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A PCAC PUZZLE : $\pi^{o} \rightarrow \gamma\gamma$ IN THE σ MODEL

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ABSTRACT

The effective coupling constant for $\pi^{o} \rightarrow \gamma\gamma$ should vanish for zero pion mass in theories with PCAC and gauge invariance. It does not so vanish in an explicit perturbation calculation in the σ model. The resolution of the puzzle is effected by a modification of Pauli-Villars-Gupta regularization which respects both PCAC and gauge invariance.

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1. INTRODUCTION

The invariant amplitude for $\pi^0 \to \gamma\gamma$ is obtained by contracting the polarization vectors of the photons with a tensor

$$T^{\mu\nu}(p,q) = \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta T(k^2) \tag{1.1}$$

where p and q are the photon momenta, which we shall always take to be on their mass shell $p^2 = 0 = q^2$. The pion momentum is $k(=p+q)$; we shall be interested in off-mass-shell values, as well as physical values $k^2 = \mu^2$. The above general form of $T^{\mu\nu}$ is dictated by Lorentz invariance and parity conservation. Gauge invariance $(p_\mu T^{\mu\nu} = T^{\mu\nu} q_\nu = 0)$ and Bose symmetry $(T^{\mu\nu}(p,q) = T^{\nu\mu}(q,p))$ are seen to hold.

Steinberger [1] calculated $T(k^2)$ in perturbation theory from the diagrams of Fig. 1 representing the virtual dissociation of the neutral pion (dotted line) through the interaction $g\bar{\psi}\gamma_5\psi\phi$ into a proton-antiproton pair (solid lines), which then radiate the photons (wavy lines). This gives in particular

$$T(0) = g\,4\pi^2 m^{-1} \tag{1.2}$$

where m is nucleon mass. The physical value $T(\mu^2)$ is only slightly different, by quantities of order $(\mu/m)^2$.

On the other hand it has been shown by Veltman and Sutherland [2] that when the off-mass-shell continuation is made with a pion field that is the divergence of the axial current (PCAC), then

$$T(0) = 0 \tag{1.3}$$

Now it happens that in the σ model which has PCAC built in as an operator equation, the calculation of $T^{\mu\nu}$ in lowest order is just that of Steinberger. We then have to reconcile (1.2) and (1.3). This is our puzzle.

The problem of course is in the same tradition as that of the photon mass, non-canonical terms in commutators - "Schwinger terms" [3),4),5)] - and violations of the Jacobi identity [4)]. It will be seen that the resolution of the puzzle lies in a proper definition of time ordered products which goes beyond the naive notion of multiplying by step functions. However in the present case the usual Pauli-Villars regularization [6)], already considered by Steinberger, does not resolve the problem. This can be traced to the fact that it simply does not respect PCAC. We develop a variation which does respect PCAC, as well as Lorentz and gauge invariance, and find that indeed the explicit perturbation calculation also then yields $T(0)=0$.

The purpose of our exercise is not so much to point out and correct the limitations of the unregulated σ model, which has little physical relevance to the process in question, but rather to demonstrate in a very simple example the unreliability of the formal manipulations common to current algebra calculations: definition of T products, Ward identities, etc. [4)].

In Section 2 we present the PCAC argument relevant to our problem. In Section 3 the perturbation calculation is performed. Section 4 is devoted to the definition of the proper regularization procedure.

2. PCAC ARGUMENT

We review briefly the PCAC argument for $T(0)=0$. Using the divergence for the axial current

$$\partial_\mu A^\mu = F\mu^2 \phi \tag{2.1}$$

which (in the case of the neutral current) is supposed to remain true even in the presence of electromagnetic fields, we have

$$
\begin{aligned}
T^{\mu\nu} &= (\mu^2 - k^2) \langle \gamma, p_\mu; \gamma, q_\nu | \phi(0) | 0 \rangle \\
&= -i (F\mu^2)^{-1} (\mu^2 - k^2) \, k_\alpha \langle \gamma, p_\mu; \gamma, q_\nu | A^\alpha(0) | 0 \rangle \\
&= (F\mu^2)^{-1} (\mu^2 - k^2) \, k_\alpha \, F^{\alpha\mu\nu}(p, q)
\end{aligned}
\tag{2.2}
$$

Now, if $(F\mu^2)^{-1}(\mu^2-k^2)F^{\alpha\mu\nu}(p,q)$ is expanded in powers of p and q, bearing in mind parity conservation and Bose symmetry and working to lowest order in electromagnetism, the expansion starts with

$$
\epsilon^{\alpha\mu\nu\omega} (p - q)_\omega \, C
$$

where C is a constant. Then from (1.1) and (2.2)

$$
T(k^2) = -2C + \cdots
$$

where the additional terms are of order k^2 and higher. However, we also have gauge invariance (again it is essential that the current is neutral)

$$
p_\mu F^{\alpha\mu\nu} = q_\nu F^{\alpha\mu\nu} = 0 \tag{2.3}
$$

This requires $C = 0$; so $T(0) = 0$.

For the subsequent discussion it is useful to note the following general form for $F^{\alpha\mu\nu}$, which is required by Lorentz invariance, parity conservation, and Bose symmetry

$$
\begin{aligned}
F^{\alpha\mu\nu}(p,q) = \; & \epsilon^{\mu\nu\omega\phi} p_\omega q_\phi k^\alpha F_1(k^2) \\
& + \left(\epsilon^{\alpha\mu\omega\phi} q^\nu - \epsilon^{\alpha\nu\omega\phi} p^\mu\right) p_\omega q_\phi \, F_2(k^2) \\
& + \left(\epsilon^{\alpha\mu\omega\phi} p^\nu - \epsilon^{\alpha\nu\omega\phi} q^\mu\right) p_\omega q_\phi \, F_3(k^2) \\
& + \epsilon^{\alpha\mu\nu\omega} \left(p_\omega - q_\omega\right) F_4(k^2)
\end{aligned}
\tag{2.4}
$$

The F_i possess no kinematical singularities. The gauge invariance constraint is

$$
F_4 = \tfrac{1}{2} k^2 F_3 \tag{2.5}
$$

Actually there exists a linear relation between the four tensors in the above decomposition; we could eliminate one of them, but it is convenient not to.

Note also that

$$
k_\alpha F^{\alpha\mu\nu} = \epsilon^{\mu\nu\omega\phi} p_\omega q_\phi \left(k^2 F_1 - 2 F_4\right) \tag{2.6}
$$

so that from (2.2) and (1.1)

$$
T(k^2) = (F\mu^2)^{-1} (\mu^2 - k^2)\left(k^2 F_1 - 2 F_4\right) \tag{2.7}
$$

Gauge invariance then gives

$$T(k^2) = (F\mu^2)^{-1}(\mu^2 - k^2)\,k^2(F_1 - F_3) \tag{2.8}$$

Since we are working to lowest order in electromagnetism $F_1 - F_3$ does not possess a dynamical singularity at $k^2 = 0$, and therefore we recover the result $T(k^2) = 0k^2$.

3. PERTURBATION THEORY ARGUMENT

The formal reasoning of the previous Section should be varifiable by explicit perturbation calculation in a model with PCAC and gauge invariance. We consider the σ model interacting with the electromagnetic field a_μ, omitting the charged pion and the neutron fields, which are not necessary for our problem. The Lagrange density is [7]

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{2}\partial_\nu a_\mu \partial^\nu a^\mu \\ & + \bar{\Psi}\left[i\not\partial + e\not a - m + g(\sigma + \phi\gamma_5)\right]\Psi \\ & + \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}(\partial\sigma)^2 - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{2}\left(\mu^2 + \tfrac{2\lambda}{f}\right)\sigma^2 \\ & - \lambda\left[(\phi^2 + \sigma^2)^2 - 2f^{-1}\sigma(\sigma^2 + \phi^2)\right]\end{aligned} \tag{3.1}$$

Here $f = g/2m$ and λ is of order g^2. The axial current is

$$A_\mu = \bar{\Psi}\, i\gamma_\mu\gamma_5\,\Psi + 2(\sigma\partial_\mu\phi - \phi\partial_\mu\sigma) - f^{-1}\partial_\mu\phi \tag{3.2a}$$

and

$$\partial_\mu A^\mu = \mu^2 f^{-1} \phi \tag{3.2b}$$

i.e., we have $F = f^{-1} = 2m/g$ in (2.1).

To lowest order in g, the relevant diagrams for $T^{\mu\nu}$ and $F^{\alpha\mu\nu}$ are given in Figs. 1 and 2 respectively, where the X represents the axial current. The diagrams have the following representation

$$\Gamma^{\mu\nu}(p,q) = ig \int d^4r \, \mathrm{Trace}\, \gamma^5 [\not{p} + \not{r} - m]^{-1} \gamma^\mu [\not{r} - m]^{-1} \gamma^\nu [\not{r} - \not{q} - m]^{-1} \tag{3.3a}$$

$$T^{\mu\nu}(p,q) = \Gamma^{\mu\nu}(p,q) + \Gamma^{\nu\mu}(q,p) \tag{3.3b}$$

$$\Gamma^{\alpha\mu\nu}(p,q) = i \int d^4r \, \mathrm{Trace}\, \gamma^5 \gamma^\alpha [\not{p} + \not{r} - m]^{-1} \gamma^\mu [\not{r} - m]^{-1} \gamma^\nu [\not{r} - \not{q} - m]^{-1} \tag{3.4a}$$

$$F^{\alpha\mu\nu}(p,q) = \Gamma^{\alpha\mu\nu}(p,q) + \Gamma^{\alpha\nu\mu}(q,p) - \frac{2mg^{-1}k^\alpha}{k^2 - \mu^2} T^{\mu\nu}(p,q) \tag{3.4b}$$

One can verify that PCAC is satisfied.

$$k_\alpha \Gamma^{\alpha\mu\nu}(p,q) = i \int d^4r \, \mathrm{Trace}\, \gamma^5 [\not{p} + \not{q}][\not{p} + \not{r} - m]^{-1} \gamma^\mu [\not{r} - m]^{-1} \gamma^\nu [\not{r} - \not{q} - m]^{-1} \tag{3.5a}$$

Decomposing $\not{p}+\not{q}$ into $2m+(\not{p}+\not{r}-m)-(\not{r}-\not{q}+m)$, we have

$$\begin{aligned} k_\alpha \Gamma^{\alpha\mu\nu}(p,q) = {} & 2mg^{-1}\Gamma^{\mu\nu} + i \int d^4r \, \mathrm{Trace}\, \gamma^5 \gamma^\mu [\not{r} - m]^{-1} \gamma^\nu [\not{r} - \not{q} - m]^{-1} \\ & - i \int d^4r \, \mathrm{Trace}\, \gamma^5 [\not{r} - \not{q} + m][\not{p} + \not{r} - m]^{-1} \gamma^\mu [\not{r} - m]^{-1} \gamma^\nu [\not{r} - \not{q} - m]^{-1} \end{aligned} \tag{3.5b}$$

In the last integral, $\not{r}-\not{q}+m$ may be taken through the γ^5, thus changing the sign of m, and then transposed to the end of the expression, cancelling the propagator.

$$k_\alpha \Gamma^{\alpha\mu\nu}(p,q) = 2mg^{-1}\Gamma^{\mu\nu} + i\int d^4r \,\mathrm{Trace}\,\gamma^5\gamma^\mu[\not{r}-m]^{-1}\gamma^\nu[\not{r}-\not{q}-m]^{-1}$$
$$+ i\int d^4r \,\mathrm{Trace}\,\gamma^5[\not{p}+\not{r}-m]^{-1}\gamma^\mu[\not{r}-m]^{-1}\gamma^\nu \tag{3.5c}$$

It is now seen that each integral must vanish since it is impossible to form a two-index pseudotensor which depends on only one vector. We find therefore

$$k_\alpha F^{\alpha\mu\nu}(p,q) = -\frac{2mg^{-1}\mu^2}{k^2-\mu^2}T^{\mu\nu}(p,q) \tag{3.5d}$$

Note that this verification of PCAC does not require any shifts of integration. Gauge invariance of $F^{\alpha\mu\nu}$ cannot be established in the same fashion.

The integrals are superficially linearly divergent. However when the trace is taken in (3.3a), no powers of r survive in the numerator and $\Gamma^{\mu\nu}$ is manifestly convergent. In the notation (1.1), we find

$$T(k^2) = 8m\pi^2 g\int_0^1 dx\int_0^{1-x} dy\,[m^2-k^2xy]^{-1} \tag{3.6}$$

and

$$T(0) = 4\pi^2 g/m \neq 0 \tag{3.7}$$

This contradicts (1.3) and shows that $T^{\mu\nu}(p,q)$ cannot be the divergence of a gauge invariant pseudotensor.

To expose the reason for this, we evaluate $F^{\alpha\mu\nu}$. When the trace is taken in (3.4a), the integral remains linearly divergent, and care must be taken to obtain an unambigous result. The method of calculation is outlined in the Appendix. A finite result is found, which in the notation (2.4) is

$$F_1 = 4\pi^2\int_0^1 dx \int_0^{1-x} dy \,[m^2 - k^2 xy]^{-1}\left[x+y-(x-y)^2 - \frac{4m^2}{k^2-\mu^2}\right]$$

$$-F_2 = F_3 = 4\pi^2\int_0^1 dx \int_0^{1-x} dy \,[m^2 - k^2 xy]^{-1}[1-x-y][x+y]$$

$$F_4 = 4\pi^2\int_0^1 dx \int_0^{1-x} dy \,[m^2 - k^2 xy]^{-1}\left[-2(m^2-k^2xy) + \tfrac{1}{2}k^2(1-x-y)(x+y)\right] \tag{3.8}$$

It can be verified that PCAC has been maintained.

$$-\frac{2mg^{-1}\mu^2}{k^2-\mu^2}T(k^2) = k^2 F_1 - 2F_4 \tag{3.9}$$

However gauge invariance has been lost

$$F_4 \neq \frac{k^2}{2} F_3 \tag{3.10}$$

It is seen that an extra term in F_4

$$4\pi^2\int_0^1 dx \int_0^{1-x} dy \,[m^2 - k^2 xy]^{-1}\left[-2(m^2-k^2xy)\right] = -4\pi^2 \tag{3.11}$$

violates gauge invariance. However that term is necessary for PCAC, (3.9).

It is remarkable that the calculation of $F^{\alpha\mu\nu}$ yields a finite expression. However it is seen from the evaluation that the finiteness of $F^{\alpha\mu\nu}$ is a consequence of the cancellation of infinite quantities, and this is the reason that a gauge non-invariant term has crept in. To exhibit this explicitly, we may contract $F^{\alpha\mu\nu}(p,q)$ with p_μ. Decomposing $\not p$ into $(\not p+\not k-m)-(\not k-m)$ yields from (3.4a)

$$p_\mu \Gamma^{\alpha\mu\nu}(p,q) = -i\int d^4r\ \mathrm{Trace}\ \gamma^5\gamma^\alpha[\not p+\not r-m]^{-1}\gamma^\nu[\not r-\not q-m]^{-1}$$
$$+i\int d^4r\ \mathrm{Trace}\ \gamma^5\gamma^\alpha[\not r-m]^{-1}\gamma^\nu[\not r-\not q-m]^{-1} \tag{3.12a}$$

The second term vanishes as in (3.5c). A similar expression may be given for $p_\mu\ \Gamma^{\alpha\nu\mu}(q,p)$. Straightforward calculation then results in

$$p_\mu F^{\alpha\mu\nu}(p,q) = 8i\epsilon^{\alpha\nu\omega\phi}p_\omega q_\phi\int d^4r\int_0^1 dx\,dy\ \delta(x+y-1)[r^2-m^2+k^2xy]^{-2}$$
$$-8i\epsilon^{\alpha\nu\omega\phi}(p_\omega+q_\omega)\int d^4r\int_0^1 dx\,dy\ \delta(x+y-1)\ r_\phi\left[(r+px-qy)^2-m^2+k^2xy\right]^{-2} \tag{3.12b}$$

If one could shift variables in the second integral, symmetric integration would render it equal to minus the first one, and $p_\mu F^{\alpha\mu\nu}$ would vanish. However, since that integral is linearly divergent, a surface term is picked up when the shift is performed. The surface term contribution is

$$p_\mu F^{\alpha\mu\nu}(p,q) = -4\pi^2\epsilon^{\alpha\nu\beta\omega}p_\beta q_\omega \tag{3.12c}$$

It can be verified that (3.12c) is exactly what is got from the explicit formulae (2.4) and (3.8).

4. REGULARIZATION

It is convenient to introduce $F'^{\alpha\mu\nu}$, defined by omitting the pion pole term [the last in (3.4b)] in $F^{\alpha\mu\nu}$. The PCAC requirement is then

$$k_\alpha F'^{\alpha\mu\nu} = \frac{2m}{g} T^{\mu\nu} \tag{4.1}$$

In the theory and approximation under consideration, $F'^{\alpha\mu\nu}$ and $T^{\mu\nu}$ are the Fourier transforms of time ordered products

$$-iF'^{\alpha\mu\nu} = \int dx\,dy\, e^{ipx} e^{iqy} \langle 0| T\left(A'^{\alpha}(0)\, j^{\mu}(x)\, j^{\nu}(y)\right)|0\rangle$$
$$T^{\mu\nu} = \int dx\,dy\, e^{ipx} e^{iqx} \langle 0| T\left(j_{\pi}(0)\, j^{\mu}(x)\, j^{\nu}(y)\right)|0\rangle \tag{4.2}$$

where

$$A'^{\alpha} = \bar{\psi}\, i\gamma_\mu \gamma_5\, \psi$$

$$j^{\mu} = \bar{\psi}\, \gamma_\mu\, \psi$$

$$j_{\pi} = g\, \bar{\psi}\, \gamma_5\, \psi$$

and the ψ's are free fields. From the free field equations of motion

$$\partial_\alpha A'^{\alpha} = \frac{2m}{g} j_{\pi} \tag{4.3}$$

The desired (4.1) then follows by the usual partial integration in (4.2), provided the equal time commutator is zero:

$$\left[A^{0}(\vec{x})\,,\;j^{\mu}(\vec{y})\right] = 0$$

It is indeed zero when evaluated from the conventional formula, obtained by simple-minded manipulation of canonical anticommutation rules,

$$\left[\psi^{+}(\vec{x})\,A\,\psi(\vec{x})\,,\;\psi^{+}(\vec{y})\,B\,\psi(\vec{y})\right]$$

$$= \psi^{+}(\vec{x})\,[A,B]\,\psi(\vec{x})\,\delta(\vec{x}-\vec{y}) \qquad (4.4)$$

where A and B are any 4×4 matrices.

The gauge invariance conditions

$$p_{\mu}F^{\alpha\mu\nu} = q_{\nu}F^{\alpha\mu\nu} = 0 \qquad (4.5)$$

$$p_{\mu}T^{\mu\nu} = q_{\nu}T^{\mu\nu} = 0 \qquad (4.6)$$

follow in the same way from the vanishing of equal time commutators involving j^{0} with j^{μ}, A^{μ}, and j_{π}.

Corresponding to the above formal arguments leading to (4.3), (4.5), (4.6) there are manipulations of the integrands for the relevant closed loop diagrams. However these manipulations may fail if the integrands do not converge well enough. We saw explicitly that the verification of (4.5) required a shift of integration variable which introduced a surface term. It was for such situations that the Pauli-Villars regularization was devised. Here we follow the approach of Gupta [8),9)], with a modification dictated by PCAC.

We are concerned with integrals

$$\int d^4k \ I(m, k)$$

which depend on the fermion mass m through the fermion propagator. The convergence of the integral is improved by the replacement

$$I(m, k) \rightarrow I_M(m, k)$$

$$I_M(m, k) = I(m, k) - I(m+M, k)$$

where M is supposed to have some large value. The convergence can be further improved by a second step

$$I_M(m, k) \rightarrow I_{MM}(m, k)$$

$$I_{MM}(m, k) = I_M(m, k) - I_M(m+M, k)$$

and so on till the required degree of convergence is obtained. The integrand is finally of the form

$$\sum_i (\pm) \ I(m_i, k) \tag{4.7}$$

where $i = 0$ is the original term, and where $m_i = m + n_i M$, and the n_i are integers. According to Gupta this can be interpreted as follows. The original currents have been replaced by more complicated structures, e.g.,

$$\bar{\psi} i\gamma_\mu \gamma_5 \psi \rightarrow \sum_i \bar{\psi}_i i\gamma_\mu \gamma_5 \psi_i \tag{4.8}$$

where the extra terms involve extra auxiliary fields ψ_i, $i \neq 0$. To get the sign change of some of the terms in (4.7), certain of these auxiliary fields are quantized with commutators rather than anticommutators - remember that with ordinary fermions a factor (-1) for each closed loop arises from the anticommutation. Of course the Bose quantization of the spin $-\frac{1}{2}$ fields involves the introduction of an indefinite metric.

In the present example the introduction of just one auxiliary mass $(m+M)$ restores the gauge invariance of $F'^{\alpha\mu\nu}$ or $F^{\alpha\mu\nu}$. In fact we saw in (3.11) that the remainder in the attempted verification was of the form

$$F_4 - \tfrac{1}{2} k^2 F_3 = -4\pi^2$$

and this, being independent of m, is immediately removed by the subtraction. However the PCAC condition (3.10) is now spoiled, if $T^{\mu\nu}$ is treated in a uniform way. In fact the contributions from the two mass values separately satisfy the equation (4.1) but with different coefficients m/g and $(m+M)/g$ respectively. The remedy is simple; to preserve PCAC the coupling constants as well as the masses must be varied for the auxiliary fields, in such a way that

$$m_i / g_i = m/g = \text{constant}$$

So we take, instead of (4.8), for j_π

$$g \bar{\Psi} \gamma_5 \Psi \rightarrow g \bar{\Psi} \gamma_5 \Psi + \sum_{i \neq 0} g_i \bar{\Psi}_i \gamma_5 \Psi_i \tag{4.9}$$

with $g_i/m_i = g/m$. The PCAC is then preserved by the regularization which has secured gauge invariance, and we would expect the formal consequence $T(0) = 0$ to be realized. Indeed it is. From the unregulated value

$$T(0) = 4\pi^2 g/m \tag{4.10a}$$

we contract the regulated value

$$T_{reg}(0) = 4\pi^2 \left(\frac{g}{m} - \frac{g_1}{m_1} \right) = 0 \tag{4.10b}$$

$$\Gamma^{\alpha\mu\nu}(p,q) + \Gamma^{\alpha\nu\mu}(q,p) = \pi^2 \int_0^1 dx \int_0^{1-x} dy \, [m^2 - k^2 xy]^{-1}$$

$$\Big\{ 2\epsilon^{\alpha\mu\nu\beta}(p-q)_\beta \left[-2m^2 + k^2\big((x+y)(1-x-y) - 2xy\big)\right] +$$

$$4 p_\beta q_\gamma \Big[(1-x-y)(x+y)\big((p-q)^\mu \epsilon^{\alpha\nu\beta\gamma} + (p-q)^\nu \epsilon^{\alpha\mu\beta\gamma}\big) +$$

$$k^\alpha \epsilon^{\mu\nu\beta\gamma} \left(x + y - (x-y)^2\right)\Big]\Big\} +$$

$$8i\epsilon^{\alpha\mu\nu\beta} \int d^4r \Big\{ \frac{r_\beta}{[(p+r)^2 - m^2][(r-q)^2 - m^2]} +$$

$$\int_0^1 dx \int_0^{1-x} dy \frac{(p-q)_\beta}{[r^2 - m^2 + k^2 xy]^2} \Big\}$$

(A.3)

It remains to perform the remaining integrals which appear to be linearly and logarithmically divergent. The first of these may be parametrized as follows

$$i \int d^4r \frac{r_\beta}{[(p+r)^2 - m^2][(r-q)^2 - m^2]} =$$

$$i \int d^4r \int_0^1 dx\,dy\, \delta(x+y-1)\, r_\beta \left[(r+px-qy)^2 - m^2 + k^2xy\right]^{-2}$$

(A.4a)

Since the integral is linearly divergent, a shift of variable picks up a surface term. Thus we find, after symmetric integration for (A.4a)

$$i\int d^4r \int_0^1 dx\,dy\;\delta(x+y-1)\,\frac{q_\beta y - p_\beta x}{[r^2-m^2+k^2xy]^2} \;-\; \frac{\pi^2}{4}\left(p_\beta - q_\beta\right) \tag{A.4b}$$

Combining (A.4b), with the second divergent integral in (A.3), we perform all the integrals over y and find for the last two terms of (A.3)

$$I = -2\pi^2 \epsilon^{\alpha\mu\nu\beta}\left(p_\beta - q_\beta\right) + 8i\,\epsilon^{\alpha\mu\nu\beta}(p-q)_\beta$$
$$\int d^4r \int_0^1 dx\;\frac{k^2x^2(1-x)}{\left(r^2-m^2+k^2x(1-x)\right)^2\left(r^2-m^2\right)} \tag{A.5}$$

Frequent use has been made of the transformation $x \to 1-x$. The r integral is now convergent, and may be evaluated with the aid of another Feynman parameter. The result, after integration over that parameter is

$$I = -2\pi^2 \epsilon^{\alpha\mu\nu\beta}\left(p_\beta - q_\beta\right) - 8\pi^2 \epsilon^{\alpha\mu\nu\beta}\left(p_\beta - q_\beta\right)$$
$$\int_0^1 dx\left[(1-x) + \frac{m^2}{k^2x}\log\left(1 - \frac{k^2}{m^2}x(1-x)\right)\right] \tag{A.6a}$$

This may also be written as

$$I = -2\pi^2 \epsilon^{\alpha\mu\nu\beta}\left(p_\beta - q_\beta\right) + 8\pi^2 \epsilon^{\alpha\mu\nu\beta}\left(p_\beta - q_\beta\right)$$
$$\int_0^1 dx \int_0^{1-x} dy\,\left[m^2 - k^2xy\right]^{-1} k^2xy \tag{A.6b}$$

When this is combined with the remaining terms in (A.3), and use is made of the definitions (3.4b) and (2.4), one obtains (3.8).

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B. Hamprecht, Nuovo Cimento 50A, 449 (1967) and
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These authors observe that the commutators as calculated by Johnson and Low depend on the order in which certain limits are performed. It was explained in [5] that different orders are appropriate for different applications.

5) J.S. Bell, Nuovo Cimento 47, 616 (1967).

The author of this paper [it will be referred to below as [5]] takes the opportunity to refute a criticism of it. It was observed in [5] that even when certain sum rules involve "effective" values for certain commutators, the "canonical" values remain appropriate for certain propagator identities or zero energy theorems. This depends on defining the propagators as limits of those in a suitably cut-off theory. It was also noted that other objects, referred to as "mutilated" propagators, differing from the originals by contact terms, satisfy identities involving the "effective" non-canonical commutators. Brandt and Orzalesi [10] consider a third way of defining propagators - let us refer to it as "smearing". In the disputed example (and indeed rather generally) "smearing" is found to be equivalent to "mutilation". The contention of Brandt and Orzalesi is that only the "smeared" propagators have physical interest. Their reasons appear to be a). a legitimate dislike for cut-offs, and b). a preference for quantities "carefully defined from a distribution theoretic point of view". With regard to a)., everyone will agree that it would be better to avoid cut-offs if possible.

Brandt and Orzalesi believe that a relevant no-cut-off formulation of the model in question was given by R. Haag and G. Luzatto [Nuovo Cimento 13, 415 (1959)]. But the final formulation of the latter authors involves as coupling parameter only the renormalized coupling constant, and it reduces trivially to free fields when this quantity is zero - as it is in the limit studied in [5]. It is to be conjectured therefore that Brandt and Orzalesi also obtained their basic material (matrix elements of products without time ordering) not from the Haag-Luzatto formulation but from the appropriate limit of the cut-off theory. With regard to b), it is rather well known that arbitrarily defined T products may differ from physical amplitudes by contact terms; see for example the discussion of Bjorken referred to in [5]. The positive reasons for taking some interest in the limiting propagators, without mutilation or smearing, were clearly stated in [5]. They are defined (and how else could one find quantities of "physical" interest in such a model?) in analogy with quantities of interest in more serious theories. Explicit reference was made to Pauli-Villars regularization of the vacuum polarization tensor of electrodynamics. The present paper provides another example.

6) A very useful and well documented account of the application of regularization to the related problems of photon-mass and the Goto-Imamura-Schwinger terms has been given by G. Källén, Lectures given at the winter schools in Karpacz and Schladming, Feb. and Mar. 1968.

7) M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

8) S.N. Gupta, Proc.Phys.Soc. A66, 129 (1953).

9) One of us (J.S. B.) acknowledges a very useful conversation with N. Kroll on the Gupta method.

10) R.A. Brandt and C.A. Orzalesi, Phys.Rev. 162, 1747 (1967).

Figure 1

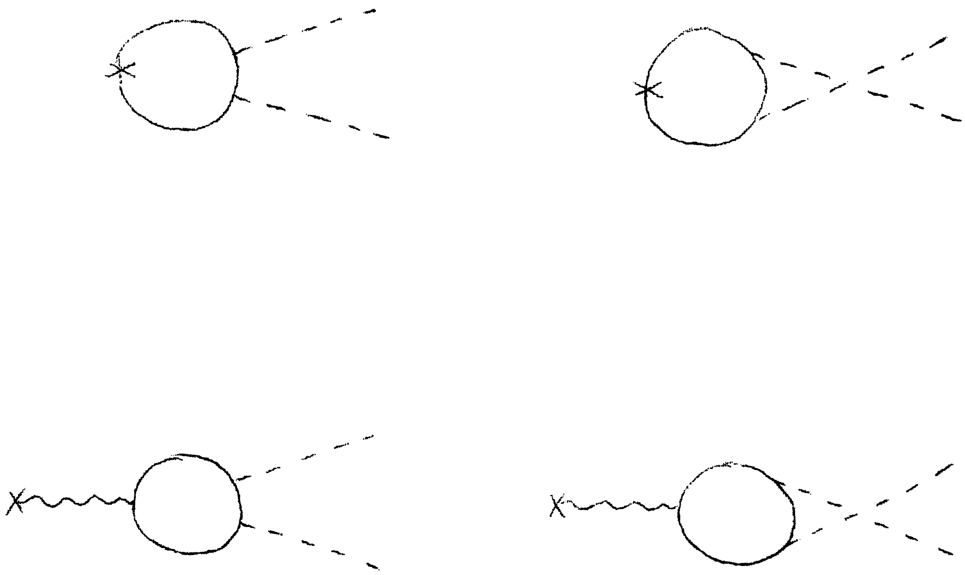

Figure 2