnetic fields as in the two-level case studied by Yu *et al.* [54].

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