THE CLASSICAL LIMIT IN NON-COMMUTATIVE GEOMETRY

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1 Introduction

In a previous paper [T] we presented a homotopy operator which microlocalizes the Hochschild homology. In this note we show how to microlocalize the cyclic homology. As a corollary of this procedure one obtains a localization of the Chern character [C], and, therefore, a local index theorem. Our procedure allows one, at the same time, to compute the cyclic homology of a potentially large class of algebras of functions. Here we are going to exemplify the procedure onto the algebra of smooth functions.

As for the Hochschild homology, see [T], the localization of the cyclic homology is done in three stages. The first step localizes the cyclic homology to *germs* of functions at the diagonal. The second step microlocalizes the cyclic chains into the *diagonal cyclic complex* consisting of *cyclic* ∞*-jets* along the diagonal. The third step consists of computing the homology of the cyclic diagonal complex of infinite jets.

Transposing the localization of the cyclic cocycles, one obtains a local Chern character for bounded Hilbert modules on smooth manifolds, whose convergence is robust, compare Connes [C], Theorem 5 and Connes-Moscovici [CM], Theorem 4.6.

Our procedure is based on an explicite homotopy operator, depending on a positive parameter $\epsilon > 0$ which deforms the cyclic chains onto chains which live on a ϵ-tubular neighborhood of the main diagonal in the various powers of the manifold. The parameter ϵ could be thought as the Planck's constant. The final homotopy operator is obtained by making the parameter ϵ converge to 0. Its existence is reduced to a converging problem involving the regularity properties of functions.

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2 General Definitions.

Recall the the *bar complex* of the associative algebra A $\{B_*(A), b'\}$ is given by $B_k = \otimes^k_{\mathbf{R}} A$, $0 \le k$, and

$$b'_k : B_k(A) \longrightarrow B_{k-1}(A)$$

$$b'_k = \sum_{i=1}^{k-1} (-1)^{i-1} b_{k,i}$$

is expressed in terms of the *face operators*

$$b_{k,i}(f_1 \otimes f_2 \otimes .... \otimes f_k) = f_1 \otimes f_2 \otimes ... \otimes f_i f_{i+1} \otimes ..... \otimes f_k,$$

for $1 \le i \le k-1$.

One defines also

$$b_{k,k}(f_1 \otimes f_2 \otimes .... \otimes f_k) = f_k f_1 \otimes f_2 \otimes .... \otimes f_{k-1}.$$

The Hochschild complex of the algebra A with coefficients in A, $\{C_*(A), b\}$ is given by

$$C_k(A) = B_{k+1}(A)$$

and

$$b_k = b'_{k+1} + (-1)^k b_{k+1,k+1}.$$

Recall that the homology of this complex, $H_*(A)$, is by definition the Hochschild homology of the algebra A. If A is a locally convex algebra, one considers the projective tensor products instead of the algebraic tensor products; this leads to the continuous Hochschild homology. If A is the algebra of smooth functions on the manifold M, then the continuous Hochschild chains of degree k will consist of smooth functions on M_{k+1}, the $(k+1)^{th}$-power of M.

Let $T_k : B_k \longrightarrow B_k$ be the *cyclic* morphism

$$T_k(f_1 \otimes f_2 \otimes .... \otimes f_k) = (-1)^{k-1} f_k \otimes f_1 \otimes f_2 \otimes .... \otimes f_{k-2} \otimes f_{k-1}.$$

The *cyclic complex* of the algebra A, $\{C_*^\lambda(A), b'\}$ is the subcomplex of the bar complex $B_*(A)$ consisting of the $T-$invariant elements

$$C_k^\lambda(A) = Ker(1 - T_k). \tag{2.1}$$

Its homology is the *cyclic homology* of the algebra A, $H_*^\lambda(A)$.

Let N_k be the T-symmetryzing operator

$$N_k = 1 + T_k + T_k^2 + T_k^3 + ... + T_k^{k-1}.$$

These operators satisfy the following identities, see e.g. Loday [L], p. 52,

$$(1 - T)b' = b(1 - T), \qquad b'N = Nb. \tag{2.2}$$

If the algebra A contains the rationals, then the sequence

$$... \xrightarrow{1-T} B_k \xrightarrow{N} B_k \xrightarrow{1-T} B_k \xrightarrow{N} B_k ....$$

is exact. Therefore, one has a 2-periodic long exact sequence of complexes.

From these, it follows also that the morphisms

$$\pi \circ \iota : \{Ker(1 - T), b'\} \longrightarrow \{Coker(1 - T), b\}, \tag{2.3}$$

(where π is the canonical projection and ι is the inclusion), define an isomorphism of complexes. In the literature this second complex is used as the definition of the cyclic complex. One sees easily that

$$(\pi \circ \iota)_k^{-1} = \frac{1}{k} N_k. \tag{2.4}$$

In the sequel we will first work with the definition (2.1) of the cyclic complex; then we will use the isomorphisms (2.3), (2.4) to transfer all homomorphisms into the second cyclic complex.

3 Localization of cyclic homology at the level of germs.

From now on A denotes the algebra of smooth functions on an arbitrary paracompact smooth manifold M.

Let d be a Riemannian distance function $d : M \times M \longrightarrow [0, \infty)$ on M.

Choose a decreasing smooth function

$$\lambda : [0, \infty) \longrightarrow [0, 1]$$

having support in the interval $[0, 1]$, and which is identically 1 on the interval $[0, 1/2]$.

Let λ_t denote the rescaled function λ

$$\lambda_t(x) = \lambda(\frac{x}{t}), \quad for \quad 0 < t.$$

Let $E_t : B_b \longrightarrow B_{k+1}$ be the *extension* operator

$$(E_t f_k)(x_1, x_2, ...x_k, x_{k+1}) = (-1)^{k-1} f_k(x_1, x_2, ...x_k) \lambda_t(d^2(x_k, x_{k+1})).$$

This operator makes sense in the projective tensor product completion of the spaces $C_k(A)$, which we convine to denote with the same symbol.

By a direct computation one verifies that

$$[(bE_t + E_t b^{'}) f_k](x_1, x_2, ...x_k) = (1 - \lambda_t(d^2(x_k, x_1)).f_k(x_1, x_2, ...x_k) \qquad (3.5)$$

Define

$$Q_t = NE_t.$$

For any $f_k \in Ker(1 - T_k)$, the above identity along with (2.2) show that

$$[(b^{'} Q_t + Q_t b^{'}) f_k](x_1, x_2, ...x_k) =$$

$$[N(bE_t + E_t b^{'}) f_k](x_1, x_2, ...x_k) =$$

$$\mathcal{L}_t(x_1, x_2, ...x_k).f_k(x_1, x_2, ...x_k),$$

where

$$\mathcal{L}_t(x_1, x_2, ...x_k) := \sum_{r=1}^{r=k} (1 - \lambda_t(d^2(x_r, x_{r+1})), \qquad (x_{k+1} := x_1).$$

The continuous function

$$\rho(x_1, x_2, ..., x_k) = Max_{1\leq r\leq k}\{d^2(x_r, x_{r+1})\}$$

is used, exclusively, to measure the square of the distance to the diagonal. In fact, we define the *s-tubular neighbourhood of the diagonal* by

$$U_s := \{(x_1, x_2, ...x_k) \mid \rho(x_1, x_2, ...x_k) < s\}.$$

The function $\mathcal{L}_t$ vanishes only on the $\frac{t}{2}$-neighbourhood of the diagonal.

On the other hand, if the function f_k vanishes on the t-neighbourhood of the diagonal, then

$$\mathcal{F}_t(f_k)(x_1, x_2, ..., x_k) := \mathcal{L}_t^{-1}(x_1, x_2, ...x_k).f_k(x_1, x_2, ..., x_k)$$

is a smooth function. Therefore, the operator $\mathcal{F}_t$ is a multiplier on B_k^t, where

$$B_k^t := \{f_k \mid f_k(U_t) = 0\}.$$

Define

$$\mathcal{K}_t = NE_t\mathcal{F}_t : B_k^t \longrightarrow B_{k+1}^t. \quad (3.6)$$

Theorem 3.1 *The operator $\mathcal{K}_t$ is a homotopy between the identity of $B_k^t \cap C_k^\lambda$ and the zero homomorphism*

$$(\mathcal{K}_t b' + b'\mathcal{K}_t)f_k = f_k. \quad (3.7)$$

Proof. The theorem follows from (3.3) along with the remark that the operator $\mathcal{F}_t$ commutes with the operator b'.

4 Microlocalization of cyclic homology.

4.1

Let

$$j_\infty(f_k) = \infty - jet\ of\ f_k\ along\ the\ diagonal$$

and let

$$J_\infty^k = \{j_\infty(f_k) \mid f_k \in C_k^\lambda.\}$$

One has the exact sequence of complexes

$$0 \longrightarrow J_k^0 \longrightarrow C_k^\lambda \xrightarrow{j_\infty} J_{k,\infty} \longrightarrow 0, \quad (4.8)$$

where J_k^0 is the kernel of j_∞.

Theorem 4.1 *The complex* $\{J_k^0, b'\}$ *is acyclic.*

Proof. Let

$$\Phi_t(x_1, x_2, ..., x_k) := \prod_{r=1}^{r=k} \lambda_{2t}(d^2(x_r, x_{r+1})). \tag{4.9}$$

This function has the properties

1. *Supp* $\Phi_t \subset U_{2t}$
2. $\Phi_t(U_t) = 1$
3. $\frac{d}{dt}\Phi_t \mid U_t = 0.$

Therefore, it is lecit to apply the homotopy operator $\mathcal{K}_t$ onto the function

$$(\frac{d}{dt}\Phi_t).f_k$$

to get

$$(\mathcal{K}_t b' + b'\mathcal{K}_t)((\frac{d}{dt}\Phi_t).f_k) = (\frac{d}{dt}\Phi_t).f_k.$$

This leads by integration to

$$\int_\epsilon^1 (\mathcal{K}_t b' + b'\mathcal{K}_t)(\frac{d}{dt}\Phi_t).f_k.dt = \Phi_1.f_k - \Phi_\epsilon.f_k,$$

or,

$$(\mathcal{H}'_\epsilon b' + b'\mathcal{H}'_\epsilon)f_k = \Phi_1.f_k - \Phi_\epsilon.f_k, \tag{4.10}$$

where

$$\mathcal{H}'_\epsilon(f_k) = \int_\epsilon^1 \mathcal{K}_t(\frac{d}{dt}\Phi_t)f_k dt.$$

If $f_k \in J_k^0$, then

$$\mathcal{H}'(f_k) := lim_{\epsilon\searrow 0}\ \mathcal{H}'_\epsilon(f_k)$$

exists, and

$$lim_{\epsilon\searrow 0}\ \Phi_\epsilon(f_k) = 0;$$

therefore,

$$(\mathcal{H}'b' + b'\mathcal{H}')f_k = \Phi_1.f_k, \tag{4.11}$$

which completes the proof of the theorem.

Corollary 4.2 *The induced morphisms*

$$(j_\infty)_{*,k} : H_k^\lambda \longrightarrow H_k(J_{*,\infty}), \qquad k = 0, 1, 2, ... \tag{4.12}$$

are isomorphisms.

4.2

We conjugate the homotopy $\mathcal{H}'_\epsilon$ and the chain homomorphism Φ_ϵ with the chain isomorphism (2.3); the resulting homomorphisms wil be denoted, respectively, by $\mathcal{H}_\epsilon$ and Ψ_ϵ.

For any $F \in C_k(A)$, let $[F] \in Coker(1 - T)$ denote its class. Then

$$\Psi_\epsilon([F]) = \frac{1}{k+1}[\Phi_\epsilon N_{k+1} F]$$

while

$$\mathcal{H}_\epsilon([F]) = \frac{1}{k+1}[\mathcal{H}'_\epsilon N_{k+1} F].$$

5 Localization of Cyclic Cohomology

The cyclic cohomology complex $\{C^*_\lambda(A), \delta\}$ is defined by

$$C^k_\lambda(A) := \{\phi | \phi \in Hom_{\mathbf{C}}(C_k(A), \mathbf{C}), \quad \phi = continuous, \quad \phi T_{k+1} = \phi\},$$

while δ is the transpose of b. Obviously, $\phi \in C^k_\lambda(A)$ iff $\phi \in Hom_{\mathbf{C}}(C_k(A), \mathbf{C})$ and ϕ factorizes to $Coker(1 - T_{k+1})$.

Let $\mathcal{H}^\epsilon$ be the homotopy $\mathcal{H}_\epsilon$ transposed

$$(\mathcal{H}^\epsilon \phi)([F]) := \phi(H_\epsilon([F])).$$

Then, for any $\phi \in C^k_\lambda(A)$ and $F \in C_k(A)$ one has the identity

$$((\mathcal{H}^\epsilon \delta + \delta \mathcal{H}^\epsilon)\phi)[F] = \phi(\Psi_1.[F] - \Psi_\epsilon.[F]) = \phi(\Phi_1.F - \Phi_\epsilon.F) \tag{5.13}$$

Corollary 5.1 *-i) For any cyclic chain ϕ of degree k and any $F \in C_k(A)$, one has*

$$((\mathcal{H}^\epsilon\delta + \delta\mathcal{H}^\epsilon)\phi)(F) = \phi \circ \Phi_1(F) - \phi \circ \Phi_\epsilon(F) \tag{5.14}$$

-ii) Any cocycle ϕ is cohomologous to

$$\phi \circ \prod_{r=0}^{r=k} \lambda_{2t}(d^2(x_r, x_{r+1})),$$

for any $\epsilon > 0$.

It is interesting to describe explicitely the homotopy operator. In fact, one has

$$(\mathcal{H}^\epsilon\phi)[F] = \phi(\frac{1}{k+1}[\mathcal{H}'_\epsilon N_{k+1}F]) =$$

$$\frac{1}{k+1}\phi(\int_\epsilon^1 [\mathcal{K}_t(\frac{d}{dt}\Phi_t)N_{k+1}F]dt) =$$

$$\frac{1}{k+1}\phi(\int_\epsilon^1 [N_{k+2}E_t(\sum_{r=0}^{r=k}(1-\lambda_t(d^2(x_r,x_{r+1})))^{-1}(\frac{d}{dt}\prod_{r=0}^{r=k}\lambda_{2t}(d^2(x_r,x_{r+1})).[N_{k+1}F]dt) =$$

$$\frac{k+2}{k+1}\phi(\int_\epsilon^1 E_t.\frac{\frac{d}{dt}\prod_{r=0}^{r=k}\lambda_{2t}(d^2(x_r,x_{r+1})}{\sum_{r=0}^{r=k} 1-\lambda_t(d^2(x_r,x_{r+1})}N_{k+1}F dt.$$

6 Local Index Theorem. The Classical Limit

6.1

Recall that the Chern character τ of the Hilbert module $\{A, \rho, H, \gamma, S\}$, due to Connes [C], is defined by the formula

$$\tau(f_0, f_1, ..., f_{2l}, f_{2l}) = Const_l.Tr\{\gamma S f_0[S, f_1][S, f_2].....[S, f_{2l}]\},$$

where $f_i \in A$. In this definition one makes the assumption that $[S, f]$ belongs to some Schatten class of compact operators, for any $f \in A$, such that the trace exists.

The basic property of τ is that it is a cyclic cocycle of the algebra A, which paired with any $e \in K^0(A)$ gives the index of the twisted operator S_e^+.

The Hilbert module $\{A, \rho, H, \gamma, S\}$ will be called *classical* Hilbert module on the closed (smooth) manifold M, if $A = C^\infty(M)$, $H = L_2(\xi)$, where ξ is a Z_2-graded complex vector bundle over M, with grading γ, and S is an elliptic pseudodifferential operator of order zero on H, $S\gamma = -\gamma S$, with $S^2 = 1$.

An important example of classical Hilbert module is provided by the signature operator. In this case, the manifold is of even dimension 2l, closed and oriented, $H = L_2(\wedge^l T^*(M))$ and $\gamma = i^l *$, where $*$ is the Hodge star operator derived from a smooth Riemannian metric on M. The operator S is $+Id$ on $Im\ d$ and the positive harmonic forms and it is $-Id$ on $\gamma Im\ d$ and the negative harmonic forms.

One may relax the regularity requirements on the manifold and the operator. For example, the signature operator makes sense on quasiconformal manifolds, see [CST], and still have a local index theorem.

Proposition 6.1 *The Chern character τ of any classical Hilbert module (not necessarily smooth) is cyclic cohomologous to the cocycle*

$$\tau_\epsilon := \tau \circ \prod_{r=0}^{r=2l+2} \lambda_{2\epsilon}(d^2(x_r, x_{r+1})),$$

for any $\epsilon > 0$.

6.2

It is concievable to expect that for geometrically defined smooth classical Hilbert modules, as for example, the signature operator, the localized Chern character τ_ϵ converges as a current, while $\epsilon \searrow 0$, to the Poincare' dual of $Ch(Symb\ S) \cup Td(T^*(M))$, up to a multiplicative constant, compare Connes [C], Theorem 5, p. 285 and Connes-Moscovici [CM], Theorem 4.6, p. 373. This is going to be discussed in more details elsewere.

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