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HIGH-ENERGY RELATIONS FOR SCATTERING OF PARTICLES AND ANTIPARTICLES WITH ARBITRARY SPIN FOLLOWING FROM ANALYTICITY AND CROSSING RELATIONS

A. Białas *) and B.E.Y. Svensson **)
CERN--Geneva

A B S T R A C T

Using analyticity and crossing relations for helicity amplitudes in two-body elementary particle reactions proceeding via strong interaction, the connection between the high energy asymptotic expansions of these amplitudes in the direct and the crossed reaction is derived. This connection is simple and generalizes in a straightforward way the corresponding relation for spinless particles. Besides allowing a proof of Pomeranchuk type theorems for differential cross-sections and polarizations, the result constitutes a very convenient parametrization of the helicity amplitudes, useful in the theoretical analysis of two-body experiments at high energy.

*) Present address : Institute of Physics, Jagellonian University, Reymonta 4, Krakow, Poland.

**) Present address : Institute of Theoretical Physics, University of Lund, Sölvegatan 14, Lund, Sweden.

65/1781/5 - TH. 633
1 December 1965

I.- INTRODUCTION

In this paper we discuss some consequences of analyticity in energy, for fixed momentum transfer, of scattering amplitudes in high energy two-body scattering of strongly interacting particles (hadrons) with arbitrary (integer or half-integer) spin. Helicity formalism [1] is used throughout. Our main result is the derivation of asymptotic relations between the helicity amplitudes for the direct reaction $a + c \rightarrow b + d$ and the crossed reaction $\bar{b} + c \rightarrow \bar{a} + d$. The relations obtained are rather simple (see Section IV, in particular Eqs. (IV.1-5)). They may be considered as a direct generalization of the asymptotic relation for the single invariant scattering amplitude in the case of spin-zero particles [2),3),4),5)]. Besides allowing a straightforward extension to particles with arbitrary spin of the (generalized) Pomeranchuk theorem [6] on the high energy connection between the differential cross sections for the two reactions, the formalism is particularly well suited for the discussion of asymptotic connections between different polarization measurements in the direct and crossed channel. Above all, however, the derived relations form the starting-point for a convenient parametrization of the helicity amplitudes in strong interaction processes, which seems to be useful in the theoretical analysis of the experimental results on two-body reactions at high energy.

The assumptions we need are standard ones [2),3),4),5)] and fairly general. First of all, we assume the helicity amplitudes to be analytic functions of the usual Mandelstam variables s, t and u; the exact analyticity properties are discussed in the remarks 1 and 2 at the end of Section III. From this condition we then derive crossing relations for helicity amplitudes. Such relations have already been given [7),8)], but they have to be modified to suit our purpose, the main point being that we need the crossing relations arising from a continuation in the <u>upper half</u> s plane.

Once we have derived appropriate crossing relations for the helicity amplitudes, we may impose the same requirements concerning boundedness, continuity and non-oscillatory behaviour of the amplitudes as one does in the treatment of spin-zero particles [2),3),4),5)], and derive the asymptotic connections mentioned above.

Although asymptotic relations for scattering of spinning particles have been considered previously [2),3)], the helicity formalism has, to our knowledge, never been applied to such problems. We would like to emphasize the power of this formalism also in the present context.

The set of assumptions which we use underlies, explicitely or tacitly, practically all current discussions of high energy two-body hadronic interactions. Thus, they have been used in more general treatments of meson-nucleon and nucleon-nucleon <u>elastic</u> scattering, like derivation of Pomeranchuk-type theorems [2),3),4),5),9)] or in the discussion of the ratio of the real to the imaginary part of the forward scattering amplitude [10)]. Moreover, most dynamical models for high energy <u>elastic</u> as well as <u>inelastic</u> two body processes (e.g., the Regge pole model [11)], the periphal model [12)]) satisfy these assumptions. It is well known that some features of these models agree remarkably well with the experimental data, whereas others seem to be incorrect. In this respect it is of great interest to analyze whether some of these model-dependent results in fact can be reproduced under more general assumptions and from this, if possible, to suggest generalizations and improvements. The asymptotic relations between helicity amplitudes, implying a consistent treatment of the spins and based on the very general assumptions of analyticity, boundedness and "reasonable" behaviour of the amplitudes in the physical region, constitute an important first step in such a program.

The complete plan of the paper is the following : in Section II, we introduce our notation and kinematical definitions. Section III is devoted to the above-mentioned modification of the crossing relations for helicity amplitudes. In Section IV, we give the asymptotic relations for these amplitudes. The connections between the crossed reaction and those reactions obtained from it by application of time reversal or charge conjugation invariance are discussed in Section V, while Section VI, finally, indicates a few applications of our asymptotic formulae, most of which will be taken up in subsequent papers [13),14)]. In Appendix A we propose another phase convention for a two-particle helicity state, and thus also for the helicity amplitudes, than the commonly used one of Jacob and Wick [1)], while Appendix B contains a direct proof of the crossing relations for helicity amplitudes, continued in the upper half s plane.

There might be readers who want to use the final results of this paper without following all steps in the derivation. For the benefit of those, we recommend the following procedure ; after a glance at the definitions and the notations in Section II, one might immediately go to the principal result, the asymptotic relations as given in Section IV, in particular Eqs. (IV.1-5). Section V may be consulted, in case of interest, to get the relations between the crossed reaction and the processes obtained from it by use of time-reversal or charge conjugation invariance.

II.- KINEMATICAL DEFINITIONS AND NOTATION

Consider the three two-particle reactions

$$a + c \rightarrow b + d \quad \text{(direct reactions, s channel),} \tag{II.1}$$

$$a + \bar{b} \rightarrow \bar{c} + d \quad \text{(t channel),} \tag{II.2}$$

$$\bar{b} + c \rightarrow \bar{a} + d \quad \text{(crossed reaction, u channel),} \tag{II.3}$$

where $\bar{a}$ denotes the antiparticle of particle a, etc. As it is indicated in Fig. 1, we let the mass and four-momentum (counted as ingoing for all four particles) be denoted by (m_a, A), (m_b, B), (m_c, C) and (m_d, D) for particles a, b, c and d respectively. All masses are assumed to be positive. From conservation of total four-momentum we have

$$A + B + C + D = 0. \tag{II.4}$$

The conventional invariant (Mandelstam) kinematical variables [15] are given as (our metric is + + + −)

$$s = -(A+C)^2 = -(B+D)^2 , \tag{II.5}$$

$$t = -(A+B)^2 = -(C+D)^2 , \tag{II.6}$$

$$u = -(A+D)^2 = -(B+C)^2 , \tag{II.7}$$

related by

$$s + u + t = m_a^2 + m_b^2 + m_c^2 + m_d^2 \equiv \sum m^2 , \tag{II.8}$$

since all particles are on their respective mass shell. Sometimes we shall find it more convenient to use, instead of s and u, the variables [3]

$$s_t = s + \frac{1}{2}t - \frac{1}{2}\sum m^2 , \tag{II.9}$$

$$u_t = u + \frac{1}{2}t - \frac{1}{2}\sum m^2 , \tag{II.10}$$

connected through the simple relation

$$s_t + u_t = 0 . \tag{II.11}$$

The physical four-momentum for particle a, i.e., its positive timelike momentum energy vector, in the s, t and u channel respectively is denoted by k_a, p_a and q_a, etc., for the other particles. Their relations to the momenta $A, \ldots, D$ are given in the following table.

Physical four-momentum of particle

Physical region of the	a	b	c	d	
s channel	$k_a = A$	$k_b = -B$	$k_c = C$	$k_d = -D$	
t channel	$p_a = A$	$p_b = B$	$p_c = -C$	$p_d = -D$	(II.12)
u channel	$q_a = -A$	$q_b = B$	$q_c = C$	$q_d = -D$	

This table says that, e.g., in the t channel we should in all above relations replace C by the negative of the physical momentum p_c for particle c.

Furthermore, in the centre of momentum system (c.m.s.) of each channel, we define a scattering angle through the equations

$$\text{s channel} \quad \cos\Theta_s = \frac{\vec{k}_c \cdot \vec{k}_d}{|\vec{k}_c|\cdot|\vec{k}_d|}, \quad \sin\Theta_s = \frac{(\vec{k}_c \times \vec{k}_d)\cdot\vec{n}_s}{|\vec{k}_c|\cdot|\vec{k}_d|}, \quad \vec{n}_s = \frac{\vec{k}_c \times \vec{k}_d}{|\vec{k}_c \times \vec{k}_d|}, \qquad \text{(II.13)}$$

$$\text{t channel}^{16)} \quad \cos\Theta_t = \frac{\vec{p}_a \cdot \vec{p}_c}{|\vec{p}_a|\cdot|\vec{p}_c|}, \quad \sin\Theta_t = -\frac{(\vec{p}_a \times \vec{p}_c)\cdot\vec{n}_t}{|\vec{p}_a|\cdot|\vec{p}_c|}, \quad \vec{n}_t = \frac{\vec{p}_a \times \vec{p}_c}{|\vec{p}_a \times \vec{p}_c|}, \qquad \text{(II.14)}$$

$$\text{u channel} \quad \cos\Theta_u = \frac{\vec{q}_c \cdot \vec{q}_d}{|\vec{q}_c|\cdot|\vec{q}_d|}, \quad \sin\Theta_u = \frac{(\vec{q}_c \times \vec{q}_d)\cdot\vec{n}_u}{|\vec{q}_c|\cdot|\vec{q}_d|}, \quad \vec{n}_u = \frac{\vec{q}_c \times \vec{q}_d}{|\vec{q}_c \times \vec{q}_d|}, \qquad \text{(II.15)}$$

where all three-momenta are referred to the respective c.m.s. Finally, we sometimes use E to denote the total energy in the c.m.s. for the two reactions (II.1 - II.3), so that $s = E^2$ in the s channel while in the u channel $E^2 = u$.

In the physical region of the s channel we have $s \geq \max\left[(m_a+m_c)^2, (m_b+m_d)^2\right]$, while the momentum transfers squared $-t$ and $-u$ vary, for fixed s, in certain intervals determined from the condition $0 \leq \theta_s \leq \pi$. In the same way, $t \geq \max\left[(m_a+m_b)^2, (m_c+m_d)^2\right]$ in the t channel where $-s$ and $-u$ are momentum transfers squared. Finally, in the u channel we have $u \geq \max\left[(m_a+m_d)^2, (m_b+m_c)^2\right]$ while $-s$ and $-t$ are momentum transfers squared.

As it is also exhibited in Fig. 1, we denote the spin and helicity of particle a by (s_a, λ_a), of its antiparticle by $(s_a, \lambda_{\bar{a}})$, etc., for the other particles. The expectation values of the transition matrix T between helicity states in the c.m.s. of the respective channels (c.m.s. helicity amplitudes) [1], as functions of the Mandelstam variables are denoted by

$$
\begin{aligned}
&\text{s channel} \quad G_{\lambda_b \lambda_d ; \lambda_a \lambda_c}(s,t,u) \, , \\
&\text{t channel} \quad F_{\lambda_{\bar{c}} \lambda_d ; \lambda_a \lambda_{\bar{b}}}(s,t,u) \, , \qquad \text{(II.16)} \\
&\text{u channel} \quad H_{\lambda_{\bar{a}} \lambda_d ; \lambda_{\bar{b}} \lambda_c}(s,t,u) \, .
\end{aligned}
$$

These helicity amplitudes are defined with respect to a co-ordinate system having its y axis parallel to respectively $\vec{n}_s$, $\vec{n}_t$ and $\vec{n}_u$ of Eqs. (II.13-15) ; our phase convention for helicity amplitudes differs from that of Jacob and Wick [1] and is given in Appendix A.

There are all together $(2\,s_a+1)\cdot(2\,s_b+1)\cdot(2\,s_c+1)\cdot(2\,s_d+1)$ helicity amplitudes in each channel. However, if some invariance principle, like space reflection (P), time reversal (T), charge conjugation (C) or identity of particles, is invoked, the number of independent amplitudes is often considerably less due to the symmetry properties of the helicity amplitudes, listed in Appendix A. In particular, we shall in this paper always assume parity conservation and make frequent use of Eq. (A.6) ; T invariance will also be assumed.

When it is unnecessary to specify explicitely all helicities, we shall denote them collectively by $\{\lambda\}$ and the helicity amplitudes by $G_{\{\lambda\}}$, etc. Moreover, we may use the relation (II.8) to reduce the number of independent Mandelstam variables to two, the particular choice being dictated by convenience in each case ; when there is no risk for confusion, we shall simply leave out the arguments. Occasionally, we shall use s_t and u_t as variables instead of s and u.

The normalization of the s channel c.m.s. helicity amplitudes is such that the s channel differential cross section, for unpolarized incident particles, is given by

$$\frac{d\sigma}{dt}(s,t) =$$

$$= \frac{1}{16\cdot\pi\cdot\lambda(s,m_a^2,m_c^2)}\cdot\frac{1}{(2s_a+1)\cdot(2s_c+1)}\cdot\sum_{\substack{\lambda_a\lambda_b\\ \lambda_c\lambda_d}}\left|G_{\lambda_b\lambda_d;\lambda_a\lambda_c}(s,t)\right|^2, \qquad \text{(II.17)}$$

where

$$\lambda(s,m_a^2,m_c^2) = (s-m_a^2-m_c^2)^2 - 4\cdot m_a^2\cdot m_c^2 \quad , \qquad \text{(II.18)}$$

arises from the flux factor. This normalization means that for scattering of spinless particles, the helicity amplitude equals the single invariant amplitude (except possibly for a phase). Analogously, the u channel differential cross section is given by

$$\frac{d\bar{\sigma}}{dt}(u,t) =$$

$$= \frac{1}{16\cdot\pi\cdot\lambda(u,m_b^2,m_c^2)}\cdot\frac{1}{(2s_b+1)\cdot(2s_c+1)}\cdot\sum_{\substack{\lambda_{\bar a}\lambda_{\bar b}\\ \lambda_c\lambda_d}}\left|H_{\lambda_{\bar a}\lambda_d;\lambda_{\bar b}\lambda_c}(u,t)\right|^2 . \qquad \text{(II.19)}$$

The t channel helicity amplitudes have a similar normalization.

III.- CROSSING RELATIONS FOR HELICITY AMPLITUDES

In this section we give a formulation of the crossing relations for helicity amplitudes, which is appropriate for the derivation of asymptotic formulae. We shall do this here by modifying those crossing relations, which already exist in the literature [7),8)].

In carrying out this modification, we follow throught the notation and method of the paper by Trueman and Wick [7)], to which we refer as T.W. These authors consider the s t crossing relations, i.e., they derive relations between the analytically continued s channel c.m.s. helicity amplitudes $G_{\{\lambda\}}$ and the t channel c.m.s. helicity amplitudes $F_{\{\lambda\}}$ in the physical region of the t channel. Although, for conventional reasons, we are more interested in the $s \leftrightarrow u$ crossing relations, we keep for the time being to this $s \leftrightarrow t$ crossing. The $s \leftrightarrow u$ crossing relations can readily be obtained at the end, essentially by relabelling the particles.

Since the final result depends critically on the path of continuation, we start by recalling how T.W. make the analytic continuation of the amplitudes $G_{\{\lambda\}}$. Their path starts in the physical region of the s channel, approaching the real s axis from above (outgoing wave condition). Since u, although allowed to vary, always stays real along the path, the variable t starts off approaching a point of the negative t axis from below according to the constraint (II.8). Furthermore, the T.W. path intersects the real s and t axes (better : the real s t u plane) and ends up in the physical region of the t channel approaching the real t axis from above (outgoing wave condition in the t channel). Consequently, the variable s, which in general is negative in the physical region of the t channel, approaches the real axis from below. Although it would require at least a three-dimensional diagram to picture this path, it suffices for our purpose to indicate it as path T.W. in the complex s plane of Fig. 2 (in the case of $m_a=m_b=\mu$, $m_c=m_d=m$) ; a more correct picture is given in Fig. T.W. 1.

The T.W. path intersects the real $s\ t\ u$ plane (the real s axis in our Fig. 2) at a point such that the path does not end up on an unphysical Riemann sheet ; this is discussed at some length in T.W. We now argue that at this point, and in a region of the real $s\ t\ u$ plane around it, the helicity amplitudes for parity-conserving reactions are either all real or all purely imaginary, depending on whether the relative parities in the initial and final states are the same or different. This is immediately clear for pion-nucleon scattering, treated explicitly in T.W. A proof, valid for parity-conserving processes in general, can be based on the expressions for the helicity amplitudes in terms of invariant amplitudes (see, e.g., the discussion resulting in Eq. (3.2) of Ref. [17]), combined with the observation that the intersection point (tacitly) is chosen in a region where the scattering angle θ_s is real but the s channel c.m.s. momenta $\sqrt{\vec{k}_a^2}$ and $\sqrt{\vec{k}_b^2}$ are purely imaginary [18].

To summarize, along the path T.W. all three variables s, t and u vary, subject to the constraint (II.8), but only s and t take complex values. At the (unphysical) point of intersection with the real $s\ t\ u$ plane, the helicity amplitudes are either all purely imaginary or all real.

The resulting $s \leftrightarrow t$ crossing relations between the analytically continued s channel c.m.s. helicity amplitudes $G_{\{\lambda\}}$ and the physical t channel c.m.s. helicity amplitudes $F_{\{\lambda\}}$ read

$$G_{\lambda_b \lambda_d ; \lambda_a \lambda_c}(s,u) \doteq$$
$$\doteq (-)^{\lambda_a - \lambda_b} \sum_{\substack{\lambda'_a \lambda_{\bar{b}} \\ \lambda_{\bar{c}} \lambda'_d}} (-)^{\lambda_{\bar{c}} - \lambda'_a} \cdot d^{s_a}_{\lambda'_a \lambda_a}(\psi_a) \cdot d^{s_b}_{\lambda_{\bar{b}} \lambda_b}(\psi_b) \cdot$$
$$\cdot d^{s_c}_{\lambda_{\bar{c}} \lambda_c}(\psi_c) \cdot d^{s_d}_{\lambda'_d \lambda_d}(\psi_d) \cdot F_{\lambda_{\bar{c}} \lambda'_d ; \lambda'_a \lambda_{\bar{b}}}(t,u) \qquad \text{(III.1)}$$

Here we have introduced the symbol "$\doteq$", which we shall often use, to mean "equal apart from a phase factor, which only depends on the spins and the intrinsic parities but not on the helicities" ; see remark 3 at the end of this Section. Furthermore, the crossing angles ψ_a, etc., entering in the rotation coefficients $d^{s_a}_{\lambda'_a \lambda_a}$, etc., are given in Eqs. (T.W. 42-43) with the following identification :

T.W. notation	ψ_1	ψ_2	χ_1	χ_2	μ_1	μ_2	m_1	m_2	Θ
Our notation	ψ_a	ψ_b	ψ_c	ψ_d	m_a	m_b	m_c	m_d	Θ_t

Equation (III.1) differs from Eq. (T.W. 41), with respect to the phases (apart from the notation). This has the following explanation. Since in our future applications we shall need the $s \leftrightarrow t$ crossing relations as well as the $s \leftrightarrow u$ ones, we found it inconvenient always to have to specify which particles are chosen as "particles 2" for each particular reaction, as is necessary if one adopts the T.W. phase convention, being the same as in the original work by Jacob and Wick [1]. We therefore decided to omit the phase factor $(-)^{s_2-\lambda_2}$ in the definition of a two-particle helicity state in Eqs. (13-14) of Ref. [1]. It is this new phase convention which causes the difference between our Eq. (III.1) and Eq. (T.W. 41). The further consequences of our redefinition of the phase are listed in Appendix A.

So far we have only reviewed the treatment in T.W. There are mainly two reasons why we have to modify these crossing relations :

a) We want one of the kinematical variables (the variable u in the $s \leftrightarrow t$ crossing) to stay constant along the path. This is impossible in the T.W. prescription, since for large enough values of u, the path would end up on an unphysical Riemann sheet.

b) For the application of the Phragmén-Lindelöf theorem [2),3),4),5),20)], which uses analyticity in the upper half complex s plane to relate the boundary values of an analytic function [(= one of the

helicity amplitudes, which from a) now is a function only of s]
on the positive and negative real axis approached from above, the path of continuation must stay in the upper half s plane.

In summary, we need the continuation for constant u along the path B.S. of Fig. 2.

To derive the crossing relations for this path B.S. continuation from the T.W. relations, we need a connection between the boundary values of $G_{\{\lambda\}}$ on the negative real s axis, approached from above and from below (cf. Fig.2) ; for the time being we neglect the condition a) above and suppose that u varies in the same way along path B.S. as along path T.W. Now, from the previous discussion, the functions $G_{\{\lambda\}}$ (or the functions $i \cdot G_{\{\lambda\}}$) are all real at that point, and in its real neighbourhood, where the T.W. path intersects the real axis. For the sake of argument we assume $G_{\{\lambda\}}$ to be real [21]. Then we conclude immediately that all $G_{\{\lambda\}}$ are real-analytic functions on the physical Riemann sheet, i.e., that for arbitrary complex s and u on this sheet

$$\left[G_{\{\lambda\}}(s^*, u^*)\right]^* = G_{\{\lambda\}}(s, u) , \qquad \text{(III.2)}$$

where a star denotes complex conjugation. In particular, for u real and $\varepsilon > 0$ tending to zero

$$\left[G_{\{\lambda\}}(s + i\varepsilon, u)\right]^* = G_{\{\lambda\}}(s - i\varepsilon, u) , \qquad \text{(III.2')}$$

which is the desired connection. We find therefore that the crossing relations for the helicity amplitudes with the analytic continuation along the path B.S. are obtained from the T.W. relations (III.1) simply by replacing $G_{\{\lambda\}}(s,u)$ on the left hand side by $G^*_{\{\lambda\}}(s,u)$.

Finally, the condition a) for the path B.S. may be restored, since for the continuation of s in the upper half plane nothing prevents u from staying constant.

Instead of deriving the crossing relations for the helicity amplitudes in our conventions via the T.W. crossing relations, one may of course redo the whole proof of T.W. using our conventions. Appendix B indicates this derivation.

The crossing relations are now essentially in a form suitable for our applications. However, we shall in fact need them in our conventions for the $s \leftrightarrow u$ crossing. This modification is mainly a matter of terminology although some care concerning the phase factors is required ; observe in this context our definitions (II.13-15). The resulting crossing relations between the s channel c.m.s. helicity amplitudes $G_{\{\lambda\}}$, ccontinued along path B.S. of Fig. 2, i.e., in the upper half s plane now for constant value of t [22], and the physical u channel c.m.s. helicity amplitudes $H_{\{\lambda\}}$, read [23]

$$G^{*}_{\lambda_b \lambda_d ; \lambda_a \lambda_c}(s,t) \doteq$$

$$\doteq (-)^{\lambda_a - \lambda_d} \cdot \sum_{\substack{\lambda_{\bar{a}} \lambda_{\bar{b}} \\ \lambda_c' \lambda_d'}} (-)^{\lambda_{\bar{b}} - \lambda_d'} \cdot d^{s_a}_{\lambda_{\bar{a}} \lambda_a}(\bar{\psi}_a) \cdot d^{s_b}_{\lambda_{\bar{b}} \lambda_b}(\bar{\psi}_b) \cdot$$

$$\cdot d^{s_c}_{\lambda_c' \lambda_c}(\bar{\psi}_c) \cdot d^{s_d}_{\lambda_d' \lambda_d}(\bar{\psi}_d) \cdot H_{\lambda_{\bar{a}} \lambda_d' ; \lambda_{\bar{b}} \lambda_c'}(u,t) . \qquad \text{(III.3)}$$

The crossing angles here are given by [observe the notation (II.15) and (II.18)].

$$\lambda^{1/2}(s, m_a^2, m_c^2) \cdot \lambda^{1/2}(u, m_a^2, m_d^2) \cdot \cos \bar{\psi}_a = \\ = -(s + m_a^2 - m_c^2) \cdot (u + m_a^2 - m_d^2) + 2 m_a^2 (m_a^2 + m_b^2 - m_c^2 - m_d^2) \, , \tag{III.4-a}$$

$$\lambda^{1/2}(s, m_b^2, m_d^2) \cdot \lambda^{1/2}(u, m_b^2, m_c^2) \cdot \cos \bar{\psi}_b = \\ = -(s + m_b^2 - m_d^2)(u + m_b^2 - m_c^2) + 2 m_b^2 \cdot (m_a^2 + m_b^2 - m_c^2 - m_d^2) \, , \tag{III.4-b}$$

$$\lambda^{1/2}(s, m_c^2, m_a^2) \cdot \lambda^{1/2}(u, m_c^2, m_b^2) \cdot \cos \bar{\psi}_c = \\ = (s + m_c^2 - m_a^2) \cdot (u + m_c^2 - m_b^2) + 2 m_c^2 \cdot (m_a^2 + m_b^2 - m_c^2 - m_d^2) \, , \tag{III.4-c}$$

$$\lambda^{1/2}(s, m_d^2, m_b^2) \cdot \lambda^{1/2}(u, m_d^2, m_a^2) \cdot \cos \bar{\psi}_d = \\ = (s + m_d^2 - m_b^2) \cdot (u + m_d^2 - m_a^2) + 2 m_d^2 \cdot (m_a^2 + m_b^2 - m_c^2 - m_d^2) \, , \tag{III.4-d}$$

$$[u \cdot \lambda(s, m_a^2, m_c^2)]^{1/2} \cdot \sin \bar{\psi}_a = m_a \cdot \lambda^{1/2}(u, m_b^2, m_c^2) \cdot \sin \Theta_u \, , \tag{III.5-a}$$

$$[u \cdot \lambda(s, m_b^2, m_d^2)]^{1/2} \cdot \sin \bar{\psi}_b = m_b \cdot \lambda^{1/2}(u, m_a^2, m_d^2) \cdot \sin \Theta_u \, , \tag{III.5-b}$$

$$[u \cdot \lambda(s, m_c^2, m_a^2)]^{1/2} \cdot \sin \bar{\psi}_c = m_c \cdot \lambda^{1/2}(u, m_d^2, m_a^2) \cdot \sin \Theta_u \, , \tag{III.5-c}$$

$$[u \cdot \lambda(s, m_d^2, m_b^2)]^{1/2} \cdot \sin \bar{\psi}_d = m_d \cdot \lambda^{1/2}(u, m_c^2, m_b^2) \cdot \sin \Theta_u \, . \tag{III.5-d}$$

The high energy limit of the expressions, which we need in the next Section, is readily derived.

The crossing relations (III.3) constitute the main result of this Section. We finally add a few remarks on their derivation, before we apply them in the next Section to obtain asymptotic formulae.

1) Our first remark concerns the analyticity properties of helicity amplitudes [17),24)]. These amplitudes may be considered as linear functions of certain invariant amplitudes (cf. for instance the treatment of pion-nucleon scattering in Appendix B, subsection c), assumed to be free from kinematical singularities. The coefficients in front of the invariant amplitudes are known functions of the variables s, t and u. But even if the invariant amplitudes have a simple singularity structure, as given, e.g., by the Mandelstam representation [15)], the singularity structure of the helicity amplitudes is in general very complicated. This forbids our writing down any kind of simple integral representation for the helicity amplitudes. However, it does not prevent us from using assumed or real knowledge on the analyticity domain of the invariant functions to establish an analyticity domain for the helicity amplitudes, the physical values of which are obtained as the values on certain parts of the boundary of this domain. It is only this fact we have used above.

2) In the T.W. derivation, the Mandelstam representation [15)] for the invariant amplitudes was assumed. In our approach, resulting in Eq. (III.3), we may relax upon this condition slightly. What we need is the analyticity of the helicity amplitudes, for fixed t, in a domain which is the upper half s plane minus any bounded region of it ; in particular we require the analyticity domain to be bounded by the positive and negative real axis outside a sufficiently large semi-circle centred at the origin. We note in this context that for scattering of spinless particles such an analyticity structure has in fact been derived from field theory [25)].

3) In the T.W. proof of the crossing relations there remains an over-all, helicity-independent phase factor undetermined, symbolized in our equations by the sign $\doteq$. This is so because in the actual proof the step taken from Eqs. (T.W.29-30) to Eq.(T.W.31) (in our Appendix B from Eqs.(B 16-17) to Eq.(B 18)) simply cannot give the phase. In fact, from this proof, one cannot even decide whether the unknown factor has modulus one ! This drawback can be overcome by observing that Eq.(T.W.31), or our Eq.(B 18), is nothing but a substitution law for helicity amplitudes. Assuming necessary analyticity properties, such a law can be proven, including all factors, using reduction technique ; in particular the proof shows that the unknown factor necessarily has modulus one. Concerning the ambiguity in its phase we shall not be concerned with this question here, however important it may be from a fundamental point of view, since in the actual applications this ambiguity either does not matter or can be settled for each particular reaction from other arguments.

IV.- ASYMPTOTIC RELATIONS BETWEEN THE HELICITY AMPLITUDES IN THE DIRECT AND CROSSED REACTION

In the preceding Section we have derived the crossing relations for the helicity amplitudes. We shall now use them in order to obtain asymptotic relations between these amplitudes. The assumptions we need are standard [2),3),4),5)] : one requires the amplitudes not to oscillate for energy tending to infinity in the physical regions of the s and u channel, and also that they are bounded by a power of s in the upper half s plane. More precisely, we can list the assumptions in the following form :

Assumption I : The helicity amplitudes are, for t fixed in the physical region, analytic functions in the upper half s plane, obeying crossing relations given by Eq. (III.3) of the preceding Section. Furthermore, they are continuous along the real axis and bounded by a power of s in the upper half s plane.

Assumption II : For t fixed in the physical region and the energy tending to infinity, the helicity amplitudes for the direct and crossed reaction can be represented by asymptotic series of the form

$$G_{\{\lambda\}}(s_t,t) = \sum_{j=0}^{\infty} g_{\{\lambda\}}^{(j)} \cdot s_t^{\alpha_j} \cdot \left[\log s_t - \frac{i\pi}{2}\right]^{\beta_j} \cdot \left[\log\left(\log s_t - \frac{i\pi}{2}\right)\right]^{\gamma_j} \cdots \quad \text{(IV.1)}$$

$$H_{\{\lambda\}}(u_t,t) = \sum_{j=0}^{\infty} h_{\{\lambda\}}^{(j)} \cdot u_t^{\alpha_j} \cdot \left[\log u_t - \frac{i\pi}{2}\right]^{\beta_j} \cdot \left[\log\left(\log u_t - \frac{i\pi}{2}\right)\right]^{\gamma_j} \cdots \quad \text{(IV.2)}$$

We prefer here the variables s_t and u_t [see Eqs. (II.9-10) for their definitions] to s and u in order to simplify some of the subsequent equations, the point being that the condition (II.11) implies the very simple transformation $s_t \to u_t = -s_t$ under crossing $s \leftrightarrow u$. In Eqs. (IV.1-2) we assume the functions

$$\alpha_j = \alpha_j(t)\,,\quad \beta_j = \beta_j(t)\,,\quad \gamma_j = \gamma_j(t)\,,\ \text{-----}\,,$$

all to be real, implying that the amplitudes $G_{\{\lambda\}}$ and $H_{\{\lambda\}}$ do not oscillate as the energy tends to infinity. The functions

$$g^{(j)}_{\{\lambda\}} = g^{(j)}_{\{\lambda\}}(t)\,,\quad h^{(j)}_{\{\lambda\}} = h^{(j)}_{\{\lambda\}}(t)\,,$$

may, in general, be complex. Finally, we assume

$$\alpha_j \geq \alpha_{j+1}\ ; \tag{IV.3-a}$$

if $\alpha_j = \alpha_{j+1}$ we further require

$$\beta_j \geq \beta_{j+1}\,, \tag{IV.3-b}$$

and, if $\alpha_j = \alpha_{j+1}$, $\beta_j = \beta_{j+1}$, then

$$\gamma_j \geq \gamma_{j+1}\,, \tag{IV.3-c}$$

etc.

Under these assumptions one is now able to prove that for all j obeying the inequality

$$\alpha_0 - \alpha_j < 1\,, \tag{IV.4}$$

there exist, for parity conserving reactions, the following simple relations between the expansion coefficients

$$g^{(j)}_{\{\lambda\}} \text{ and } h^{(j)}_{\{\lambda\}}$$

of Eqs. (IV.1-2)

$$g^{(j)}_{\lambda_b\lambda_d;\lambda_a\lambda_c} \doteq h^{(j)\,*}_{-\lambda_a\lambda_d;-\lambda_b\lambda_c} \cdot e^{-i\pi\alpha_j} \quad \text{if } \alpha_0-\alpha_j < 1 \; . \qquad \text{(IV.5)}$$

The proof of the formulae (IV.5) proceeds along very similar lines as the proof for the spinless case [2),3),4),5)]. Therefore we do not give it here. We emphasize that the relations (IV.5) are valid only under the condition (IV.4). For higher terms in the expansion corresponding relations still do exist but they are more complicated. The reason for this is that the crossing relations for the helicity amplitudes are diagonal only if Eq. (IV.4) is fulfilled ; otherwise they mix together different helicity amplitudes. Probably, a more suitable choice of amplitudes could simplify the formulae also in higher approximations.

V.- SOME APPLICATIONS OF TIME REVERSAL AND CHARGE CONJUGATION INVARIANCE

The asymptotic formulae derived in Section 4 relate the helicity amplitudes at high energy for the direct reaction (II.1) and the crossed reaction (II.2). By invoking invariance under time reversal, we may relate the helicity amplitudes

$$H^{(T)}_{\{\lambda\}} = H^{(T)}_{\lambda_{\bar{b}}\lambda_c;\,\lambda_{\bar{a}}\lambda_d}$$

for the process

$$\bar{a} + d \to \bar{b} + c \qquad \text{(time-reversed crossed reaction)} \tag{V.1}$$

to the helicity amplitudes

$$H_{\{\lambda\}} = H_{\lambda_{\bar{a}}\lambda_d;\,\lambda_{\bar{b}}\lambda_c}$$

for the crossed reaction (II.2) ; application of Eq. (A.9) gives immediatly

$$H^{(T)}_{\lambda_{\bar{b}}\lambda_c;\,\lambda_{\bar{a}}\lambda_d} = (-)^{\lambda_{\bar{b}}-\lambda_c-\lambda_{\bar{a}}+\lambda_d} \cdot H_{\lambda_{\bar{a}}\lambda_d;\,\lambda_{\bar{b}}\lambda_c} . \tag{5.2}$$

Together with the asymptotic formulae (IV.1-5), this equation establishes a connection between the helicity amplitudes for the reactions (II.1) and (V.1) at high energy.

In an analogous way one may derive relations between the helicity amplitudes for the crossed reaction and the amplitudes
$H^{(C)}_{\{\lambda\}} = H^{(C)}_{\lambda_a\lambda_{\bar{d}};\,\lambda_b\lambda_{\bar{c}}}$

for the reaction

$$b + \bar{c} \rightarrow a + \bar{d} \qquad \text{(charge-conjugated crossed reaction),} \tag{V.3}$$

if charge conjugation invariance is valid. From Eq. (A.11) we get

$$H^{(C)}_{\lambda_a \lambda_{\bar{d}}\,;\,\lambda_b \lambda_{\bar{c}}} = \gamma_a \cdot \gamma_b \cdot \gamma_c \cdot \gamma_d \cdot H_{\lambda_{\bar{a}} = \lambda_a,\, \lambda_d = \lambda_{\bar{d}}\,;\,\lambda_{\bar{b}} = \lambda_b,\, \lambda_c = \lambda_{\bar{c}}} \,, \tag{V.4}$$

where γ_a is the charge parity of particle a (or $\bar{a}$), etc., for the other particles.

Finally, invariance under C.P.T., i.e., combination of charge conjugation, parity and time-reversal, may in an obvious way be used to get relations between the helicity amplitudes for the reaction

$$a + \bar{d} \rightarrow b + \bar{c} \qquad \text{(C.P.T. reversed crossed reaction),} \tag{V.5}$$

and those of the crossed reaction.

VI.- SOME APPLICATIONS OF THE ASYMPTOTIC FORMULAE

So far we have developed a rather intricate mathematical formalism. What, then, have we gained from this work in terms of physical insight ? We would like to end this paper by pointing out in answer to this question, some applications of the derived formulae. However, the detailed applications, studying particular reactions, require papers of their own and will therefore be postponed to subsequent publications, in which we shall treat pion-nucleon elastic and charge-exchange scattering [13] and vector meson production in meson-nucleon collision [14].

The first and obvious conclusions one can draw from the asymptotic relations between the helicity amplitudes for the direct and crossed reactions as given in Eqs. (IV.1-5) is the asymptotic connection between the differential cross-sections for the two processes. From the definitions (II.17,19) we get immediatly

$$\lim_{\substack{s \to +\infty \\ t \text{ fixed}}} (2s_a+1) \cdot \frac{d\sigma}{dt}(s,t) = \lim_{\substack{s \to +\infty \\ t \text{ fixed}}} (2s_b+1) \cdot \frac{d\bar{\sigma}}{dt}(s,t) \quad . \qquad \text{(VI.1)}$$

This is then the generalization of the Pomeranchuk theorem [6] to the differential cross-section in collisions of hadrons with arbitrary spins. By applying the relations (V.2) and (V.4), further similar relations are easily derived.

Besides the asymptotic relations for the cross-sections, one may also be interested in analogous relations for different kinds of polarization parameters. All such measurements are most conveniently expressed in terms of spin space density matrices, in our approach referred to the helicity basis. To give an example of the procedure, let us compare the helicity density matrix $\rho_{\lambda_d \lambda_d'}$ for particle d in the direct reaction with the corresponding quantity $\bar{\rho}_{\lambda_d \lambda_d'}$ in the crossed reaction, assuming the initial states to be unpolarized and parity to be conserved. From their definitions these density matrices are, if we impose the normalization that the traces should be unity, given by

$$I_0 \cdot \rho_{\lambda_d \lambda_d'} = \sum_{\lambda_a \lambda_b \lambda_c} G_{\lambda_b \lambda_d ; \lambda_a \lambda_c} \cdot G^*_{\lambda_b \lambda_d' ; \lambda_a \lambda_c} , \qquad \text{(VI.2-a)}$$

$$I_0 = \sum_{\substack{\lambda_a \lambda_b \\ \lambda_c \lambda_d}} \left| G_{\lambda_b \lambda_d ; \lambda_a \lambda_c} \right|^2 , \qquad \text{(VI.2-b)}$$

$$\bar{I}_0 \cdot \bar{\rho}_{\lambda_d \lambda_d'} = \sum_{\lambda_{\bar{a}} \lambda_{\bar{b}} \lambda_c} H_{\lambda_{\bar{a}} \lambda_d ; \lambda_{\bar{b}} \lambda_c} \cdot H^*_{\lambda_{\bar{a}} \lambda_d' ; \lambda_{\bar{b}} \lambda_c} , \qquad \text{(VI.3-a)}$$

$$\bar{I}_0 = \sum_{\substack{\lambda_{\bar{a}} \lambda_{\bar{b}} \\ \lambda_c \lambda_d}} \left| H_{\lambda_{\bar{a}} \lambda_d ; \lambda_{\bar{b}} \lambda_c} \right|^2 . \qquad \text{(VI.3-b)}$$

In the very high energy limit, only the first terms in the expansions (IV.1-2) need to be considered and they give limiting values to ρ, $\bar{\rho}$ which satisfy, from (IV.5)

$$\lim_{E \to \infty} \rho^*_{\lambda_d \lambda_d'} = \lim_{E \to \infty} \bar{\rho}_{\lambda_d \lambda_d'} . \qquad \text{(VI.4)}$$

In particular, the polarizations P_n and $\bar{P}_n$ along the scattering plane normal, defined in Eqs. (II.13,15), are given in terms of the density matrices by [1)]

$$P_n = \sum_{\lambda_d} \left[(s_d + \lambda_d)\cdot(s_d - \lambda_d + 1)\right]^{1/2} \cdot \operatorname{Im}\left(\rho_{\lambda_d - 1, \lambda_d}\right) , \tag{VI.5}$$

$$\bar{P}_n = \sum_{\lambda_d} \left[(s_d + \lambda_d)\cdot(s_d - \lambda_d + 1)\right]^{1/2} \cdot \operatorname{Im}\left(\bar{\rho}_{\lambda_d - 1, \lambda_d}\right) , \tag{VI.6}$$

from which we conclude

$$\lim_{E\to\infty} P_n = - \lim_{E\to\infty} \bar{P}_n \ . \tag{VI.7}$$

In a similar way any kind of polarization measurements may be discussed and possible connections between them for the crossed and direct reactions may be established.

It should be observed that the relations of this section are all independent of the over-all phase in the crossing relations. Moreover, we emphasize that the connections (VI.6, 4 and 7) are derived taking into account only the first term in each of the expansions (IV.1-2). In the present day experimental situation, this does not seem to be a good approximation [4),10)]. On the contrary, at least a second term should in general also be taken into account. It is in this respect that we hope our formalism to be most useful. In our opinion, the simple parametrization, in terms of the parameters α (maybe also β, γ, ...) g and h of Eqs. (IV.1-2), connected through the equalities (IV.5), is the main domain of application of our formulae. We shall illustrate this point in the forthcoming publications already referred to [13),14)].

ACKNOWLEDGEMENTS

The authors would like to thank Professors L. Van Hove and C.G. Wick as well as Drs. A. Kotanski, J. Namysłowski and R. Stora for very helpful discussions.

We would like also to express our gratitude to Professors Ch. Peyrou and L. Van Hove for kind hospitality extended to us in the CERN Track Chamber (A.B.) and Theoretical Study (B.S.) Divisions.

APPENDIX A

SOME CONSEQUENCES OF OUR PHASE CONVENTION FOR A TWO-PARTICLE HELICITY STATE

Since for reasons given in Section III we deviate from the generally accepted phase convention of Jacob and Wick [1] (J.W.) for a two-particle helicity state, we list here the formulae of J.W. for which our convention introduces deviations. Moreover, we add the transformation property of a two-particle state, in particular a particle-antiparticle state, under charge conjugation. To make the comparison with J.W. as easy as possible, we use in this Appendix the notation of J.W. throughout.

The J.W. Convention	Our Convention

a) Two-particle state

The J.W. Convention	Our Convention	
(J.W. 13-14) $\psi_{p\lambda_1\lambda_2} = \psi_{p\lambda_1} \cdot (-)^{s_2-\lambda_2} \cdot e^{-i\pi J_y^{(2)}} \psi_{p\lambda_2}$)	$\psi_{p\lambda_1\lambda_2} = \psi_{p\lambda_1} \cdot e^{-i\pi J_y^{(2)}} \psi_{p\lambda_2}$.	(A.1)

b) Helicity amplitude

Denoting the helicity amplitudes for the reaction $a + b \rightarrow c + d$ by

The J.W. Convention	Our Convention	
(J.W. 26-31) $f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta,\varphi)$,	$F_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta,\varphi)$,	(A.2)

we have

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b} = (-)^{s_d-\lambda_d} \cdot (-)^{s_b-\lambda_b} \cdot F_{\lambda_c\lambda_d;\lambda_a\lambda_b} \qquad \text{(A.3)}$$

In general, we shall assume the scattering plane to be the x-z plane with the y axis parallel to $\vec{p}_a \times \vec{p}_c$, the cross product of the c.m.s. momenta for particle a and c, as we have indicated in Section II. This implies that we have chosen $\varphi = 0$ in Eq. (A.2).

The subsequent formulae in J.W. up to and including (J.W. 39') stay unchanged, since they do not make explicit use of the phase convention.

c) Parity conservation

(J.W. 40') $$P\,\psi_{p\lambda_1\lambda_2} = \eta_1\eta_2 (-)^{s_1+s_2-\lambda_1+\lambda_2} \cdot e^{i\pi J_y}\,\psi_{p,-\lambda_1,-\lambda_2}, \qquad \Big| \qquad P\,\psi_{p\lambda_1\lambda_2} = \eta_1\eta_2 (-)^{s_1-s_2-\lambda_1+\lambda_2} \cdot e^{i\pi J_y}\,\psi_{p,-\lambda_1,-\lambda_2}, \qquad \text{(A.4)}$$

This means a different over-all phase $(-)^{2s_2}$; the only essential change, besides obvious replacements in (J.W. 40-41), is thus that η_g of (J.W. 43) has to be redefined :

(J.W. 43) $$\eta_g = \frac{\eta_c\cdot\eta_d}{\eta_a\cdot\eta_b}\cdot(-)^{s_c+s_d-s_a-s_b}, \qquad \Big| \qquad \eta_g = \frac{\eta_c\cdot\eta_d}{\eta_a\cdot\eta_b}\cdot(-)^{s_c-s_d-s_a+s_b}. \qquad \text{(A.5)}$$

For $\varphi = 0$, i.e., the scattering plane being the x-z plane, the implication of space reflection invariance is thus

(J.W. 44) $$f_{-\lambda_c,-\lambda_d;-\lambda_a,-\lambda_b}(\Theta,\varphi=0) = \eta_g\,(-)^{\lambda_c-\lambda_d-\lambda_a+\lambda_b}\cdot f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\Theta,\varphi=0), \qquad \Big| \qquad F_{-\lambda_c,-\lambda_d;-\lambda_a,-\lambda_b}(\Theta,\varphi=0) = \eta_g\cdot(-)^{\lambda_c-\lambda_d-\lambda_a+\lambda_b}\cdot F_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\Theta,\varphi=0). \qquad \text{(A.6)}$$

The subsequent formulae up to and including (J.W. 44") remain unaltered.

d) Identical particles

(J.W. 45) $P_{12}\psi_{p\lambda_1\lambda_2} = (-)^{2s-\lambda_1+\lambda_2} \cdot e^{i\pi J_y}\psi_{p\lambda_2\lambda_1}$,

$P_{12}\psi_{p\lambda_1\lambda_2} = (-)^{2s} \cdot e^{i\pi J_y}\psi_{p\lambda_2\lambda_1}$, (A.7)

(J.W. 46) $P_{12}|J,M;\lambda_1\lambda_2\rangle = (-)^{J-2s} \cdot |J,M;\lambda_2\lambda_1\rangle$,

$P_{12}|J,M;\lambda_1\lambda_2\rangle = (-)^{J-2s+\lambda_1-\lambda_2} \cdot |J,M;\lambda_2\lambda_1\rangle$. (A.8)

There is also a corresponding change in (J.W. 47-48).

e) Time reversal

The Eqs. (J.W. 49-55) stay unchanged. For completeness we quote here the implication of time reversal invariance for the relation between the helicity amplitudes $F_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ for the reaction $a + b \rightarrow c + d$ and the amplitudes $F^{(T)}_{\lambda_a\lambda_b;\lambda_c\lambda_d}$ for the time-reversed reaction $c + d \rightarrow a + b$ [cf. Eq. (J.W. 55)], if time reversal holds :

$$F^{(T)}_{\lambda_a\lambda_b;\lambda_c\lambda_d} = (-)^{\lambda_c-\lambda_d-\lambda_a+\lambda_b} \cdot F_{\lambda_c\lambda_d;\lambda_a\lambda_b} , \qquad (A.9)$$

a result which is equally valid in the J.W. convention. We have here adopted what we think is the most consistent definition, namely to define the y axis in the c.m.s. of the reaction $c + d \rightarrow a + b$ to be parallel to $\vec{p}_c \times \vec{p}_a$, i.e., opposite to the y axis for the reaction $a + b \rightarrow c + d$.

f) Charge conjugation

With the antiparticle of particle 1 denoted by $\bar{1}$, etc., the effect of charge conjugation C is, for both phase conventions,

$$C\,\psi_{p\lambda_1\lambda_2}(1,2) = \gamma_1\cdot\gamma_2\cdot\psi_{p\lambda_1\lambda_2}(\bar{1},\bar{2})\,, \tag{A.10}$$

where γ_1 and γ_2 are the charge parities of particles 1 and 2, respectively. In particular, if $2=\bar{1}$, Eq. (A.10) reads

$$\begin{aligned} C\,\psi_{p\lambda_1\lambda_{\bar{1}}}(1,\bar{1}) &= \psi_{p\lambda_1\lambda_{\bar{1}}}(\bar{1},1) = \\ &= (-)^{2s_1}\,P_{1,\bar{1}}\,\psi_{p\lambda_{\bar{1}}\lambda_1}(1,\bar{1})\,, \end{aligned} \tag{A.11}$$

where s_1 is the spin of particle 1 and where $P_{1,\bar{1}}$ is the operator interchanging particles 1 and $\bar{1}$. The corresponding transformation of the partial wave helicity states is readily obtained [cf. Eqs. (A.7-8)].

APPENDIX B

CROSSING RELATIONS FOR HELICITY AMPLITUDES CONTINUED ALONG A PATH IN THE UPPER HALF s PLANE

We give here a direct derivation of the crossing relations with our conventions for the analytical continuation, i.e., having the path entirely in the upper half s plane and having u fixed (in the $s \to t$ crossing), as explained in Section III. The proof is copied from the proof of Trueman and Wick [7] (T.W.) ; to facilitate comparison with T.W. we adopt throughout this Appendix the notation of T.W. Only those points in the derivation, for which our convention gives rise to changes, will be indicated.

In the formulae below it must also be noted that our choice of phase for a two-particle helicity state (see Appendix A) deviates from the one of T.W., the latter being the same as that introduced by Jacob and Wick [1].

As in T.W., the only case we shall treat explicitly is the one in which particles a and b of Fig. 1 are spinless ; moreover, the masses are assumed to be pairwise equal : $m_a = m_b = \mu$, $m_c = m_d = m$. The generalization to all particles having spins and to unequal masses should be obvious.

a) Relation between our notation and that of T.W.

Besides the relations already given in connection with Eq. (III.1), we have

T.W. notation	α	β	λ	μ	σ_1	σ_2	s_1	s_2	q_1	q_2	p_1	p_2
Our notation	λ_a	λ_b	λ_c	λ_d	s_a	s_b	s_c	s_d	k_a	k_b	k_c	k_d

T.W. notation	$q_1' = q_1$	$q_2' = -Q_2$	$p_1' = -P_1$	$p_2' = p_2$	$\mathcal{S}_1^2$	$\mathcal{S}_2^2$
Our notation	p_a	$-p_b$	$-p_c$	p_d	$\lambda(s, m_a^2, m_c^2)$	$\lambda(s, m_b^2, m_d^2)$

b) Continuation of kinematical quantities

Defining all roots of positive numbers to be positive and letting an arrow provisionally mean "continued along the path B.S. of Fig. 2 to the end point" (note that our convention is to have u fixed along the path), we find

$$\sqrt{s} \to i \cdot \sqrt{-s} \; ; \; \sqrt{-t} \to i \cdot \sqrt{t} , \tag{B.1}$$

$$\sqrt{-s \cdot t} \to - \sqrt{-s \cdot t} , \tag{B.2}$$

$$\mathcal{S} \to - \mathcal{S} , \tag{B.3}$$

$$\sqrt{\vec{p}_1^{\,2}} = \sqrt{\varepsilon_1^2 - m^2} \to - \sqrt{\vec{P}_1^{\,2}} , \tag{B.4}$$

$$\sin \Theta_{s/2} \to \frac{\sqrt{-s \cdot t}}{\mathcal{S}} , \tag{B.5}$$

$$\sin \Theta_s \to 4 \cdot i \cdot \frac{\sqrt{-s \cdot t} \cdot p \cdot q \cdot \sin \Theta_t}{\mathcal{S}^2} , \tag{B.6}$$

$$\cos \Theta_{s/2} \to 2 \cdot i \cdot \frac{p \cdot q \cdot \sin \Theta_t}{\mathcal{S}} . \tag{B.7}$$

Here, Eq. (B.4) refers to the discussion in T.W. of Eqs. (T.W. 23-24).

To arrive at the correct signs in Eqs. (B.6-7) it is essential to observe that $\emptyset$ of Eq. (T.W. A.3) is defined entirely in terms of the vector components, the continuation of which are given also in our conventions by Eq. (T.W. 17) ; therefore we, too, have $\emptyset = -2E\, p\, q \sin \Theta_t$ as in Eq. (T.W. A.5).

c) π N crossing via the invariant amplitudes

Let the s channel scattering amplitude for π N elastic scattering be $\bar{u}(p_2)\, T\, u(p_1)$ where, as in (T.W. 1),

$$T \equiv T(s,u) = -A(s,u) + \frac{1}{g_\mu} \cdot i\gamma(q_1+q_2) \cdot B(s,u) \,. \tag{B.8}$$

In the t channel ($\pi\pi \to N\bar{N}$), the amplitude is $\bar{u}(p_2)\,\bar{T}\,v(P_1)$ where

$$\bar{T} \equiv \bar{T}(t,u) = -\bar{A}(t,u) + \frac{1}{g_\mu} \cdot i\gamma(q_1 - Q_2) \cdot \bar{B}(t,u) \,, \tag{B.9}$$

(cf. Eq. (T.W. 17) and (T.W. 22') for the notation). Choosing the Dirac γ matrices to be Hermitian, we then have the following connection between the values of the functions A and B at the end point of path B.S. and the invariant amplitudes in the t channel [27]:

$$A^*(s,u) = \bar{A}(t,u) \;; \quad B^*(s,u) = \bar{B}(t,u) \,. \tag{B.10}$$

As usual, a star denotes complex conjugation. We emphasize that the value of the variable u stays constant during the continuation.

By repeating the derivation in T.W. leading up to Eq. (T.W. 13) and observing Eqs. (B.1-7,10), we find for the values of the s channel c.m.s. helicity amplitudes $G_{\mu\lambda}(s,u)$ at the end point of path B.S. in Fig. 2, neglecting an over-all factor i,

$$G^*_{++}(s,u) \doteq \sin\chi \cdot F_{++}(t,u) + \cos\chi \cdot F_{+-}(t,u) \,, \tag{B.11a}$$

$$G^*_{+-}(s,u) \doteq \cos\chi \cdot F_{++}(t,u) - \sin\chi \cdot F_{+-}(t,u) \,, \tag{B.11b}$$

$$\operatorname{tg}\chi = \frac{m \cdot q \cdot \sin\Theta_t}{E \cdot (p - q \cdot \cos\Theta_t)} \,, \tag{B.11c}$$

where $F_{\mu\lambda}(t,u)$ are the t channel c.m.s. helicity amplitudes.

Equations (B.11) are the desired crossing relations for the helicity amplitudes in this particular case. We note the similarity between our formulae and (T.W. 11',13), the only difference being that, due to our path B.S. continuation as manifested in particular in Eq. (B.10), the amplitudes $G_{\mu\lambda}$ enter complex conjugated. In this comparison it should be observed that the different phase conventions cause no trouble due to the particular choice of "particle 2" made in T.W.

d) "Geometrical" derivation of the crossing relations

To give a derivation of the crossing relations valid for any spin we follow T.W. and continue the s channel c.m.s. helicity amplitudes by writing, as in Eq. (T.W. 20),

$$G_{\mu\lambda}(s,u) = \sum_{\mu'\lambda'} \mathcal{U}_{\mu\mu'}(\ell_\beta^{-1}; p_2) \cdot G_{\mu'\lambda'}(p_2,q_2\,;\,p_1,q_1) \cdot \mathcal{U}_{\lambda'\lambda}(p_1; \ell_\beta) . \tag{B.12}$$

Each factor in this equation is then continued separately.

Consider first the "generalized" helicity amplitudes $G_{\mu\lambda}(p_2,q_2\,;\,p_1,q_1)$. To continue these quantities, we use the second argument of T.W., i.e., the one starting at Eq. (T.W. 25), and write

$$\begin{aligned} & G^*_{\mu'\lambda'}(\ell^{-1}p_2, \ell^{-1}q_2\,;\,\ell^{-1}p_1, \ell^{-1}q_1) = \\ & = \sum_{\mu\lambda} \mathcal{U}^*_{\mu'\mu}(\ell^{-1}; p_2) \cdot G^*_{\mu\lambda}(p_2,q_2\,;\,p_1,q_1) \cdot \mathcal{U}^*_{\lambda\lambda'}(p_1;\ell) , \end{aligned} \tag{B.13}$$

where ℓ is an arbitrary, real Lorentz transformation, which is kept fixed along the path of continuation. The relation corresponding to (T.W. 28) turns out to be

$$\mathcal{U}^*_{\lambda\lambda'}(p_1;\ell) \longrightarrow \mathcal{U}^*_{\lambda\lambda'}(-P_1;\ell) = (-1)^{\lambda-\lambda'} \cdot \mathcal{U}_{\lambda'\lambda}(\ell^{-1}; P_1) . \tag{B.14}$$

To arrive at this result it should be observed that the rotation angles involved often may be expressed directly in terms of the three-vectors (scalar or vector products) and that, consequently, the continuation of these angles follows directly from (T.W. 17), independent of the particular path of continuation chosen in the s plane.

Furthermore, for the uncrossed particle, i.e., particle d of Fig. 1 in the $s \to t$ crossing, we have

$$U^{*}_{\mu'\mu}(\ell^{-1}; p_2) \to U^{*}_{\mu'\mu}(\ell^{-1}; p_2' = p_2) = (-)^{\mu'-\mu} \cdot U_{-\mu',-\mu}(\ell^{-1}; p_2), \tag{B.15}$$

from the properties of rotation coefficients. Thus, instead of Eq. (T.W. 29) we get at the end point of the continuation along path B.S.

$$\begin{aligned} &G^{*}_{\mu'\lambda'}(\ell^{-1}p_2, -\ell^{-1}Q_2; -\ell^{-1}P_1, \ell^{-1}q_1) = \\ &= \sum_{\mu\lambda} (-)^{\lambda-\lambda'+\mu'-\mu} \cdot U_{-\mu',-\mu}(\ell^{-1}; p_2) \cdot U_{\lambda'\lambda}(\ell^{-1}; P_1) \cdot G^{*}_{\mu\lambda}(p_2, -Q_2; -P_1, q_1) . \end{aligned} \tag{B.16}$$

Comparing this relation to the transformation law for the t channel "generalized" helicity amplitudes

$$\begin{aligned} &F_{\mu'\lambda'}(\ell^{-1}p_2, \ell^{-1}P_1; \ell^{-1}q_1, \ell^{-1}Q_2) = \\ &= \sum_{\mu\lambda} U_{\mu'\mu}(\ell^{-1}; p_2) \cdot U_{\lambda'\lambda}(\ell^{-1}; P_1) \cdot F_{\mu\lambda}(p_2, P_1; q_1, Q_2), \end{aligned} \tag{B.17}$$

[observe the difference between this formula and Eq. (T.W. 30) due to the different phase conventions], we may conclude, as in T.W., that <u>if</u> there exists a direct connection between $G^{*}_{\mu\lambda}$ and $F_{\mu\lambda}$ it must be, apart from a helicity independent phase factor, which in this treatment remains undetermined [28])

$$G^{*}_{\mu\lambda}(p_2, -Q_2; -P_1, q_1) \doteq (-)^{\mu-\lambda} \cdot F_{-\mu,\lambda}(p_2, P_1; q_1, Q_2), \tag{B.18}$$

or, since we assume parity conservation, from Eq. (A.6) [23],

$$G^{*}_{\mu\lambda}(p_2, -Q_2; -P_1, q_1) \doteq F_{\mu,-\lambda}(p_2, P_1; q_1, Q_2) \quad . \tag{B.19}$$

It remains to consider the continuation of the rotation coefficients in Eq. (B.12). Since at every stage of the continuation the four vectors $\vec{p}_1$, $\vec{q}_1$, $\vec{p}_2$ and $\vec{q}_2$ may be assumed to lie in one plane, which we have chosen as the x-z plane, these coefficients depend only on one angle each, namely an angle of rotation around the y axis. We denote this angle for particle c by χ_1', for particle d by χ_2'. These primed angles are defined in the same way as the corresponding unprimed angles in T.W. as long as the two velocity points O and C in T.W. (see Fig. T.W. 4) are connected by a real Lorentz transformation. Due to different conventions concerning the analytical continuation, however, the final values of the primed and the unprimed angles may differ (see Eqs. (B.20,22) below).

Since the rotation coefficients in Eq. (B.12) then are real at the initial and final points, the complex conjugation causes no trouble ; we may simply disregard it. The values of the crossing angles χ_1' and χ_2' at the end point of the continuation require, on the other hand, careful examination. Consider first the cosinus. The expressions (T.W. 34) remain unchanged, since they refer to the case when the two velocity points O and C of T.W. are connected by a real Lorentz transformation. Observing the Eqs. (B.3-4) and their generalizations to unequal masses we then find, at the end point of path B.S.,

$$\cos \chi_1' = \cos \chi_1 \quad , \tag{B. 20-a}$$

$$\cos \chi_2' = -\cos \chi_2 \quad , \tag{B. 20-b}$$

where the expressions for the unprimed angles in terms of the Mandelstam variables and the masses can be found in Eq. (T.W. 42).

As to the sinus of the crossing angles, finally we observe that Eq. (T.W. 37) may be written

$$\mathcal{S}_1 \cdot \left(p_{10}^2 - m_1^2 \right)^{1/2} \cdot \sin\chi_1' = 2 \cdot m_1 \cdot \left[(-\vec{p}_1) \times \vec{q}_1 \right] \cdot \vec{e}_y \, , \qquad \text{(B.21)}$$

where $\vec{e}_y$ is the unit vector parallel to the normal $\vec{p}_2 \times \vec{q}_1$ of the previously defined x-z plane ; $\vec{e}_y$ is invariant under the continuation. Continuing Eq. (B.21) to the end point of path B.S., observing Eq. (T.W. 17), we now obtain

$$\sin\chi_1' = \sin\chi_1 \, , \qquad \text{(B.22a)}$$

$$\sin\chi_2' = - \sin\chi_2 \, , \qquad \text{(B.22b)}$$

where $\sin\chi_1$ and $\sin\chi_2$ are given in Eq. (T.W. 43).

Collecting all results we find that the final value of the anatically continued s channel c.m.s. helicity amplitudes $G_{\mu\lambda}$ (s,u), in the path B.S. continuation of Fig. 2, is related to the t channel c.m.s. helicity amplitudes $F_{\mu\lambda}$ (t,u) through the crossing relations

$$\begin{aligned} G^*_{\mu\lambda}(s,u) &\doteq \\ &\doteq \sum_{\mu'\lambda'} (-)^{\lambda'-\mu'} \cdot d^{s_2}_{\mu'\mu}(\chi_2') \cdot d^{s_1}_{\lambda'\lambda}(\chi_1') \cdot F_{\mu'\lambda'}(t,u) \doteq \\ &\doteq \sum_{\mu'\lambda'} (-)^{s_1-\lambda'} \cdot d^{s_2}_{\mu'\mu}(\chi_2) \cdot d^{s_1}_{\lambda'\lambda}(\chi_1') \cdot F_{\mu'\lambda'}(t,u) \, . \end{aligned} \qquad \text{(B.23)}$$

To obtain the second equality in Eq. (B.23), we used the symmetry relation

$$d^{s_2}_{\mu'\mu}(\pi + \chi_2) = (-)^{s_2+\mu'} \cdot d^{s_2}_{-\mu'\mu}(\chi_2) \, , \qquad \text{(B.24)}$$

for the d-functions.

By evaluating the crossing angles as in Eq. (T.W. 38), we rediscover Eq. (B.11) from Eq. (B.23).

The crossing relations in the general case of unequal masses and all particles having spins are readily obtained (see Section III.).

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16) The definition of θ_t is chosen to conform with the conventions of Ref. [7].

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18) This is the region of the real s t u plane which would correspond to the decay process $d \to a + c + \bar{b}$, if this reaction were physically possible (cf. Ref. [19]).

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21) Since the crossing relations are up to a common phase equalities, this does not limit the proof.

22) This value of t should of course be in the physical region of both the s and the u channel.

23) Observe that the use of the sign $\doteq$ allows factors like $(-)^{2\lambda_a}$ to be discarded.

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26) In this proof one also sees where assumptions like parity conservation and time-reversal invariance enter.

27) Cf. the corresponding relation for the $s \rightarrow u$ crossing in Ref. [2),3)].

28) See remark 3 at the end of Section III.

FIGURE CAPTIONS

Figure 1 Illustrating the notation for the reactions (II.1-3).

Figure 2 The path of continuation in the complex s plane used in the derivation of the crossing relations for helicity amplitudes in the Trueman-Wick conventions [7] (path T.W.) and in our convention (path B.S.) for masses $m_a = m_b = \mu$, $m_c = m_d = m$. See the text for further details.

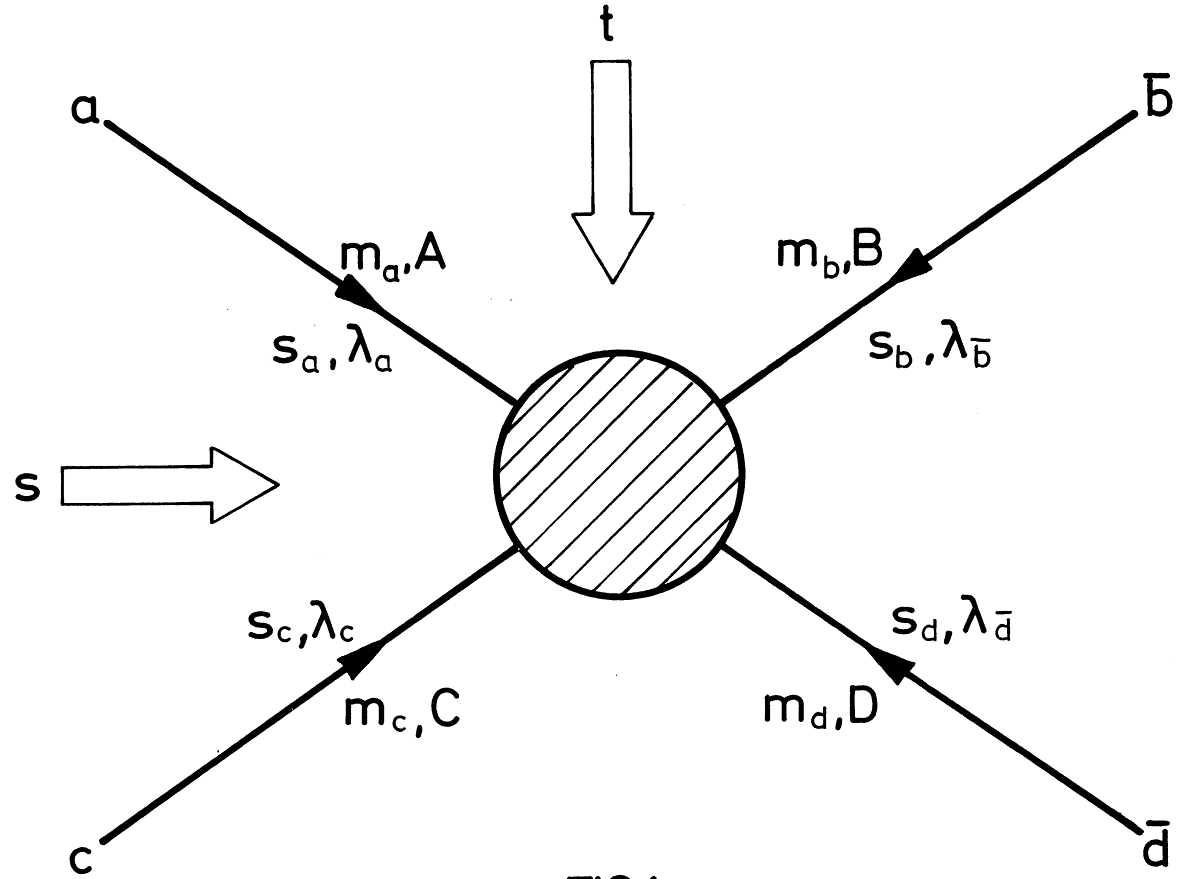
t
a
$\bar{b}$
m_a, A
m_b, B
s_a, λ_a
$s_b, \lambda_{\bar{b}}$
s
s_c, λ_c
$s_d, \lambda_{\bar{d}}$
m_c, C
m_d, D
c
$\bar{d}$

FIG.1

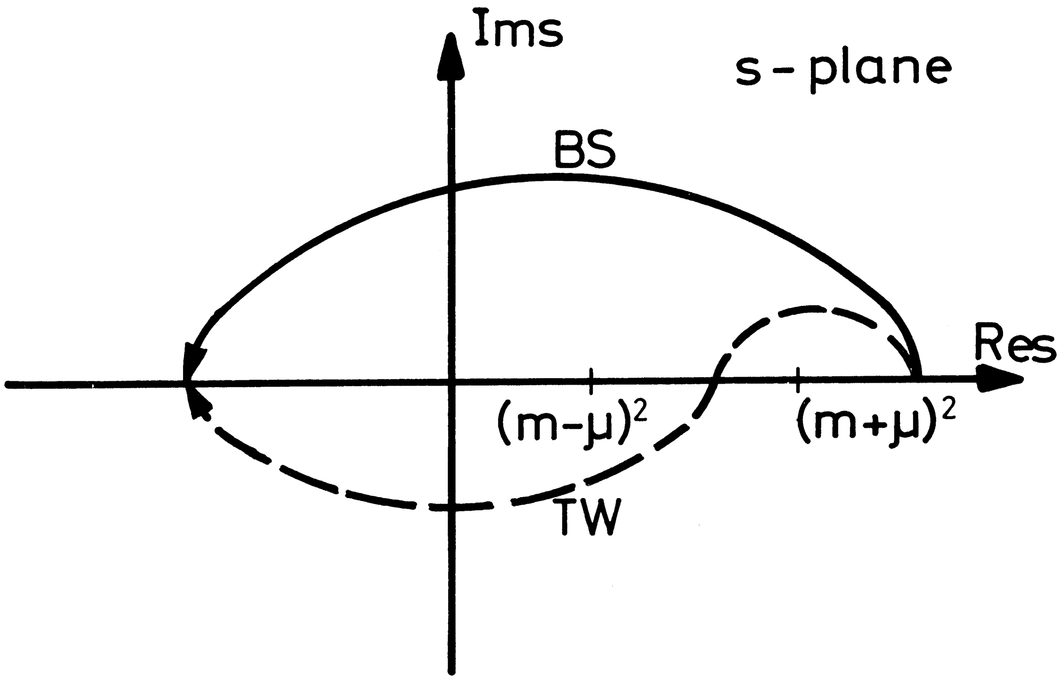
Ims
s - plane
BS
Res
$(m-\mu)^2$
$(m+\mu)^2$
TW

FIG. 2