SISSA 139/FM/98

One Dimensional Many-Body Problems with Point Interactions

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Abstract

The integrability of one dimensional quantum mechanical many-body problems with general contact interactions between two particles characterized by boundary conditions is studied. It is shown that besides the pure (repulsive or attractive) δ-function interaction there are other kinds of singular point interactions which give rise to two new one-parameter families of integrable quantum mechanical one dimensional many-body systems. The bound states and scattering matrices are calculated for both bosonic and fermionic statistics.

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A quantum particle moving in a local singular potential concentrated at one point ($x = 0$) has been extensively discussed in the literature, see eg. [1] and references therein. In one space dimension the problem is generally characterized [6] (see also [1-5]) by one of the two types of, respectively, nonseparated and separated boundary conditions imposed on the (scalar) wave function ϕ at $x = 0$:

1. (generic four dimensional family)

$$\phi(0_+) = e^{i\theta} a\phi(0_-) + e^{i\theta} b\phi'(0_-) \ , \quad \phi'(0_+) = e^{i\theta} c\phi(0_-) + e^{i\theta} d\phi'(0_-), \tag{1}$$

where

$$0 \leq \theta < \pi; \quad a, b, c, d \in I\!R \ \text{ with } \ ad - bc = 1. \tag{2}$$

2. (special two dimensional family)

$$\phi'(0_+) = h^+\phi(0_+) \ , \quad \phi'(0_-) = h^-\phi(0_-) \tag{3}$$

where

$$h^{\pm} \in I\!R \cup \{\infty\}. \tag{4}$$

(Note that the values $\theta = b = 0$, $a = d = 1$ in (1) correspond to positive (resp. negative) δ-function potential for $c > 0$ (resp. $c < 0$), while in (3) $h^+ = \infty$ or $h^- = \infty$ correspond to Dirichlet boundary conditions and $h^+ = 0$ or $h^- = 0$ correspond to Neumann boundary conditions).

We consider here wave functions ϕ that are not just scalars but have n components ('spin' states). Then the above boundary conditions are not the most general, it is possible to make (1) and (3) 'spin' dependent or even to mix different 'spins; but we are not considering such possibilities in the present paper.

The one dimensional N-boson and N-fermion problems with δ-function interactions have been investigated in [7] and [8] respectively in terms of Yang-Baxter equation. In this paper we study the integrability of one dimensional systems of N identical particles with more general contact interactions described by the boundary conditions (1) and/or

(3) that are imposed on the relative coordinates of the particles. We show that besides the delta-interaction for two other one parameter (sub)families, one with nonseparated boundary conditions (1) and another with separated boundary conditions (3), the N-particle system also satisfies a Yang-Baxter relation. The wave functions can be obtained by Bethe's hypothesis. The scattering matrix can be explicitly given.

We first consider the case of two particles ($N = 2$) with coordinates x_1, x_2 and momenta k_1, k_2 respectively. Each particle has n-'spin' states designated by s_1 and s_2, $1 \leq s_i \leq n$. For $x_1 \neq x_2$, these two particles are free. The wave functions ψ are symmetric (resp. antisymmetric) with respect to interchange $(x_1, s_1) \leftrightarrow (x_2, s_2)$ for bosons (resp. fermions). In the region $x_1 < x_2$, the Bethe's hypothesis states that the wave function is

$$\psi = \alpha_{12} e^{i(k_1 x_1 + k_2 x_2)} + \alpha_{21} e^{i(k_2 x_1 + k_1 x_2)}, \tag{5}$$

where α_{12} and α_{21} are $n^2 \times 1$ column matrices. In the region $x_1 > x_2$,

$$\psi = (P^{12}\alpha_{12}) e^{i(k_1 x_2 + k_2 x_1)} + (P^{12}\alpha_{21}) e^{i(k_2 x_2 + k_1 x_1)}, \tag{6}$$

where according to the symmetry or antisymmetry conditions, $P^{12} = p^{12}$ for bosons and $P^{12} = -p^{12}$ for fermions, p^{12} being the operator on the $n^2 \times 1$ column that interchanges $s_1 \leftrightarrow s_2$. We now transform to the center of mass coordinate X and the relative coordinate x defined as $X = (x_1 + x_2)/2$, $x = x_2 - x_1$. Set $k_{12} = (k_1 - k_2)/2$. Substituting (5) and (6) into the boundary conditions at $x = 0$, we get respectively for (1)

$$\left\{ \begin{array}{l} \alpha_{12} + \alpha_{21} = e^{i\theta} a P^{12}(\alpha_{12} + \alpha_{21}) + i e^{i\theta} b k_{12} P^{12}(\alpha_{12} - \alpha_{21}), \\ i k_{12}(\alpha_{21} - \alpha_{12}) = e^{i\theta} c P^{12}(\alpha_{12} + \alpha_{21}) + i e^{i\theta} d k_{12} P^{12}(\alpha_{12} - \alpha_{21}) \end{array} \right. \tag{7}$$

and for (3)

$$\left\{ \begin{array}{l} i k_{12}(\alpha_{21} - \alpha_{12}) = h_+(\alpha_{12} + \alpha_{21}) , \\ i k_{12} P^{12}(\alpha_{12} - \alpha_{21}) = h_- P^{12}(\alpha_{12} + \alpha_{21}) . \end{array} \right. \tag{8}$$

As far as the system (7) is concerned, eliminating the term $P^{12}\alpha_{12}$ we obtain the relation

$$\alpha_{21} = Y_{21}^{12} \alpha_{12} , \tag{9}$$

where

$$Y_{21}^{12} = \frac{2 i e^{i\theta} k_{12} P^{12} + i k_{12}(a - d) + (k_{12})^2 b + c}{i k_{12}(a + d) + (k_{12})^2 b - c}. \tag{10}$$

The system (8) instead is contradictory unless

$$h_+ = -h_- \doteq h \in I\!R \cup \{\infty\} \tag{11}$$

and in this case it also leads to equation (9), where this time

$$Y_{21}^{12} = \frac{ik_{12} + h}{ik_{12} - h} \, . \tag{12}$$

For $N \geq 3$ and $x_1 < x_2 < ... < x_N$, the wave function is given by

$$\psi = \alpha_{12...N} e^{i(k_1 x_1 + k_2 x_2 + ... + k_N x_N)} + \alpha_{21...N} e^{i(k_2 x_1 + k_1 x_2 + ... + k_N x_N)} + (N! - 2) \text{ } \textit{other terms}. \tag{13}$$

The columns α have $n^N \times 1$ dimensions. The wave functions in the other regions are determined from (13) by the requirement of symmetry (for bosons) or antisymmetry (for fermions). Along any plane $x_i = x_{i+1}$, $i \in 1, 2, ..., N-1$, from similar considerations as above we have

$$\alpha_{l_1 l_2 ... l_i l_{i+1} ... l_N} = Y_{l_{i+1} l_i}^{ii+1} \alpha_{l_1 l_2 ... l_{i+1} l_i ... l_N} \, , \tag{14}$$

where

$$Y_{l_{i+1} l_i}^{ii+1} = \frac{2ie^{i\theta} k_{l_i l_{i+1}} P^{ii+1} + ik_{l_i l_{i+1}}(a-d) + (k_{l_i l_{i+1}})^2 b + c}{ik_{l_i l_{i+1}}(a+d) + (k_{l_i l_{i+1}})^2 b - c} \tag{15}$$

for nonseparated b.c. and

$$Y_{l_{i+1} l_i}^{ii+1} = \frac{ik_{l_i l_{i+1}} + h}{ik_{l_i l_{i+1}} - h} \tag{16}$$

for separated b.c. Here $k_{l_i l_{i+1}} := (k_{l_i} - k_{l_{i+1}})/2$ play the role of spectral parameters and $P^{ii+1} = p^{ii+1}$ for bosons and $P^{ii+1} = -p^{ii+1}$ for fermions, where p^{ii+1} is the operator on the $n^N \times 1$ column that interchanges $s_i \leftrightarrow s_{i+1}$.

For consistency [7] Y must satisfy the Yang-Baxter equation with spectral parameter [9], i.e.,

$$Y_{ij}^{m,m+1} Y_{kj}^{m+1,m+2} Y_{ki}^{m,m+1} = Y_{ki}^{m+1,m+2} Y_{kj}^{m,m+1} Y_{ij}^{m+1,m+2} \, ,$$

or

$$Y_{ij}^{mr} Y_{kj}^{rs} Y_{ki}^{mr} = Y_{ki}^{rs} Y_{kj}^{mr} Y_{ij}^{rs} \tag{17}$$

if m, r, s are all unequal, and

$$Y_{ij}^{mr} Y_{ji}^{mr} = 1, \qquad Y_{ij}^{mr} Y_{kl}^{sq} = Y_{kl}^{sq} Y_{ij}^{mr} \tag{18}$$

if m, r, s, q are all unequal.

Clearly, Y given by (15) satisfies the relation (18) for all θ, a, b, c, d. But calculations show that Y satisfies the relations (17) only when $\theta = 0$, $a = d$ and $b = 0$, that is, according to the constraint (2), $\theta = 0$, $a = d = \pm 1$, $b = 0$, c arbitrary. The case $a = d = 1$, $\theta = b = 0$ corresponds to δ-function interactions, which has been investigated in [7, 8]. The case $a = d = -1$, $\theta = b = 0$, which we shall refer to as 'anti-δ' interaction, is related to another singular interactions between any two particles (for $a = d = -1$ and $\theta = b = c = 0$ see [2]) and has not been studied before for the N-particle case. Concerning Y given by (16), it satisfies both the relations (18) and (17) for arbitrary h.

We have thus shown that regarding the N-particle (either boson or fermion) problem in addition to the known one parameter family there are two new one parameter families, which can be also dealt with via the Bethe hypothesis. Hence, altogether there are three integrable one parameter families with contact interactions of type delta, anti-delta and separated one, described respectively by one of the following conditions on the wave function along the plane $x_i = x_j$ for any pair of particles with coordinates x_i and x_j respectively,

$$\phi(0_+) = +\phi(0_-), \quad \phi'(0_+) = c\phi(0_-) + \phi'(0_-) \ , \ c \in \mathbb{R} \ ; \tag{19}$$

$$\phi(0_+) = -\phi(0_-), \quad \phi'(0_+) = c\phi(0_-) - \phi'(0_-) \ , \ c \in \mathbb{R} \ ; \tag{20}$$

$$\phi'(0_+) = h\phi(0_+), \quad \phi'(0_-) = -h\phi(0_-) \ , \ h \in \mathbb{R} \cup \{\infty\} \ . \tag{21}$$

The wave function is given by (13) with the α's determined by (14) and initial conditions, with respectively

$$Y^{ii+1}_{l_{i+1}l_i} = \frac{i(k_{l_i} - k_{l_{i+1}})P^{ii+1} + c}{i(k_{l_i} - k_{l_{i+1}}) - c} \ ; \tag{22}$$

$$Y^{ii+1}_{l_{i+1}l_i} = -\frac{i(k_{l_i} - k_{l_{i+1}})P^{ii+1} + c}{i(k_{l_i} - k_{l_{i+1}}) + c} \ ; \tag{23}$$

and

$$Y^{ii+1}_{l_{i+1}l_i} = \frac{i(k_{l_i} - k_{l_{i+1}}) + 2h}{i(k_{l_i} - k_{l_{i+1}}) - 2h} \ . \tag{24}$$

We now study further the new one dimensional N-particle systems having two particle point interactions. We refer to [10] for the discussion of selfadjointness question in the three-body case. In the following we shall proceed without paying attention to this problem. First we investigate the question of bound states. Concerning the 'anti-δ' boundary conditions (20) when $c > 0$ the system has a single bound state (differently from the pure δ-function interaction case where the N-particle system has a single bound state when $c < 0$). For $N = 2$, the space part of the unique bound state has the form, in the relative coordinate $x = x_2 - x_1$,

$$\psi_2 = [\theta(-x) - \theta(x)]e^{-\frac{c}{2}|x|}, \tag{25}$$

where $\theta(x) = (1 + sgn(x))/2$ is the step function, see [5]. The eigenvalue corresponding to the bound state (25) is $-c^2/2$. By generalization we get the bound state for the N-particle system

$$\psi_N = \alpha \prod_{k>l} [\theta(x_k - x_l) - \theta(x_l - x_k)]e^{-\frac{c}{2}\sum_{i>j}|x_i - x_j|}, \tag{26}$$

where α is the spin wave function. It can be checked that ψ_N satisfy the boundary condition (20) at $x_i = x_j$ for any $i \neq j \in 1, ..., N$, by noting that in this case either $x_k > x_i, x_j$ or $x_k < x_i, x_j$ for $k \neq i, j$. The spin wave function α here satisfies $P^{ij}\alpha = -\alpha$ for any $i \neq j$, that is, $p^{ij}\alpha = -\alpha$ for bosons and $p^{ij}\alpha = \alpha$ for fermions.

It is worth to mention that ψ_N is of the form (13) in each of the above regions. For instance comparing ψ_N with (13) in region $x_1 < x_2 ... < x_N$ we get

$$k_1 = -i\frac{c}{2}(N - 1),\ k_2 = k_1 + ic,\ k_3 = k_2 + ic, ..., k_N = -k_1. \tag{27}$$

The energy of the bound state ψ_N is

$$E = -\frac{c^2}{12}N(N^2 - 1). \tag{28}$$

It is interesting that, although the interactions between two particles are different from the ones in the pure δ-function case, the binding energies (28) of the bound states are of a similar form as the ones in the pure δ-function interaction case considered in [8].

Concerning the separated boundary conditions (21) it turns out that when $h < 0$ the system has a $2^{N(N-1)/2}$-times degenerate bound state. For $N = 2$, the space part of the orthogonal basis (label $\pm$) in the doubly degenerate bound state subspace has the form, again in the relative coordinate $x = x_2 - x_1$,

$$\psi_{2,\pm} = (\theta(x) \pm \theta(-x))e^{h|x|}. \tag{29}$$

The eigenvalue corresponding to the bound states (29) is $-h^2$. By generalization we get the $2^{N(N-1)/2}$ bound states for the N-particle system

$$\psi_{N,\underline{\epsilon}} = \alpha_{\underline{\epsilon}} \prod_{k>l} (\theta(x_k - x_l) + \epsilon_{kl}\theta(x_l - x_k))e^{h\sum_{i>j}|x_i - x_j|}, \tag{30}$$

where $\alpha_{\underline{\epsilon}}$ is the spin wave function and $\underline{\epsilon} \equiv \{\epsilon_{kl} \ : \ k > l\}$; $\epsilon_{kl} = \pm$, labels the $2^{N(N-1)/2}$-fold degeneracy.

It can be checked that also $\psi_{N,\underline{\epsilon}}$ satisfies the boundary condition (21) at $x_i = x_j$ for any $i \neq j \in 1, ..., N$. The spin wave function α here satisfies $P^{ij}\alpha = \epsilon_{ij}\alpha$ for any $i \neq j$, that is, $p^{ij}\alpha = \epsilon_{ij}\alpha$ for bosons and $p^{ij}\alpha = -\epsilon_{ij}\alpha$ for fermions.

Again $\psi_{N,\underline{\epsilon}}$ is of the form (13) in each region. For instance comparing $\psi_{N,\underline{\epsilon}}$ with (13) in the region $x_1 < x_2 ... < x_N$ we get

$$k_1 = ih(N-1), \ k_2 = k_1 - 2ih, \ k_3 = k_2 - 2ih, ..., k_N = -k_1. \tag{31}$$

The energy of the bound state ψ_N is

$$E = -\frac{h^2}{3}N(N^2 - 1) \ . \tag{32}$$

Now we pass to the scattering matrix. For real $k_1 < k_2 < ...k_N$, in each coordinate region such as $x_1 < x_2 < ...x_N$, the following term in (13) is an outgoing wave

$$\psi_{out} = \alpha_{12...N}e^{k_1 x_1 + ... + k_N x_N}. \tag{33}$$

An incoming wave with the same exponential as (33) is given by

$$\psi_{in} = [P^{1N}P^{2(N-1)}...]\alpha_{N(N-1)...1}e^{k_N x_N + ... + k_1 x_1} \tag{34}$$

in the region $x_N < x_{N-1} < ... < x_1$. The scattering matrix is defined by $\psi_{out} = S\psi_{in}$. From (14) we have

$$\alpha_{12...N} = [Y_{21}^{12}Y_{31}^{23}...Y_{N1}^{(N-1)N}]\alpha_{2...N1} = ...$$

$$= [Y_{21}^{12}Y_{31}^{23}...Y_{N1}^{(N-1)N}][Y_{32}^{12}Y_{42}^{23}...Y_{N2}^{(N-2)(N-1)}]...[Y_{N(N-1)}^{12}]\alpha_{N(N-1)...1} \equiv S'\alpha_{N(N-1)...1},$$

where $Y^{ii+1}_{l_{i+1}l_i}$ is given by (23) or (24). Therefore

$$S = S'P^{N1}P^{(N-1)2}...P^{1N} = S'[P^{12}][P^{23}P^{12}][P^{34}P^{23}P^{12}]...[P^{(N-1)N}...P^{12}].$$

Defining

$$X_{ij} = Y^{ij}_{ij}P^{ij} \tag{35}$$

we obtain

$$S = [X_{21}X_{31}...X_{N1}][X_{32}X_{42}...X_{N2}]...[X_{N(N-1)}]. \tag{36}$$

The scattering matrix S is unitary and symmetric due to the time reversal invariance of the interactions. $< s_1's_2'...s_N'|S|s_1s_2...s_N >$ stands for the S matrix element of the process from state $(k_1s_1, k_2s_2, ..., k_Ns_N)$ to state $(k_1s_1', k_2s_2', ..., k_Ns_N')$.

The momenta (27) ($c > 0$) and (31) are imaginary for bound states. The scattering of clusters (bound states) can be discussed in a similar way as in [8]. For instance for the scattering of a bound state of two particles ($x_1 < x_2$) on a bound state of three particles ($x_3 < x_4 < x_5$), the scattering matrix is $S = [X_{32}X_{42}X_{52}][X_{31}X_{41}X_{51}]$. One thing should be noted here is that the space part of the wave function (26) is antisymmetric under interchange of any two particles, a situation which is different from the case of the pure δ-function interaction case. Therefore the spin part of the wave function is antisymmetric for bosons and symmetric for fermions.

An interesting question is if, and up to what extend, the two new families we found are really physically different from those already known due to Yang [7]. For that aim consider the unitary transformation $\mathcal{U}$ in $L^2(R)$ (a 'local kink-type gauge transformation') that consists of the multiplication by $\text{sgn}(x)$. Under this transformation a separated point interaction (1) with parameters θ, a, b, c, d goes to a separated point interaction with parameters $\theta, -a, -b, -c, -d$. In particular the δ interaction with arbitrary strength c goes to the anti-δ interaction and thus these two Hamiltonians are formally unitarily equivalent. The physical status of such a discontinuous gauge transformation of the 'kink' type should however be further clarified. On the other hand the nonseparated point interaction (3) remains invariant under $\mathcal{U}$. In fact its Hamiltonian can not be unitarily equivalent to the one for the δ or anti-δ case as seen by the fact that their spectra are different (there are two (degenerate) bound states for the interaction (3) !). We note however that a separated point interaction can be regarded as consisting of two independent subsystems, each of them having the same spectrum (taking h = c/2) as the δ or anti-δ, but living on different spaces (halfaxes instead of the whole R), which again excludes the unitary equivalence of the associated Hamiltonians.

Let us pass now to the many-body case, $N > 1$. It can be checked that under the "kink type" gauge transformation

$$\mathcal{U} = \prod_{i>j} \text{sgn}(x_i - x_j)$$

the N-boson (resp. fermion) delta type contact interaction goes over to the N-fermion (resp. boson) anti-δ interaction. In particular, in the region $x_1 < x_2 < \ldots < x_N$, the action of the unitary transformation $\mathcal{U}$ is just the identity, which corresponds to the fact that (22) exchanges with (23) if we simultaneously send $c \to -c$ and $P^{ii+1} \to -P^{ii+1}$. In other regions, it is the factors P^{ii+1} which guarantees the correct transformations rules (see eg. (6) for the case $N = 2$). We can thus affirm that for N-body systems there is a sort of duality [4] between bosons (resp. fermions) with δ interaction of strength c and fermions (resp. bosons) with anti-δ interaction of strength $-c$. Indeed the exchange of these two interactions type together with the exchange of statistics (boson and fermion) should give the "same physics". These two situations are in fact unitarily equivalent though we do not know what may be the physical significance of such a gauge transformation $\mathcal{U}$ which is non smooth and does not factorize through one particle Hilbert spaces.

We close with a remark that the equality of the binding energies of the bound states for anti-δ and δ has a simple consequence. If we take the particles to be fermions, the number of 'spin' states to be $n = 4$ and c to be 2 in suitable units, we have that the model has $SU(4)$ symmetry and the binding energies E can be compared with the binding energies

[4] After completing this work we became aware of [11], where another kind of duality is observed between the system of two bosons (resp. fermions) with delta-interaction and two-fermions (resp. bosons) with delta'-interaction. Note however that the N-body system with δ' interaction is not included in our list (19,20,21) of exactly solvable models.

of the ground states for light nuclei. For instance, taking $N = 3$ we get $E = 8$ which is the nuclear binding energy (in Mev) of He^3 and H^3 (as pointed out in [8]). We remark that also in our 'separated case' (32) by tuning the parameter h it is possible to get the above nuclear binding energies.

In our opinion an interesting future task is to include into considerations also the three-body interaction and to see wether there are any consequences of such an inclusion for the integrability of this quantum mechanical system.

ACKNOWLEDGEMENTS: We would like to thank P. Kulish, P. Kurasov and V. Rittenberg for helpful comments.

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, New York: Springer, 1988.

[2] P. Kurasov, J. Math. Analys. Appl. 201, 297-333(1996).
P. Chernoff and R. Hughes, J Func. Anal. 111, 97-117(1993).

[3] Yu.A Kuperin, K.A. Makarov and B.S. Pavlov, Teoret. Mat. Fiz. 63, 78-87(1985)

[4] I. Segal, Int. Ser. Numer. Math. 20, 54-61(1971).

[5] S. Albeverio, Z. Brzeźniak and L. Dąbrowski, J. Phys. A 27, 4933-4943(1994).

[6] S. Albeverio, L. Dąbrowski and P. Kurasov, Lett. Math. Phys. 45, 33-47(1998).

[7] C.N. Yang, Phys. Rev. Lett. 19, 1312(1967).
C.N. Yang, Phys. Rev. 168, 1920(1968).

[8] C.H. Gu and C.N. Yang, Commun. Math. Phys. 122, 105-116(1989).

[9] Z.Q. Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras*, World Scientific, 1993.

[10] S. Albeverio and R. Kurasov, *Singular perturbations of differential operators and solvable Schrödinger type operators*, Cambridge Univ. Press., to appear.
R. Kurasov, Rev. Math. Phys. 9, 853-906(1997).
V.S. Buslaev, S.P. Merkuriev and S.P. Salikov, Probl. Mat. Fiz., vup. 9, 14-30(1979); Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 84, 16-22(1979) (in Russian).

[11] T. Cheon and T. Shigehara, *Some aspects of generalized contact interaction in one-dimensional quantum mechanics*, to appear in Proc. Int. Conf. Math. Res. Quant. Mech. (QMath7), Prag, 1998.