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GROUP-THEORETICAL TECHNIQUES FOR THE CONSTRUCTION OF REGGE AND MULTI-REGGE MODELS

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A B S T R A C T

The Regge and multi-Regge models for high-energy scattering and production processes are treated in the framework of the Bali-Chew-Pignotti formalism. The compatibility of these models with the analytic properties of the amplitude is discussed and shown to require the existence of families of Regge trajectories. One of the formalisms which permit to build families of trajectories which satisfy all the analyticity and factorization requirements is discussed in its main aspects. It exploits the properties of the matrix elements of the representations of the Lorentz group and works for both the Regge and the multi-Regge models. The other methods, both analytic and group-theoretical, developed in order to solve the same problems are briefly discussed, together with their main results. A short review of the most interesting phenomenological applications of these results is given.

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1. INTRODUCTION

The Regge and multi-Regge models provide at present some of the most useful tools for the interpretation of the high energy scattering and production processes [1]. General treatments of these formalisms can be found in recent review papers [2)-5)]. Here we deal with some kinematical details of these models, i.e., with the behaviour of the Regge parameters at small momentum transfer. In the next Section we try to give an idea of the large amount of work which has been dedicated to this problem, especially in the last two years. Though many very different formalism have been used, the final results agree rather well. In the central part of this paper we necessarily concentrate our attention on one of the many possible points of view, i.e., the one which seems to permit a unified treatment of the Regge and the multi-Regge models.

In Sections 3.-5., we describe the multi-Regge formalism in the form introduced by Bali, Chew and Pignotti [6], with some formal modification useful for the following developments. The usual Regge pole model appears as a special case. In Section 6. we discuss the difficulties which arise when one takes into account some analytic properties of the amplitude. In Sections 7. and 8. we describe the general lines of a model developed by Cosenza, Sciarrino and Toller [7)-9)], in which these difficulties are eliminated. The simplest results for two-body reactions, which coincide with the results obtained by other authors, are described in Section 2. The results for the multi-Regge case have not yet been written in a sufficiently compact and useful form and therefore we do not give them here in detail. In Section 9. we give a short summary of the main phenomenological applications.

2. VARIOUS FORMALISMS AND SOME RESULTS

During the development of the Regge pole model for high energy scattering, it has been recognized [10)-18)] that it was necessary to refine the model in order to take into account the fundamental analytic properties of the amplitude. In particular, it has been realized that, when the interacting particles have non-vanishing spins [10)-12)] or when the mass of an outgoing particle is different from the mass of the corresponding incoming particle (unequal mass reactions) [13)-15)], a single Regge pole contribution has not the right analytic properties at vanishing momentum transfer $(t=0)$. In both cases it has been necessary to introduce "families" of Regge trajectories and to impose some relations between their trajectory functions and residues, in such a way that the unwanted singularities of the corresponding contributions to the amplitude cancel each other. The situation is known as "conspiracy" [16)].

The existence of families of Regge trajectories correlated at $t=0$ has also been suggested by Domokos and Suranyi [19)], on the ground of the extra symmetry [20)] which the Bethe-Salpeter or the multiperipheral [21)] equations acquire at vanishing four-momentum. In the case we are considering, these equations are symmetric with respect to the subgroup of the Lorentz group which contains the transformations which do not change the four-momentum transfer Q. If we consider forward elastic scattering, we have $Q=0$ and the symmetry group is the Lorentz group $O(3,1)$. It follows that the high energy behaviour of the amplitude is determined by the poles in the complex plane of a Casimir operator of this group. The contribution of one of these "Lorentz poles" can be decomposed into a family of Regge pole contributions. By means of a suitable analytic continuation, it is possible to use, instead of $O(3,1)$, the corresponding compact group $O(4)$. The connection between these two formalisms is discussed in Ref. 22).

In successive works [23)-26)], the $O(3,1)$ and the $O(4)$ expansions have been extended to elastic forward amplitudes involving particles with arbitrary spins, taking into account also the invariance with respect to parity and PCT. The families of Regge

trajectories obtained in this way have all the properties necessary to satisfy the analyticity requirements mentioned above, but these formalisms do not determine their contributions to unequal mass or non-forward scattering.

In these simple "Lorentz pole" models, the different kinds of families of conspiring Regge trajectories are classified according to the "Lorentz quantum numbers" which label the different irreducible representations of $O(3,1)$. This classification has been confirmed by all the following much more complete investigations and has a general validity. Using the notations of Ref. 24), the Lorentz quantum numbers are:

a) a complex parameter λ ;

b) a number M, which can assume the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$;

c) the Lorentz signature τ , which can assume two values such that

$$\tau^2 = (-1)^{2M} \quad ; \tag{2.1}$$

d) if $M = 0$, there is also the "Lorentz natural parity" σ , which can assume the two values ± 1.

Moreover, a Lorentz pole is characterized by the internal quantum numbers B (baryon number), Y (hypercharge), I (isotopic spin) and G (G parity, defined only if $B = Y = 0$). All the trajectories belonging to a family have the same internal quantum numbers as the corresponding Lorentz pole. They are all boson trajectories if M is integral and all fermion trajectories if M is half integral.

The trajectories belonging to a family are labelled by a number ν, which takes the values $0, 1, 2, \ldots$. If $M = 0$, there is one trajectory for each value of ν and all the trajectories have the same natural parity σ [*)] as the corresponding Lorentz pole.

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*) We call "natural parity" the product of parity and signature.

If $M>0$, there are two trajectories for each value of ν with different natural parity. In this case, σ is no more a Lorentz quantum number, but it is a parameter which labels the different trajectories of the family. The $t=0$ intercepts of the trajectory functions $\ell_{\nu\sigma}(t)$ are given by

$$\ell_{\nu\sigma}(0) = \lambda - \nu - 1 \tag{2.2}$$

and the corresponding signatures τ_ν are *)

$$\tau_\nu = \tau\,(-1)^\nu . \tag{2.3}$$

In the last two years a great effort has been made in order to understand the properties of the families of Regge trajectories for not necessarily vanishing momentum transfer and for arbitrary masses and spins of the interacting particles. A considerable part of this work [7)-9),26)-45)] makes use of some properties of the representations of the groups $O(3,1)$ or $O(4)$. Though a great variety of different points of view has been developed, we may try to individuate the following three kinds of papers:

a) papers based on the $O(4)$ expansion of the Bethe-Salpeter equation **) and on a perturbative treatment [26), 33)-37)] of the part of the kernel which is not $O(4)$ invariant;

b) papers based on the connection [see Refs. 18), 46)] between Regge pole contributions and Born terms in field theory [38)-40)];

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*) In order to obtain more compact formulae, we define the signature as i^{2j}, where j is the spin of a particle lying on the trajectory. Therefore, with our notations, signatures and parities of fermion trajectories are imaginary.

**) We do not deal here with the more detailed "dynamical" properties of the Regge trajectories which are suggested by the Bethe-Salpeter model.

c) papers based directly on the postulates of S matrix theory, Refs. 41)-45), 7)-9), i.e., essentially on

i) analyticity: the amplitude must have some analytic properties which we shall discuss later;

ii) factorization: all the Regge trajectories must have well defined quantum numbers and factorized residues. This is a consequence of unitarity [see Ref. 47), where reference to the original papers can be found]. For the reaction

$$(1)+(2)\rightarrow(3)+(4)\,, \tag{2.4}$$

the residue must have the form

$$\beta^{(1,3)}_{\lambda_1\lambda_3}(t)\;\beta^{(2,4)}_{\lambda_2\lambda_4}(t)\,, \tag{2.5}$$

where, for instance, the first factor is the same for all the reactions involving the particles (1) and (3). We have indicated by λ_i the t channel helicities.

The behaviour of the trajectory functions in a neighbourhood of $t=0$ has been investigated for the first time by Domokos and Surányi [26),33)] by means of a perturbative treatment of the Bethe-Salpeter equation. They found that the derivatives at $t=0$ of the trajectory functions were not independent. For instance, the following "mass formulae" were obtained

$$\ell_\nu(t)=\lambda-1-\nu+\left[A+B(\lambda-\nu-1)(\lambda-\nu)\right]t+$$

$$+O(t^2)\,,\quad M=0\,, \tag{2.6}$$

$$l_{\nu\sigma}(t) = \lambda - 1 - \nu + \sigma A(\lambda - \nu - \tfrac{1}{2})\, t^{\frac{1}{2}} +$$

$$+\left[B + C(\lambda-\nu-1)(\lambda-\nu) + A^2(\lambda-\nu-\tfrac{1}{2})\right] t + O(t^{\frac{3}{2}}), \quad M = \tfrac{1}{2}, \tag{2.7}$$

$$l_{\nu\sigma}(t) = \lambda - 1 - \nu + \left[A + (B+\sigma C)(\lambda-\nu-1)(\lambda-\nu)\right] t +$$

$$+ O(t^2), \quad M = 1, \tag{2.8}$$

In general, for $M > 0$ we have *)

$$l_{\nu,+1}(t) - l_{\nu,-1}(t) = O(t^M). \tag{2.9}$$

It is rather difficult to obtain sufficiently general information on the Regge residues from the Bethe-Salpeter model. Important results about the residues have been obtained from the models belonging to the classes b) and c) described above. The models based on field theoretical propagators give perhaps the most powerful results [39]. However, in the following we deal only with models based on S matrix postulates, which present the advantage of being based on a set of clearly stated and well justified assumptions and can therefore be treated in a mathematically rigorous way.

Unfortunately, the presently available models do not give a complete description of all the possible choices of residues and trajectory functions which satisfy the conditions stated above. For instance, the models developed in Refs. 7)-9), 41)-43) describe only families of parallel trajectories given by

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*) P. Surányi, private communication.

$$\ell_{\nu\sigma}(t) = \ell_0(t) - \nu = \lambda(t) - \nu - 1 \,. \tag{2.10}$$

The Regge residues are described by formulae which have the following general structure

$$\beta^{(1,3)\,\nu\sigma}_{\lambda_1\lambda_3}(t) = \sum_{\alpha} \Gamma^{\nu\sigma}_{\lambda_1\lambda_3\alpha}\left(t, \ell_0(t)\right) \hat{\gamma}^{(1,3)}_{\alpha}(t), \tag{2.11}$$

where the functions Γ are known kinematical factors which depend on the Lorentz quantum numbers, on the masses, spins and parities of the particles (1) and (3), on the momentum transfer t and on the pole position $\ell_0(t)$. The functions $\hat{\gamma}_\alpha^{(1,3)}(t)$, are arbitrary independent functions analytic apart from "dynamical" singularities. We call them "reduced Lorentz residues".

From the equations of the form (2.11) one can obtain the $t = 0$ behaviour of the various residue functions. For instance, in the unequal mass case the residues which do not vanish identically have the behaviour [7]

$$\beta^{(1,3)\,\nu\sigma}_{\lambda_1\lambda_3}(t) \sim t^{\frac{1}{2}\left[-\lambda(t)+1+|M-|\lambda_1-\lambda_3||\right]}, \quad M_1 \neq M_3 \,. \tag{2.12}$$

For the equal mass vertices which have some practical interest, i.e., those involving nucleons and spinless mesons, one gets the results described in the Table. A similar table for arbitrary spins can be found in Ref. 7). The parametrizations of the type (2.11) imply also some constraints between different residues and between their derivatives at $t = 0$. It is not yet clear whether these parametrizations provide all the possible sets of residue functions consistent with parallel trajectories, but, as we shall see, there is some reason to be optimistic.

One of the most interesting features of the Lorentz pole models with parallel trajectories is the possibility of writing the sum of the Regge pole contributions in compact form in terms of the Lorentz residues and of the matrix elements of the representations of $O(3,1)$. This is important, because in the unequal mass case near the forward direction the non-leading trajectories give appreciable contributions even at high energy. From this compact formula, one can derive the following property of the s channel helicity amplitudes in high energy exactly forward scattering [7]

$$f^{s}_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,\theta_s=0) = \delta_{\lambda_1-\lambda_3,\lambda_2-\lambda_4}\, O\left(s^{\lambda(0)-1-|M-|\lambda_1-\lambda_3||}\right). \tag{2.13}$$

We see that, as pointed out by Sawyer [31], M is just the absolute value of the s channel helicity flip which dominates at high energy in forward scattering.

Very recently, Lorentz pole models with non-parallel trajectories have been constructed [44),45]. At their present stage, these models deal only with zero spin particles and with families with $M=0$; however, they can be probably extended to more general reactions. In these models, the Regge trajectory functions are given by a set of implicit equations which automatically take into account the conditions (2.6) and also conditions on higher derivatives. The Regge residues are still given by a formula similar to Eq. (2.11); however, the kinematic factors Γ in this more general case contain all the trajectory functions and their derivatives.

There is another large class of papers [47)-73)] in which some properties of the Regge parameters near $t=0$ are derived, starting from analyticity and factorization. These works make use of purely analytic techniques, i.e., no use is made of the properties of the representations of $O(3,1)$ or $O(4)$. In most cases, they start from the analytic properties of the helicity amplitudes which have been investigated in great detail in the recent years [74)-84)]. General discussions of the principles of this method can be found in Refs. 16),47).

The main difference between the group theoretical models and the analytic techniques consists in the different way of presenting the results. The group theoretical models provide parametrizations of the residues [see Eq. (2.11)] which automatically satisfy all the analyticity constraints. The analytic techniques permit to write explicitly a set of necessary constraints. In other words, the first method gives "sufficient" conditions, while the second method gives "necessary" conditions. It is clear that a comparison between the results of the two techniques permits to control their efficiency.

In general, there is a good agreement between the results of group theoretical and analytic techniques. The simplest papers using the analytic method [48)-52)] deal only with the leading trajectories; they show the necessity of introducing the quantum number M which can be defined independently of any group theoretical consideration [85)]. Also the necessity of parity doublets for $M > 0$, the property (2.9) and the behaviour of the residues of the leading trajectories can be obtained in this way.

Some more refined papers [58)-63)] deal also with non-leading trajectories. They find that the classification and the composition of the families of trajectories coincide necessarily with the situation suggested by group theory and described above. The leading power of t contained in the residues is found to be in accord with Eq. (2.12) and the Table. The most recent and refined works [64)-73)], in which analyticity and factorization are exploited completely, studying at the same time reactions with different external masses and with different spins, have given rather complete results. For instance, the mass formulae (2.6)-(2.8) have been rederived in this way. Also the connections between the leading terms of the residues of the different Regge trajectories have been obtained and shown to be in accord with the group theoretical models.

The most general results have been obtained by Weis [65)], who has considered arbitrary values of M, general spins and masses. By means of an elegant use of the crossing relations, he has been able

to find the most general form of the leading terms in the expansion in powers of t of the residues and of the trajectory functions, and also to obtain important informations on the non-leading terms. He has also shown that in the Lorentz pole models of Refs. 7), 41) the leading terms of the residues have just the most general form. There is no indication against the assumption that also the non-leading terms are the most general, if one assumes parallel trajectories.

It seems reasonable to believe that the parallel trajectory approximation is a good one for small momentum transfers. Therefore, we may conclude that the Lorentz pole models with parallel trajectories can be used, after some simplification, in the phenomenology of high energy nearly forward (or backward) reactions.

Another direction of development of the Regge pole idea has been its application to the high energy production processes, which has given rise to the multi-Regge models [5),6),86)-103)]. It is certain that also in this case analyticity and factorization give rise to strong constraints on the residue and trajectory functions which appear in the model *). However, the investigation of this problem has been initiated only very recently. Some study on the crossing relations has been carried out [96)-98)] and some properties of the residues have been found by means of field theoretical models [99)-103)].

A promising approach is the extension to the multi-particle reaction of the Lorentz pole models with parallel trajectories. This extension has been completely carried out [8),9)] but some further work is necessary in order to write the results in a sufficiently

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*) It is clear that the trajectory functions in a multi-Regge model have to satisfy the constraints obtained in the simple Regge model. It seems very improbable that the analyticity of the multi-particle amplitudes gives rise to new constraints on the trajectories.

simple and useful form. Nevertheless, we think that it is useful to describe a Lorentz pole model directly in its multi-particle form. This does not imply much greater difficulties and, looking at the problems in their most general form, sometimes helps in understanding their real structure.

3. THE SCATTERING AMPLITUDE

We consider the following scattering (or production) process involving n particles with non-vanishing masses *)

$$(1)+(2)\rightarrow(3c)+(4c)+\dots+(nc)\,, \qquad n\geqslant 4\,, \tag{3.1}$$

where (ic) indicates the antiparticle corresponding to the particle (i). It is very convenient to describe the connected part of the corresponding scattering amplitude by means of the function

$$M'_{m_1\dots m_n}(a_1,a_2;a_3,\dots,a_n)\,, \tag{3.2}$$

where the arguments a_i are elements of the homogeneous Lorentz group or, more exactly, of the corresponding universal covering group [7].

The Lorentz transformation a_i transforms quantities measured in one of the rest systems of the particle (i) into quantities measured in the "laboratory system" $\mathcal{L}$. For instance, the laboratory four-momentum of the particle (i) is given by

$$P^{(i)}=L(a_i)\,Q^{(i)}\,, \tag{3.3}$$

where $L(a_i)$ is the Lorentz 4×4 matrix and

$$Q^{(i)}=(M_i,0,0,0)\,, \qquad i=1,\dots,n\,. \tag{3.4}$$

In other words, the argument a_i individuates not only the four-momentum of the particle (i), but also a well defined rest system. The index m_i represents the z component of the spin of the particle (i) measured in the rest system defined by the element a_i **).

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*) The discussion of Ref. 8) holds if at least one of the particles has a non-vanishing mass.

**) This formalism for the description of the spin of a relativistic particle has been proposed by Moussa and Stora [104].

In order to obtain a more compact formalism, we write the amplitude (3.2) in the new form

$$(-1)^{j_3-m_3}\cdot\ldots\cdot(-1)^{j_n-m_n}\cdot$$

$$\cdot M_{m_1,m_2,-m_3,\ldots,-m_n}(a_1,a_2,a_3 t,\ldots,a_n t), \tag{3.5}$$

where t represents the Lorentz transformation which inverts all the space-time co-ordinates and j_i is the spin of the particle (i). We assume that the function M (for different values of its arguments) describes all the reactions which can be obtained from the reaction (3.1) by means of the substitution rule. An outgoing particle is characterized by the fact that the corresponding argument of the M function contains time inversion. The function M is more useful for general theoretical considerations; when one treats in detail a given reaction, it is more convenient to use the function M' [see Eq. (3.2)], which has, as we have said, a direct simple physical meaning.

For given values of the four-momenta, the rest systems of the particles can be chosen in many different ways. This ambiguity is taken into account by n covariance conditions which the function M has to satisfy. They have the form

$$M_{m_1\ldots m_i\ldots m_n}(a_1,\ldots,a_i h,\ldots,a_n)=\sum_{m'_i} R^{\pi_i j_i}_{m'_i m_i}(h)\cdot$$

$$\cdot M_{m_1\ldots m'_i\ldots m_n}(a_1,\ldots,a_i,\ldots,a_n)\,,\quad h\in H_+\,, \tag{3.6}$$

where H_+ is the rotation group (containing space inversion) and $R^{\pi j}_{mm'}(h)$ is the spin rotation matrix corresponding to spin j and parity π.

The function M is defined only for values of the arguments which satisfy the four-momentum conservation

$$\sum_{i=1}^{n} L(a_i)\, Q^{(i)} = 0 . \tag{3.7}$$

Moreover, it satisfies the Lorentz invariance condition

$$M_{m_1 \dots m_n}(a a_1, \dots, a a_n) = M_{m_1 \dots m_n}(a_1, \dots, a_n), \tag{3.8}$$

where a is an arbitrary Lorentz transformation.

As shown in Ref. 8), our function M is strictly connected with the more familiar spinor M function [105] which depends on the four-momenta of the interacting particles.

4. THE GROUP THEORETICAL VARIABLES

The function M described above is not directly useful for the construction of a multi-Regge model. In order to write simple multi-Regge formulae, the amplitude has to be written as a function of a suitable set of variables, which have been introduced by Bali, Chew and Pignotti (BCP) [6] (see also [97]). In this Section we introduce these variables in a form which is suitable for the following developments.

As it is well known, the same amplitude can be described by means of several different multi-Regge models, which are useful in different regions of the space of the outgoing momenta. Each model can be associated to a given "coupling scheme", which is a simply connected (tree-like) graph whose external lines represent the particles involved in the process we are considering *) (see Fig. 1). We assume that all the vertices are composed by three lines (BCP consider a more general situation).

We label the external lines by means of the integers $1,2,\ldots,n$ and the internal lines by means of the integers $(n+1),\ldots,(2n-3)$. For each internal line we choose a standard orientation. When we want to indicate an internal line with the orientation opposite to the standard one, we put a bar over its symbol. The vertices of a coupling scheme are indicated by means of the symbols of the three corresponding incoming lines (some of these symbols may contain a bar). For instance, the coupling scheme represented in Fig. 1 contains the vertices $(13\bar{6})$, $(46\bar{7})$ and (257).

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*) As explained in Ref. 106), a coupling scheme describes a possible way of building an invariant (under the Poincaré group), starting from the Hilbert space vectors which describe the states of the particles involved in the reaction considered. For a detailed treatment of the four-particle case, see Refs. 107)-111).

For each external line we consider an incoming four-momentum $P^{(i)}$ $(i \leqslant n)$. Also for the internal lines we define the four-momenta $P^{(i)}$ $(i > n)$, in such a way that four-momentum is conserved at each vertex. For instance, for the vertex $(13\bar{6})$ we have

$$P^{(1)} + P^{(3)} + P^{(\bar{6})} = 0 . \tag{4.1}$$

We are using the convention

$$P^{(\bar{i})} = -P^{(i)} . \tag{4.2}$$

These equations can be used to express the four-momenta $P^{(i)}$ as sums of external four-momenta.

We define also the variables

$$W_i = (P^{(i)})^2 , \quad i = 1, \cdots, (2n-3) . \tag{4.3}$$

Of course, we have

$$W_i = M_i^2 , \quad i = 1, \ldots, n . \tag{4.4}$$

Moreover, for each value of W_i we choose a four-vector $Q^{(i)}$ with the property

$$(Q^{(i)})^2 = W_i , \tag{4.5}$$

for $i \leqslant n$, the choice is given by Eq. (3.4). For $i > n$ we could take

$$\begin{cases} Q^{(i)} = (W_i^{\frac{1}{2}}, 0, 0, 0) , & W_i \geqslant 0 , \\ Q^{(i)} = (0, 0, 0, (-W_i)^{\frac{1}{2}}) , & W_i \leq 0 , \end{cases} \tag{4.6}$$

but in the following we shall use also a different choice. We define the little groups $H^{(i)}$ containing all the Lorentz transformations h with the property

$$L(h)\,Q^{(i)} = Q^{(i)}. \tag{4.7}$$

In general this group depends on W_i.

Now we consider many systems of reference. One of them is the "laboratory system" $\mathcal{S}$, in which the four-momenta of the particles are given by Eq. (3.3). The other systems of reference are associated with the vertices of the coupling scheme. For instance, in the system $\mathcal{S}(46\bar{7})$, which corresponds to the vertex $(46\bar{7})$, the relevant four-momenta have the standard forms

$$\begin{cases} P^{(4)} = L(b_4)\,Q^{(4)}, \\ P^{(6)} = L(b_6)\,Q^{(6)}, \\ P^{(\bar{7})} = L(b_{\bar{7}})\,Q^{(7)}. \end{cases} \tag{4.8}$$

The boosts b_4, b_6, $b_{\bar{7}}$ are chosen once for all, according to suitable conventions. Each of them depends only on the three variables W_4, W_6, W_7. They must satisfy the condition

$$L(b_4)Q^{(4)} + L(b_6)Q^{(6)} + L(b_{\bar{7}})Q^{(7)} = 0. \tag{4.9}$$

Let i denote an internal line going from the vertex $(\bar{i}\ldots)$ to the vertex $(i\ldots)$. We call $\hat{h}_i$ the Lorentz transformation which transforms quantities measured in the system $\mathcal{S}_{(\bar{i}\ldots)}$ into quantities measured in the system $\mathcal{S}_{(i\ldots)}$. According to our convention on the orientation of the internal lines, we put

$$\hat{h}_{\bar{i}} = \hat{h}_i^{-1}. \tag{4.10}$$

For instance, the four-vector $P^{(6)} = -P^{(\bar{6})}$, which in the system $\mathcal{S}(13\bar{6})$ (see Fig. 1) has the form

$$-L(b_{\bar{6}})\,Q^{(6)}\,, \tag{4.11}$$

in the system $\mathcal{S}_{(46\bar{7})}$ takes the form

$$-L(\hat{h}_6 b_{\bar{6}})\,Q^{(6)}. \tag{4.12}$$

Comparing with Eq. (4.8), we see that

$$L(b_6)\,Q^{(6)} = -L(\hat{h}_6 b_{\bar{6}})\,Q^{(6)}, \tag{4.13}$$

and therefore

$$h_6 = t\,b_6^{-1}\,\hat{h}_6\,b_{\bar{6}} \in H^{(6)}, \tag{4.14}$$

or, in general

$$\hat{h}_i = b_i\,t\,h_i\,b_{\bar{i}}^{-1},\quad h_i \in H^{(i)},\quad i > n. \tag{4.15}$$

Note that Eq. (4.10) can be written as

$$h_{\bar{i}} = t\,h_i^{-1}\,t. \tag{4.16}$$

Let now i denote an external line connected with the vertex $(i\ldots)$. We call $\hat{h}_i$ the Lorentz transformation which transforms quantities measured in the system $\mathcal{S}$ into quantities measured in the system $\mathcal{S}_{(i\ldots)}$. For instance, the four-vector $P^{(4)}$, which in the system $\mathcal{S}$ has the form $L(a_4)Q^{(4)}$, in the system $\mathcal{S}_{(46\bar{7})}$ is given by

$$L(\hat{h}_4 a_4)\,Q^{(4)}. \tag{4.17}$$

Comparing with Eq. (4.8) we obtain

$$L(b_4)\,Q^{(4)} = L(\hat{h}_4 a_4)\,Q^{(4)}, \tag{4.18}$$

and therefore

$$h_4 = t\,b_4^{-1}\,\hat{h}_4\,t\,a_4 \in H^{(4)} = H_+ \tag{4.19}$$

where the factors t have been introduced for convenience. In general we have

$$\hat{h}_i = b_i\,t\,h_i\,a_i^{-1}\,t, \quad h_i \in H_+, \quad i \leq n. \tag{4.20}$$

Now we consider a "continuous path" in the coupling scheme which connects two external lines and we assume that all the internal lines are orientated towards the last line of the path. As an example, we consider the path 1, 6, 7, 5 in Fig. 1. Then, from the group property of the Lorentz transformations, we have immediately:

$$\hat{h}_5 = \hat{h}_7\,\hat{h}_6\,\hat{h}_1 \tag{4.21}$$

and, using Eqs. (4.15) and (4.20), we obtain

$$a_5^{-1}a_1 = \left[h_5^{-1}\,t\,b_5^{-1}\right]\cdot\left[b_7\,t\,h_7\,b_7^{-1}\right]\cdot$$

$$\cdot\left[b_6\,t\,h_6\,b_6^{-1}\right]\cdot\left[b_1\,t\,h_1\right]. \tag{4.22}$$

It is a clear that in a similar way we can express all the ratios $a_r^{-1}a_s$ in terms of the group theoretical variables $\{W_i, h_i\}$. In Ref. 8), the set of equalities of the type (4.22) has been used as a definition of the group theoretical variables.

It is important to remark that the elements h_i are not uniquely determined by the boosts a_i. In fact, for instance, the

system $\mathcal{G}_{(46\bar{7})}$ is not uniquely determined by the conditions (4.8). Another system obtained from it by means of the Lorentz transformation k could work equally well provided that

$$\begin{cases} L(k\,b_4)\,Q^{(4)} = L(b_4)\,Q^{(4)}, \\ L(k\,b_6)\,Q^{(6)} = L(b_6)\,Q^{(6)}, \\ L(k\,b_{\bar{7}})\,Q^{(7)} = L(b_{\bar{7}})\,Q^{(7)}. \end{cases} \tag{4.23}$$

We call $K^{(46\bar{7})}$ the group containing all the Lorentz transformations which satisfy the conditions (4.23). Other similar "covariance groups" can be defined for the other vertices. It is clear that such a change of the reference system $\mathcal{G}_{(46\bar{7})}$ implies the following transformation

$$\begin{cases} \hat{h}_4 \to k\,\hat{h}_4, \\ \hat{h}_6 \to k\,\hat{h}_6, \\ \hat{h}_{\bar{7}} \to k\,\hat{h}_{\bar{7}}, \quad k \in K^{(46\bar{7})}, \end{cases} \tag{4.24}$$

or, using the variables h_i

$$\begin{cases} h_4 \to b_4^{-1}\,k'\,b_4\,h_4, \\ h_6 \to b_6^{-1}\,k'\,b_6\,h_6, \\ h_{\bar{7}} \to b_{\bar{7}}^{-1}\,k'\,b_{\bar{7}}\,h_{\bar{7}}, \quad k' = t\,k\,t \in K^{(46\bar{7})}. \end{cases} \tag{4.25}$$

A transformation of this kind does not influence the left-hand side of Eq. (4.22).

Now we may describe the amplitude by means of a function T of the new variables defined by

$$M_{m_1 \dots m_n}(a_1, \dots, a_n) =$$

$$= \sum_{m'_1 \dots m'_n} R^{\pi_1 j_1}_{m'_1 m_1}(h_1) \cdot \dots \cdot R^{\pi_n j_n}_{m'_n m_n}(h_n) \cdot$$

$$\cdot T_{m'_1 \dots m'_n}(W_{n+1}, \dots, W_{2n-3}, h_{n+1}, \dots, h_{2n-3}), \quad h_i \in H^{(i)}. \tag{4.26}$$

From Eq. (3.8) we see that the left-hand side depends only on the ratios $a_r^{-1} a_s$, which are given by Eq. (4.22). The dependence of the left-hand side on the elements h_i $(i \leqslant n)$ is completely determined by the conditions (3.6) and this fact has been taken into account in writing the right-hand side of Eq. (4.26).

If (irs) is a vertex, from the invariance of the elements a_i under the transformation (4.25), and from Eq. (4.26) we see that the function T does not change if we perform the following set of operations $(k \in K^{(irs)})$

a) if i is an internal line, perform the following substitution

$$h_i \to b_i^{-1} k b_i h_i, \quad i > n, \tag{4.27}$$

b) if i is an external line, perform the following linear transformation on the spin index m_i

$$T_{m_i} \to \sum_{m'_i} R^{\pi_i j_i}_{m'_i m_i}(b_i^{-1} k b_i) T_{m'_i}, \quad i \leq n, \tag{4.28}$$

c) perform the same operations for the lines r and s.

Similar covariance properties correspond to the other vertices. We call them the "vertex conditions". They are the only conditions imposed by Lorentz invariance on the function T.

5. THE MULTI-REGGE MODEL

Our main purpose is to describe Regge or multi-Regge models, which approximate the amplitude when all the group elements h_i $(i > n)$ go to infinity. This is possible (in the physical region) only when the groups $H^{(i)}$ $(i > n)$ are non-compact. For W_i positive, $H^{(i)}$ is isomorphic to the compact group $O(3)$. Therefore, we have to assume that all the variables W_i $(i > n)$ are negative *). In this case, all the groups $H^{(i)}$ $(i > n)$ are isomorphic to $O(2,1)$, which contains all the Lorentz transformations which do not operate on the z co-ordinate.

Now we have to choose the boosts b_i and $b_{\bar{i}}$ in a suitable way. We shall indicate by $u_z(\mu)$ a rotation of an angle μ around the z axis and by $a_z(\zeta)$ a pure Lorentz transformation with velocity $\tanh\zeta$ along the z axis. For the four-vectors $Q^{(i)}$ it is convenient to use the form (4.6). Then we have $H^{(i)} = O(2,1)$. Moreover, we may choose the boosts in the subgroup containing the Lorentz transformations which act only on the z and time co-ordinates. We consider only the two cases which are of interest for the Regge and multi-Regge models.

a) Vertices with two external lines, as the vertex $(13\bar{6})$ of Fig. 1. We put

$$\begin{cases} b_1 = a_z(\zeta_1), \\ b_3 = a_z(\zeta_3)\, u_z(\pi)\, t, \\ b_{\bar{6}} = e, \end{cases} \tag{5.1}$$

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*) We consider $W_i = 0$ as a limit point. The existence of a regular limit of this kind is the origin of the conspiracy conditions.

where e represents the unit and

$$\begin{cases} \sinh \zeta_1 = (M_1^2 - M_3^2 + W_6)\left[2 M_1 (-W_6)^{\frac{1}{2}}\right]^{-1}, \\ \sinh \zeta_3 = (M_1^2 - M_3^2 - W_6)\left[2 M_3 (-W_6)^{\frac{1}{2}}\right]^{-1}. \end{cases} \tag{5.2}$$

b) Vertices with two internal lines, as the vertex $(46\bar{7})$ of Fig. 1. We take

$$\begin{cases} b_4 = e, \\ b_6 = a_2(\zeta_6), \\ b_{\bar{7}} = a_2(\zeta_{\bar{7}})\, u_2(\pi)\, t, \end{cases} \tag{5.3}$$

where

$$\begin{cases} \sinh \zeta_6 = (W_7 - W_6 - M_4^2)\left[2 M_4 (-W_6)^{\frac{1}{2}}\right]^{-1}, \\ \sinh \zeta_{\bar{7}} = (W_7 - W_6 + M_4^2)\left[2 M_4 (-W_7)^{\frac{1}{2}}\right]^{-1}. \end{cases} \tag{5.4}$$

The function T defined by means of these conventions coincides (apart from some signs) with the amplitude introduced by BCP [6]. Following these authors, we define a multi-Regge term as a contribution to this function in which the dependence on the group elements h_i is given by the "functions of the second kind" on the group $O(2,1)$ *).

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*) Every matrix element of an irreducible representation of $O(2,1)$ can be decomposed into the sum of two functions of the second kind, which have simple asymptotic properties. For details, see Refs. 7), 107), 112)-114).

For the simple Regge pole mode, this group theoretical approach has been discussed and justified in Refs. 107)-109); the connection with the usual Regge pole formalism has been treated in Refs. 110), 111).

The rules for the construction of a multi-Regge term, with factorized residues are the following:

a) given a coupling scheme, for each vertex (irs) write the residue

$$\beta^{(irs)}_{m_i m_r m_s}(W_i, W_r, W_s) , \tag{5.5}$$

b) for each internal line r write the expression

$$I_{m_r m_{\bar{r}}}(h_r) = (2\ell_r(W_r)+1)\,\mathrm{cotg}\,[\pi(\ell_r(W_r)-m_r)]\cdot$$
$$\cdot \tilde{A}^{\tau_r^2 \sigma_r, \bar{\tau}_r, -\ell_r(W_r)-1}_{m_r, -m_{\bar{r}}}(h_r)\; i^{2m_{\bar{r}}} , \tag{5.6}$$

where $\ell_r(W_r)$ is the trajectory function, τ_r is the signature and σ_r is the natural parity. These parameters are just the quantum numbers which label the irreducible representations of $O(2,1)$. $\tilde{A}$ indicates one of the corresponding functions of the second kind;

c) for each internal fermion line which is not directed towards the external line 1, write a factor (-1);

d) sum over the indices m_r, $m_{\bar{r}}$, $r = (n+1),\ldots,(2n-3)$.

These rules do not take into account the "vertex conditions" given in the preceding Section. It is easy to see that, with the conventions we have chosen, all the covariance groups $K^{(irs)}$ coincide with the group K generated by the elements $u_z(\mu)$ and $s' = su_y(\pi)$, where s represents the inversion of the three space co-ordinates. Imposing on a multi-Regge term the vertex condition corresponding to a given vertex, and using the properties

$$\begin{cases} A^{\sigma\tau\ell}_{m m'}(u_z(\mu)\, h) = \exp(-i m \mu)\, A^{\sigma\tau\ell}_{m m'}(h), \\ A^{\sigma\tau\ell}_{m m'}(s' h) = \sigma (-i)^{2m} A^{\sigma\tau\ell}_{-m, m'}(h), \end{cases} \tag{5.7}$$

we see that the only effect is a set of constraints on the residue function [see Eq. (5.5)] corresponding to the same vertex. These constraints are

$$\beta^{(i\,2\,3)}_{m_i m_2 m_3}(W_i, W_2, W_3) = 0 \quad \text{if} \quad m_i + m_2 + m_3 \neq 0, \tag{5.8}$$

$$\beta^{(i\,2\,3)}_{-m_i, -m_2, -m_3}(W_i, W_2, W_3) = \sigma_i \sigma_2 \sigma_3\, \beta^{(i\,2\,3)}_{m_i m_2 m_3}(W_i, W_2, W_3), \tag{5.9}$$

where σ_i is the natural parity of the Regge pole if i is an internal line and

$$\sigma_i = \Pi_i\, i^{2 j_i} \tag{5.10}$$

if i is an external line.

6. THE ANALYTICITY CONDITIONS

After this introduction to the BCP formalism, we can meet our main problem, i.e., to investigate the analytic properties of a multi-Regge term. We have to remember that the basic analytic properties of the amplitude are expressed in terms of the spinor function [105] (which depends on the four-momenta of the external particles). Its singularities are called dynamical because they have a direct physical meaning. Therefore, we need the connection between the analytic properties of the spinor amplitude and of the function T introduced above. Of course, the function T has to be defined also for complex values of the arguments. Therefore, we introduce the complex Lorentz group, the complex little groups and the complex covariance groups. Now, the boosts b_i belong to the complex Lorentz group and are defined also for complex values of the variables W_i. All the preceding formulae can be extended to the complex case.

We shall use the following result which has been proved in Refs. 7), 8).

Proposition: If all the boosts b_i and $b_{\bar{i}}$ are entire functions of the variables W_i, then the function T is free of kinematic singularities and constraints. This means that, if D is a Lorentz invariant open subset of the space where the spinor function is defined and D' is the corresponding subset of the space where T is defined, there is a one to one correspondence between the spinor functions Lorentz covariant and analytic in D and the functions T which satisfy the vertex conditions and are analytic in D'.

This result cannot be directly applied to the function T of the preceding Section, because the boosts defined by Eqs. (5.1)-(5.4) contain singularities. In fact, the boosts (5.1) (vertex with two external lines) are singular for

$$\begin{cases} W_6 = 0, \\ W_6 = (M_1 \pm M_3)^2. \end{cases} \tag{6.1}$$

The points defined by the last equation are just thresholds and pseudo-thresholds. The boosts (5.3) (vertex with one external line) are singular for

$$\begin{cases} W_6 = 0, \\ W_7 = 0, \\ \pm W_6^{\frac{1}{2}} \pm W_7^{\frac{1}{2}} = M_4. \end{cases} \tag{6.2}$$

As shown in Fig. 2, the last equation (6.2) represents a parabola in the plane (W_6, W_7). We call it the "threshold parabola".

In order to use the proposition stated above, we have to define some new boosts $\tilde{b}_i$ and $\tilde{b}_{\bar{i}}$ which depend analytically on the variables W_i. From Eq. (4.9) we see that also the four-vector $Q^{(i)}$ $(i > n)$ should be an analytic function of W_i. Therefore, we have to modify also our choice (4.6) introducing the new four-vectors

$$\begin{cases} \tilde{Q}^{(i)} = \left(\frac{1}{2}(1+W_i), 0, 0, \frac{1}{2}(1-W_i)\right), \quad i > n, \\ \tilde{Q}^{(i)} = Q^{(i)}, \quad i \leq n. \end{cases} \tag{6.3}$$

It is possible to show that, if the equalities (6.1) or (6.2) do not hold, the connection between the old and the new boosts can be written as

$$\tilde{b}_i = a_{(irs)}\, b_i\, u_i^{-1} \tag{6.4}$$

where $a_{(irs)}$ is the same for all the boosts corresponding to the same vertex (irs). All the elements in Eq. (6.4) depend on W_i, W_r, W_s. Moreover, the elements u_i have the property

$$L(u_i)\, Q^{(i)} = \tilde{Q}^{(i)}. \tag{6.5}$$

By means of the new boosts, according to the general formalism, we define the function $\tilde{T}$, which is free of kinematic singularities and constraints. The connection between the functions T and $\tilde{T}$ has been given in Ref. 8) and is

$$\tilde{T}_{m_1 \dots m_n}(W_{n+1}, \dots, W_{2n-3}, \tilde{h}_{n+1}, \dots, \tilde{h}_{2n-3}) =$$

$$= \sum_{m'_1 \dots m'_n} R^{\pi_1 j_1}_{m'_1 m_1}(t u_1^{-1} t) \cdot \dots \cdot R^{\pi_n j_n}_{m'_n m_n}(t u_n^{-1} t) \cdot$$

$$\cdot T_{m'_1 \dots m'_n}(W_{n+1}, \dots, W_{2n-3}, h_{n+1}, \dots, h_{2n-3}), \tag{6.6}$$

where

$$h_i = t u_i^{-1} t \tilde{h}_i u_{\bar{i}} . \tag{6.7}$$

From Eq. (6.4) we see that at least some of the elements u_i have to be singular at the points described by Eqs. (6.1) or (6.2).

Using Eqs. (6.6) and (6.7) and the rules a)-d) given in Section 5, we may write down a multi-Regge contribution to the function $\tilde{T}$. It turns out that it has unwanted singularities due to the elements u_i which appear both as arguments of the matrices R of Eq. (6.6) and in the arguments (6.7) of the $O(2,1)$ functions $\tilde{A}$ which appear in Eq. (5.6). These are singularities in the variables W_i and one could hope to cancel them by means of a proper choice of the residues (5.5) *).

However, when one of the variables W_i is small, another kind of singularity becomes dangerous. These are singularities in

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*) This situation is called "evasion" 16).

the variables $\tilde{h}_i$ for fixed values of the variables W_i and therefore they cannot be avoided by means of a proper choice of the residue functions. These singularities are due to the fact that the functions $\tilde{A}(h)$ continued analytically in the complex little group are singular for values of the argument such that *)

$$L_{tt}(h) = \pm 1 \,. \tag{6.8}$$

Singularities of this kind are unavoidable in a Regge or multi-Regge contribution. However, as we are interested in the high energy behaviour, these singularities are harmless if they can be confined in a low energy region. Unfortunately, this is not the case. For instance, we have (see Fig. 1)

$$s_{34} = (p^{(3)} + p^{(4)})^2 = (Q^{(4)} - L(\tilde{b}_4^{-1}\tilde{b}_6\tilde{h}_6\tilde{b}_6^{-1}\tilde{b}_3)Q^{(3)})^2 =$$

$$= (Q^{(4)} - L(b_4^{-1}b_6h_6b_6^{-1}b_3)Q^{(3)})^2 \,, \tag{6.9}$$

and it is easy to show that if we take, for instance **), $h_6 = e$, we have in general

$$\lim_{W_6 \to 0} s_{34} = \infty \,, \tag{6.10}$$

so that the function $\tilde{A}(h_6)$ has singularities at arbitrarily large "subenergy" s_{34}.

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*) $L_{tt}(h)$ means the element of the 4×4 matrix $L(h)$ which connects time components to time components. In the usual Regge formalism, Eq. (6.8) means simply $\cos\theta_t = \pm 1$.

**) e represents the unit element. Clearly, $L_{tt}(e) = 1$.

This is just the difficulty which was treated for backward elastic scattering by Freedman and Wang [13] and has forced these authors to introduce families of Regge trajectories.

7. A LORENTZ POLE MODEL

The difficulty mentioned above was avoided by Freedman and Wang [13] introducing an auxiliary expansion in terms of Khuri poles [115], which does not present complications near $W = 0$. We are now considering a more general and difficult problem and, in order to reach some conclusion, we have to choose a more refined expansion, in terms of the matrix elements of the representations of the Lorentz group (Lorentz pole expansion). The advantages of this auxiliary expansion are the following:

a) a Lorentz pole contribution with factorized residues can always be decomposed into a family of Regge pole contributions with factorized residues [7),23),24)];

b) the treatment of the vertex conditions [see Eqs. (4.27), (4.28)] is facilitated by the group properties of the representation matrix elements.

If we assume that all the Lorentz poles have $M = 0$, a "multi-Lorentz" contribution to the function $\tilde{T}$ can be constructed by means of the following rules, which are perfectly analogous to the rules given above for the construction of a multi-Regge contribution to T. If some of the Lorentz poles have $M > 0$, some additional indices are required [9].

a) For every vertex (irs), write a Lorentz residue of the form

$$\gamma^{(irs)}_{j_i m_i j_r m_r j_s m_s}(W_i, W_r, W_s) . \tag{7.1}$$

If i is an external line, j_i simply means the spin of the corresponding particle.

b) For each internal line r, write the expression

$$\mathcal{J}_{j_r m_r j_{\bar r} m_{\bar r}}(\tilde h_r) = -[\lambda_r(W_r)]^2 \cdot$$

$$\cdot\, \tilde{\mathcal{A}}^{\sigma_r,\tau_r,0,-\lambda_r(W_r)}_{j_r,m_r,j_{\bar r},-m_{\bar r}}\left(c_r^{-1}\tilde h_r c_{\bar r}\right)\,(-1)^{j_{\bar r}-m_{\bar r}}\,,$$

(7.2)

where $\tilde{\mathcal{A}}$ represents the "function of the second kind" on the group $O(3,1)$ *). The function $\lambda_r(W_r)$ represents the "Lorentz trajectory". The element c_r depends analytically on W_r, W_i, W_s, where (irs) is the vertex which contains r. This element and the element $c_{\bar r}$, which has similar properties, have to be chosen in a suitable way, as we shall see later. τ_r and σ_r are Lorentz quantum numbers.

c) Write a factor (-1) for each fermionic internal line which is not directed towards the external line 1.

d) Sum over the indices j_r, m_r, $j_{\bar r}$, $m_{\bar r}$, $r=(n+1),\ldots,(2n-3)$.

We remark that in this way we obtain a function defined when the elements $\tilde h_i$ are arbitrary Lorentz transformations. However, this function has a physical meaning only when $\tilde h_i \in \tilde H^{(i)}$. As a consequence, we have that the Lorentz pole parametrization is not unique, i.e., different Lorentz residues could give rise to Lorentz pole contributions which coincide in the physically meaningful region.

We remark also that we have included in our rules a "factorization rule" for the Lorentz residues. This rule has no direct physical justification [44] similar to the one available for Regge residues [47]. In a non-factorized model, one should replace the

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*) Also the matrix elements of the irreducible representations of $O(3,1)$ can be decomposed into the sum of two functions of the second kind, which have simple asymptotic properties. For details, see Refs. 23), 24), 114), 116).

product of the residue functions (7.1) by a single function which depends on the same indices and parameters. In dealing with this factorization problem, one has to take into account the fact that in general the Lorentz pole parameters are not uniquely determined by the physical amplitude. The real problem is to know whether, between all the possible Lorentz pole parametrizations of a family of parallel Regge trajectories, there is one using a single Lorentz pole (one for each internal line) with well defined quantum numbers and factorized residues. The answer is yes for exactly forward elastic scattering [64)-73)] and a similar result seems probable in some more general cases. If the Regge trajectories are not parallel, it is clear that many "counterconspiring" Lorentz poles must exist in the λ plane, at least for $W_i \neq 0$.

Our program consists of building first a simple model satisfying all the necessary factorization and analyticity conditions and then eventually of generalizing it more and more. Therefore, we start by considering families of multi-Regge terms which can be described by a single multi-Lorentz contribution with factorized Lorentz residues. This factorization assumption, also if not necessary, is very useful because it implies as a direct consequence the (necessary) factorization property of the Regge residues.

We have to check that the Lorentz pole contribution described above satisfies the following conditions:

a) it has no unwanted singularities if the subenergies [see Eq. (6.9)] are sufficiently large;

b) it must be possible to decompose it into a family of multi-Regge contributions with factorized residues and well defined quantum numbers;

c) it has to satisfy the covariance "vertex" conditions given in Section 4.

The first condition is satisfied if the elements c_r and $c_{\bar{r}}$ are properly chosen. In fact, the representation matrix elements which appear in Eq. (7.2), continued in the complex Lorentz group, are singular for [7]

$$L_{tt}\left(c_r^{-1}\,\tilde{h}_r\,c_{\bar{r}}\right) = \pm 1 \,. \tag{7.3}$$

Then, for instance, if we consider the singularities in the variable $\tilde{h}_6$ (see Fig. 1), we may take

$$\begin{cases} c_6 = \tilde{h}_6^{-1}\,\tilde{h}_4 \,, \\ c_{\bar{6}} = \tilde{h}_{\bar{6}}^{-1}\,\tilde{h}_3 \,, \end{cases} \tag{7.4}$$

and from Eq. (6.9), we see that Eq. (7.3) for $r=6$ is equivalent to

$$S_{34} = (M_3 \pm M_4)^2 \,, \tag{7.5}$$

so that the singularities appear only for small values of the subenergy S_{34}. Other choices of c_r and $c_{\bar{r}}$ give a similar result, but are more convenient for other reasons, Ref. 7).

In conclusion, if we neglect the points where $\lambda_r(W_r)$ is integral, our multi-Lorentz term is analytic when all the subenergies are large. Of course, one can introduce "dynamical singularities" in the Lorentz trajectories or in the Lorentz residues. Other dynamical singularities can arise from a divergence in the infinite sums which appear in the multi-Lorentz terms, but one can easily choose the Lorentz residues in such a way that these singularities do not appear, if they are not desired.

$\tilde{a}$ At the points where $\lambda_r(W_r)$ is integral, the functions develop some singularities which have to be cancelled by suitable factors contained in the Lorentz residues or by means of a cancellation between different Lorentz poles. This problem, discussed in detail in Ref. 117), is similar to the one that appears in the usual Regge pole formalism [1)-4)], where phenomena like sense or nonsense choosing, compensating trajectories and ghost killing mechanisms take place. It is interesting to remark that when the behaviour of the Lorentz residues is consistently chosen, all the Regge trajectories of the corresponding family follow well defined ghost killing mechanisms [117)].

The decomposition of a Lorentz pole contribution into Regge pole contributions [7),23),24),114),118)-120)] is just a complicated mathematical exercise, which can immediately be generalized to the multi-Regge case [9)]. The result of this decomposition is an infinite sum of multi-Regge terms labelled by the set of integral non-negative parameters $\nu_{n+1},\ldots,\nu_{2n-3}$. The interpretation is that the Lorentz pole exchanged along the internal line r generates a family of Regge trajectories labelled by the parameter ν_r; the same thing happens for the Lorentz poles exchanged along the other internal lines. Then, choosing in all the possible ways for each internal line a member of the corresponding family of trajectories, we build all the multi-Regge terms which appear in the sum. As we have already said, it follows from the calculation that the Regge residues are factorized if the Lorentz residues have the same property. Therefore, condition b) is automatically satisfied. The trajectory quantum numbers are just those described in Section 2.

The condition c) is perhaps the most delicate in this formalism and we shall discuss it in the next Section.

8. THE VERTEX CONDITIONS

It is convenient (and possible [7),9)]) to choose the element c_r in such a way that *)

$$c_r^{-1}\,\tilde{l}_r^{-1}\,\tilde{k}\,\tilde{l}_r c_r \in H_+^c \quad \text{if} \quad \tilde{k} \in \tilde{K}^{(irs)c}. \tag{8.1}$$

We remark also that from the definition of the function $\tilde{\alpha}$ we have [24)]

$$\tilde{\alpha}^{\sigma\tau 0\lambda}_{j m j' m'}(u a) = \sum_{m''} R^{\sigma(-1)^j, j}_{m m''}(u)\, \tilde{\alpha}^{\sigma\tau 0\lambda}_{j m'' j' m'}(a),$$

$$u \in H_+^c\,, \quad L_{tt}(a) \neq \pm 1. \tag{8.2}$$

It follows immediately that the vertex conditions described at the end of Section 4, when applied to a Lorentz contribution, are equivalent to a set of constraints on the residue function (7.1). For the vertex $(13\bar{6})$ (see Fig. 1) the constraint takes the form

$$\sum_{m_1' m_3' m_{\bar{6}}'} R^{\pi_1 j_1}_{m_1' m_1}(\tilde{l}_1^{-1}\tilde{k}\tilde{l}_1)\, R^{\pi_3 j_3}_{m_3' m_3}(\tilde{l}_3^{-1}\tilde{k}\tilde{l}_3)\cdot$$

$$\cdot R^{\pi_{\bar{6}} j_{\bar{6}}}_{m_{\bar{6}}' m_{\bar{6}}}(c_{\bar{6}}^{-1}\tilde{l}_{\bar{6}}^{-1}\tilde{k}\tilde{l}_{\bar{6}} c_{\bar{6}})\, \gamma^{(13\bar{6})}_{m_1' m_3' j_{\bar{6}} m_{\bar{6}}'}(W_6) =$$

$$= \gamma^{(13\bar{6})}_{m_1 m_3 j_{\bar{6}} m_{\bar{6}}}(W_6)\,, \quad \tilde{k} \in \tilde{K}^{(13\bar{6})c}\,, \quad \pi_{\bar{6}} = \sigma_6(-1)^{j_{\bar{6}}}. \tag{8.3}$$

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*) The superscript c means that the group is complex.

The problem is now to find a parametrization of the type

$$\gamma^{(13\bar{6})}_{m_1 m_3 j_{\bar{6}} m_{\bar{6}}}(W_6) =$$

$$= \sum_{\alpha} \Xi^{\alpha}_{m_1 m_3 j_{\bar{6}} m_{\bar{6}}}(W_6)\, \hat{\gamma}^{(13\bar{6})}_{\alpha}(W_6), \tag{8.4}$$

such that, if we assume that the functions $\hat{\gamma}$ are arbitrary independent analytic functions, we obtain all the analytic solutions of Eq. (8.3). The functions $\hat{\gamma}$ are just the "reduced Lorentz residues" which appear also in Eq. (2.11). The functions Ξ are known kinematic factors.

This problem has been solved for vertices with two external particles in Ref. 7) *) and for vertices with one external particle in Ref. 9). In both cases the most difficult step is to choose the boosts $\tilde{b}_i$ and the elements c_i in such a way that the arguments of the three representation matrices R which appear in Eq. (8.3) turn out to be identical. Then, the procedure consists of adding vectorially the three angular momenta described by the indices of the residue. The resultant angular momentum $j_{(13\bar{6})}$ can be interpreted as an "orbital angular momentum" which has to be added to the spins of the particles (1) and (3) in order to obtain an object with total angular momentum $j_{\bar{6}}$. The quantity $j_{(13\bar{6})}$, which in Eq. (8.4) is hidden in the summation index α, plays the main role in determining the pseudothreshold behaviour of the residues. This is in accord with the treatments of pseudothresholds [81)-83)] based on a generalization of the well-known fact that the threshold behaviour of partial wave amplitudes is determined by orbital angular momentum.

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*) This solution does not eliminate the singularities at the threshold $W_6 = (M_1+M_3)^2$.

In conclusion, we can say that it is possible to build "multi-Lorentz" models, which satisfy all the analyticity and factorization requirements. They can describe only families of parallel trajectories, but present the important advantage of providing a compact expression for the sum of all the multi-Regge contributions belonging to a family. This compact expression, suitably modified and simplified, can be useful for phenomenological purposes.

9. APPLICATIONS

The information on the small t behaviour of the Regge parameters obtained by both the analytic and the group theoretical methods has found some phenomenological application in the following fields:

a) resonance spectroscopy: the mass formula (2.7) has been tentatively used for large positive t, in order to classify the known baryon resonances by means of a small number of families of Regge trajectories [121),122)]. For some suggestions about meson trajectories, see Refs. 123),124);

b) high energy multi-particle production: the relative scarcity of experimental information does not permit at present a clear test of the kinematical details of the multi-Regge models, which, on the other hand, are just now beginning to be understood theoretically;

c) high energy elastic or quasi-elastic forward (or backward) reactions: as this is the most important application, we shall discuss it in some more detail.

From the discussion of Section 2, it follows that one has to classify the known Regge trajectories according to the Lorentz quantum numbers which describe their behaviour near $t = 0$. It is necessary to check that this classification is consistent with all the existing data and eventually to use it to obtain predictions.

Unfortunately, for the low-lying trajectories, this task is rather difficult and the results are uncertain, due to the presence of an unknown contribution of Regge cuts, which, being not subject to factorization constraints, do not necessarily possess well-defined Lorentz quantum numbers.

From the properties of the Lorentz pole contributions [24], we see that the trajectories which appear in the asymptotic expansion of spin-averaged total cross-sections have necessarily the Lorentz quantum numbers $M = 0$ and $\sigma = +1$. The trajectories P, P', ρ, ω, A_2 are certainly of this kind. Of course, their SU_3 partners must have the same Lorentz classification. For reactions dominated by the exchange of these trajectories, the high energy forward amplitude is essentially non-spin flip.

The first suggestion for the existence of $M = 1$ Lorentz poles has been found studying p-n charge exchange scattering [125),126)]. The differential cross-section for this reaction presents a forward peak which, being very narrow, can be explained only by pion exchange. However, due to the Lorentz pole selection rules [24], an $M = 0$ pion cannot contribute to forward nucleon nucleon scattering and so it has been suggested that the pion belongs to a Lorentz family with $M = 1$. This is not the only possible explanation of the data [126] but it is accurate, simple and natural.

However, the most suitable reaction for the investigation of $M = 1$ Lorentz poles is pion photoproduction. In fact, at the pion-photon vertex the helicity flip is necessarily ± 1, so that only Lorentz poles with $M = 1$ can give an important contribution to the high energy forward amplitude [55),127),128)], [see Eq. (2.13)]. Also in this case a narrow peak due to pion exchange is observed and the data are well explained assuming an $M = 1$ pion trajectory [129),130)]. A stronger evidence in favour of the $M = 1$ pion exchange in photoproduction has been obtained by means of the "finite energy sum rules" [124), 131)-134)]. The most refined form of this technique, using "continuous moment sum rules" [124),133),134)] permits, to a certain extent, to suggest that the contribution considered is due to a pole and not to a Regge cut. A complete discussion of the reactions mentioned above, taking into account the factorization constraints, can be found in Ref. 135).

The results mentioned above seem to be in favour of the $M = 1$ pion, but one has to remark that the photoproduction high energy data can perfectly be explained by a model involving an $M = 0$ pion plus cut contributions [136),137)]. Very recently it has been shown [138)] that a model of this kind can be consistent also with the continuous moment sum rules.

The simple $M = 1$ pion model seems to be in contradiction with the experimental data on ρ meson production [139),140)]. However, it has been shown that these difficulties disappear when the contribution of other trajectories (e.g., the A_1) are considered [141)].

It has also been suggested [142),143)] that the B trajectory is generated by a Lorentz pole with $M = 1$ and forms a parity doublet with a trajectory ρ' having the same quantum numbers as the ρ trajectory. Also this assumption seems to be corroborated by photoproduction sum rules [144)] and also (indirectly) by pion-nucleon scattering sum rules [145)-147)] and by other considerations [135)].

A joint study of many resonance production reactions connected by factorization has been performed recently by Devenish [148)]. Also in this case, the assignment $M = 1$ for both the pion and the B trajectory seems to be favoured.

These assignments of the Lorentz quantum numbers are in accord with an elegant and symmetric scheme proposed by Ahmadzadeh, Refs. 149),150), which takes into account also SU_3 symmetry and exchange degeneracy.

The main objection against the assignment $M = 1$ for the pion trajectory comes from more theoretical considerations. It has been shown by Mandelstam [151)] that many results of the theory of current algebra and partially conserved axial current can be obtained from the assumption that the pion, in the limit of vanishing mass, decouples from pairs of equal mass particles (or complex systems of particles), but does not decouple from pairs of unequal mass particles.

It is a very general property of the Lorentz pole formalism that a single Lorentz pole can never generate a zero mass particle coupled with equal mass vertices [117]. This is due to the fact that the ratio between the residues at $t = 0$ of different trajectories (which is determined by the formalism) is such that, if one trajectory generates a zero mass particle, some other trajectories of the same family generate poles of the amplitude which cannot be interpreted as due to particle exchange and are therefore unacceptable.

If we are not willing to introduce a new family of trajectories which compensate these unwanted singularities [152], the first Mandelstam requirement is certainly satisfied. From Eq. (2.12), we see immediately that if $M > 0$, in the unequal mass case the residue with $\lambda_1 - \lambda_3 = 0$ (i.e., the "sense" residue, which describes the coupling of the physical pion) vanishes identically at $t = 0$. Therefore [151),153)] an $M > 0$ pion decouples also from unequal mass vertices and we have to assume that the pion has $M = 0$.

In order to explain the photoproduction data, if we want to avoid strong Regge cut contributions *), we have to introduce two different trajectories with the pion quantum numbers [153),155)-157)], one with $M = 0$ and the other with $M = 1$. These trajectories must have very similar $t = 0$ intercepts and therefore rather complicated mixing phenomena are expected [155)-157)].

If we want to preserve SU_3 symmetry and exchange degeneracy as in the Ahmadzadeh scheme, we have to introduce many other trajectories. A scheme of this kind has been discussed by Fox and Sertorio [158]; these authors have obtained in this way a satisfactory description of many quasi-elastic processes.

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*) It has been suggested [154] that the discontinuity of a Regge cut is large only when a Regge pole is present in the unphysical sheet. In this case the cut contribution can be approximated by means of the pole contribution and has well defined Lorentz quantum numbers.

ACKNOWLEDGEMENTS

A large part of the present paper summarizes a research carried out in collaboration with Drs. A. Sciarrino and G. Cosenza.

TABLE

Vertex	Regge residue	Regge pole quantum numbers		$t=0$ behaviour	
		σ	$\tau_\nu^{G(-1)^I}$	ν even	ν odd
$\pi - \pi$ K - K	$\beta^{\nu\sigma}_{oo}(t)$	+1	+1	$t^{\frac{1}{2}M}$	$t^{\frac{1}{2}(M+1)}$
N - N	$\beta^{\nu\sigma}_{\frac{1}{2}\frac{1}{2}}(t)$	+1	+1	$t^{\frac{1}{2}M}$	$t^{\frac{1}{2}(M+1)}$
		-1	+1	$t^{\frac{1}{2}\lvert M-1\rvert}$	$t^{\frac{1}{2}M}$
	$\beta^{\nu\sigma}_{\frac{1}{2},-\frac{1}{2}}(t)$	+1	+1	$t^{\frac{1}{2}\lvert M-1\rvert}$	$t^{\frac{1}{2}(\lvert M-1\rvert+1)}$
		-1	-1	$t^{\frac{1}{2}M}$	$t^{\frac{1}{2}\lvert M-1\rvert}$

Behaviour near $t=0$ of some equal mass residues. For values of the quantum numbers which do not appear in the Table, the residues vanish identically.

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FIGURE CAPTIONS

Figure 1:

A coupling scheme corresponding to a multi-Regge model.

Figure 2:

The two co-ordinate axes and the parabola represent the pairs (W_6, W_7) such that the boosts defined by Eq. (5.3) are singular.

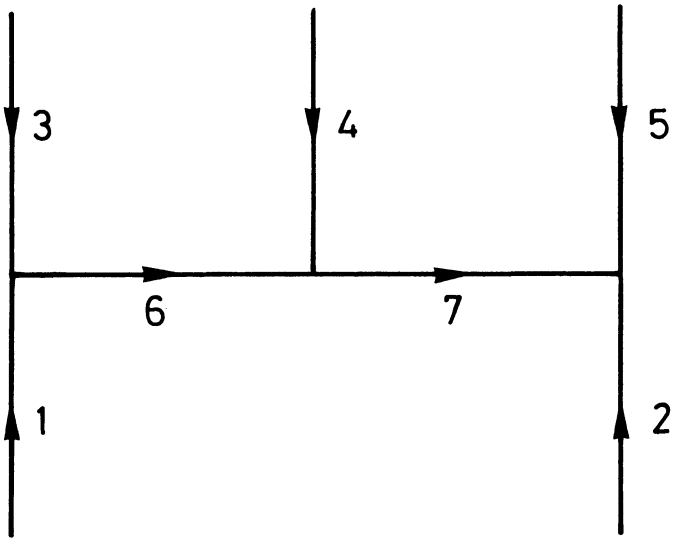

FIG.1

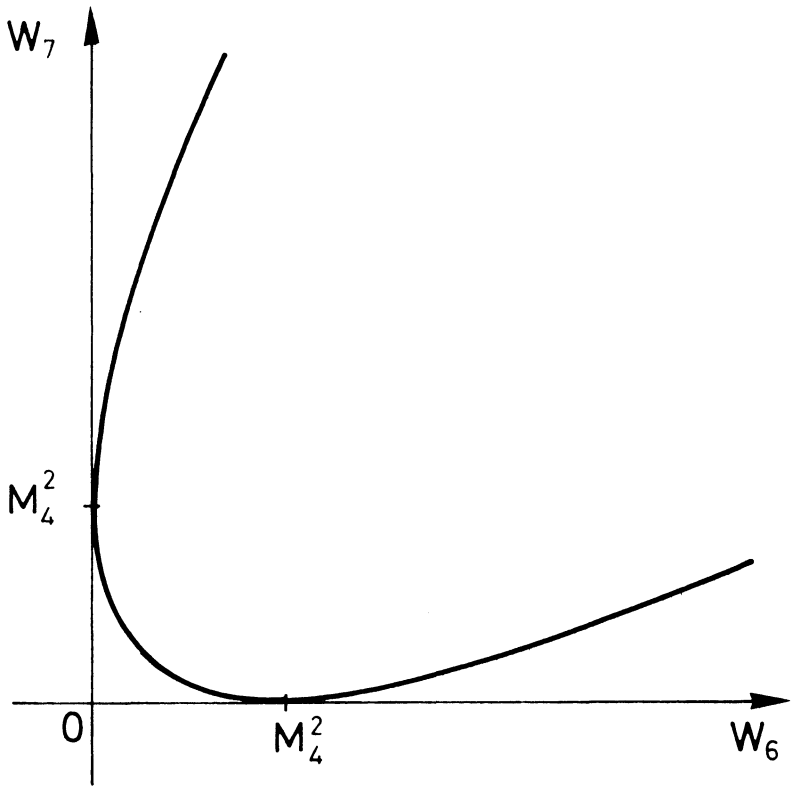

FIG.2