Archives

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A NEW RIGOROUS INEQUALITY ON THE $\pi_o \pi_o \rightarrow \pi_o \pi_o$ S WAVE AMPLITUDE

A. Martin
CERN - Geneva

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Some time ago, a number of rigorous inequalities on the scattering amplitude were obtained by Martin [1] and Common [2]. These looked like mathematical curiosities. However, Wanders and his collaborators [3] were able to show that they were in fact useful in practical calculations of the pion-pion S wave. For this reason, I wish to present here a refined version of one of these inequalities.

Let me summarize the previous linear inequalities which were obtained from axiomatic analyticity and positivity properties of the absorptive parts in all channels.

We have, noting the $\pi_0 \pi_0 \to \pi_0 \pi_0$ s wave as $f_0(s)$, with s square of the c.m. energy, and the pion mass taken equal to unity :

$$f_0(4) > f_0(0) > f_0\left(2\left[1+\frac{1}{\sqrt{3}}\right]\right) \simeq f_0(3.15) \quad [1] \qquad (1)$$

$$f_0(0) > \frac{1}{2}\int_2^4 f_0(s)\,ds \quad [1] \qquad (2)$$

$f_0(s)$ has a unique minimum [2),4)] for

$$1.02 < s < 1.7 \qquad (3)$$

Here, we want to derive an inequality of the type (1), but, however, more stringent. The derivation of the right-hand side of inequality (1) was as follows :

i) $$f_0(0) > F_0\left(s=0, \cos\theta_s = \frac{1}{\sqrt{3}}\right) = F_0\left(s=0, t = 2\left[1+\frac{1}{\sqrt{3}}\right]\right)$$

This inequality becomes an equation if $\ell = 4$ and higher waves are neglected.

ii) $$F_0\left(s=0,\ t=2\left(1+\frac{1}{\sqrt{3}}\right)\right) > f_0\left(2\left(1+\frac{1}{\sqrt{3}}\right)\right)$$

Here the inequality does not become an equality when $\ell = 4$ is neglected. One has to neglect also $\ell = 2$.

What we want to do is to improve this situation.

As was noted in Ref. 1), we can write dispersion relations for fixed s, $0 < s < 4$, as

$$F(s, \cos\theta_s) = f_0(s) + \frac{1}{\pi}\int_{z_0(s)}^{\infty} dz \left[\frac{1}{z-\cos\theta_s} + \frac{1}{z+\cos\theta_s} - \log\frac{z+1}{z-1}\right] A(s,z) \tag{4}$$

where, from unitarity in the t and u channels, A is a positive quantity. $-z_0(s)$ is given by $t = 4$, i.e.,

$$z_0(s) = \frac{4+s}{4-s} \tag{5}$$

The question is to decide, for a given s, for which value of $\cos\theta_s$ the bracket in the integral (4) is positive for all z and for which value it is negative for all z.

The following is easy to prove : for given $\cos\theta_s$ the bracket is a decreasing function of z if $\cos\theta_s^2 < \frac{1}{3}$. If $\cos\theta_s^2 > \frac{1}{3}$ the bracket has a unique minimum when z varies from 1 to ∞.

Therefore the conclusion is that :

$$\alpha) \qquad F(s, \cos\theta_s) < f_0(s) \tag{6}$$

if $\cos\theta_s^2 < \frac{1}{3}$, which was already established in Ref. 1),

$$\beta) \qquad F(s, \cos\theta_s) > f_0(s)$$

if the bracket is positive at the lower limit of integration over z in (4), i.e.,

$$F(s, \cos\theta_s) > f_0(s)$$

if

$$\cos\theta_s^2 > z_0\left[z_0 - \frac{2}{\log\frac{z_0+1}{z_0-1}}\right] = \phi(s) \tag{7}$$

with $z_0 = (4+s/4-s)$, which for z_0 large enough $(s \to 4)$ reduces to

$$\cos\theta_s^2 > \frac{1}{3} + \frac{4}{45 z_0^2} + \cdots\cdots \tag{8}$$

It is the combination of (6) and (7) which will give rise to interesting consequences : we choose $\bar{s}$ and $\bar{t}$ such that

$$\begin{aligned} \cos\theta_s &= -\frac{1}{\sqrt{3}} \\ \cos\theta_t &= \sqrt{\phi(t)} \end{aligned} \tag{9}$$

It may be seen graphically that this corresponds to a situation where t is already big enough, so that the search for solutions should be made around

$$\cos\theta_s = -\frac{1}{\sqrt{3}} \qquad \cos\theta_t = \frac{1}{\sqrt{3}}$$

which corresponds to s = 0.2143. The exact solution of (9) is

$$\bar{s} = 0{,}2134$$
$$\bar{t} = 2{,}9863$$

Hence, from properties (6) and (7), we get

$$f_0(0{,}2134) \geq F_0(\bar{s},\bar{t}) \geq f_0(2{,}9863) \qquad (10)$$

Since $|\cos\theta_t|$ is very close to $1/\sqrt{3}$, this inequality becomes practically an equality if one neglects $\ell = 4$ and higher waves in the s and t channels.

To demonstrate how tight this inequality is we can get another one in the opposite direction : we have

$$f_0(\bar{s}) < F\left(\bar{s}, \cos\theta_s = -\sqrt{\phi(\bar{s})}\right)$$

which corresponds to $\bar{\bar{t}} = 3.205$. It can be checked that the corresponding $|\cos\theta_t|$ is less than $1/\sqrt{3}$. Therefore

$$f_0(3.205) > f_0(0.2134) > f_0(2{,}9863) \qquad (11)$$

Comparison with inequality (1) will demonstrate the progress made.

R E F E R E N C E S

1) A. Martin, Nuovo Cimento 47, 265-281 (1967).

2) A.K. Common, Nuovo Cimento 53A, 946 (1968).

3) G. Auberson, O. Piguet and G. Wanders, Communication to the XIVth International Conference on High Energy Physics, Vienna (1968), quoted in the rapporteur's talk of A. Wightman - (to be published).

4) F.K. Cheung, private communication I am grateful to Dr. Cheung for pointing out a numerical mistake in Ref. 1).