# JOINT DECAY DISTRIBUTION IN DOUBLE RESONANCE PRODUCTION

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## ABSTRACT

The joint decay distribution for reactions like $K^+p \to K^*N^*$ and $\bar{p}p \to \bar{N}^*N^*$ is derived in terms of the joint spin space density matrix, using helicity formalism. The predictions of the peripheral model for these distributions are briefly discussed, and a general test of the peripheral model is given for the case of parity violation in one of the decays.

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## 1. INTRODUCTION

In recent years, considerable progress has been achieved on quasi-two-particle reactions in which both outgoing particles are unstable. Examples of such reactions are

$$K^- + p \rightarrow \omega^0 + \Lambda^0 \tag{1}$$

$$K^+ + p \rightarrow K^* + N^* \tag{2}$$

$$\bar{p} + p \rightarrow \overline{N^*} + N^* \tag{3}$$

$$\bar{p} + p \rightarrow \overline{\Lambda^0} + \Lambda^0 \tag{4}$$

Once the spins and decay properties of the unstable particles are known, the angular distributions of their decay products provide valuable information on the production process.

The joint decay distributions for reactions of the types (1), (2) and (4) have been derived by Byers and Yang [1], Schlein [2], and Durand and Sandweiss [3], respectively. Gottfried and Jackson [4] gave the separate decay distributions of K* and N*, using the helicity formalism. Li and Martin [5] finally gave the joint decay distribution for reaction (2) in the case of vector exchange.

Here we shall derive the general formulae for the joint decay distribution for reactions like (2) and (3) for unpolarized initial state and fixed scattering angle, using the helicity formalism. For reaction (2), the result is of course equivalent to that obtained by Schlein [2], but it seemed worth while to reformulate the problem in the helicity formalism and using co-ordinate systems appropriate for the comparison with the peripheral model [4].

It is clear that by considering the joint decay distribution one can get more information than by studying each decay separately: the latter procedure determines 6 production parameters (3 from each decay), while the former gives 13 additional production parameters [6] (a production parameter is a bilinear combination of production matrix elements).

The plan of the paper is the following:

In Section 2 we present the general formalism which then is applied to reaction (2) in Section 3 and to reaction (3) in Section 4. Section 5 treats a different problem, namely the predictions of the peripheral model for the production parameters. In the case of reactions like (1) or (4), a test of the peripheral model is given, independent of the quantum numbers of the exchanged particles. This is a test on the phases of the amplitudes.

## 2. GENERAL FORMALISM

### a) Joint density matrix

The processes to be studied are of the general type

$$a + b \rightarrow c + d , \tag{5}$$

where a denotes the incident particle and b the target proton. Particle d is the outgoing isobar, while c represents the vector meson in reaction (2) and the anti-isobar in reaction (3). We shall use the notation of Jacob and Wick [7] to which we refer as JW. The production amplitudes in the helicity representation are denoted by

$$\langle \lambda_c \, \lambda_d \, | \, T(x) \, | \, \lambda_a \, \lambda_b \rangle , \tag{6}$$

where x is the cosine of the cms production angle.

From parity conservation, one has (JW.44)

$$\langle -\lambda_c , -\lambda_d | T(x) | -\lambda_a , -\lambda_b \rangle = \eta \cdot (-1)^{\lambda_c - \lambda_d - \lambda_a + \lambda_b} \langle \lambda_c \lambda_d | T(x) | \lambda_a \lambda_b \rangle , \tag{7}$$

where $\eta = \pm 1$ is independent of the helicities.

Next, we choose as spin quantization axis (z axis) for c the direction of the momentum of a in the rest frame of c. The y axis is taken to be the normal to the production plane. This defines in the rest frame of c a right-handed co-ordinate system which we call the c reference system. In the same way we define the d reference system as the co-ordinate system in the rest frame of d which has its z axis along the momentum of b in this frame and its y axis along the normal to the production plane. The advantage of this choice for the discussion of the peripheral model is explained in Ref. [4]. Denoting by m and n the spin components along the respective quantization axes, we have

$$\langle m\, n \,|\, T(x) \,|\, \lambda_a \lambda_b \rangle =$$

$$= \sum_{\lambda_c \lambda_d} d^{s_c}_{m \lambda_c}(\psi_c)\, d^{s_d}_{n \lambda_d}(\psi_d)\, (-1)^{s_d - \lambda_d} \langle \lambda_c \lambda_d \,|\, T(x) \,|\, \lambda_a \lambda_b \rangle, \tag{8}$$

where $s_c$ denotes the spin of particle c etc. The sign $(-1)^{s_d - \lambda_d}$ in Eq. (8) is due to the phase conventions [see the discussion following (JW.13)]. The angles appearing in the rotation functions are defined through the equations

$$\operatorname{tg} \psi_c = \frac{M_c}{E_c} \sqrt{1 - x^2} \left( x - \frac{v_c}{v_a} \right)^{-1}, \quad 0 \leq \psi_c < \pi, \tag{9}$$

$$\operatorname{tg} \psi_d = \frac{M_d}{E_d} \sqrt{1 - x^2} \left( x - \frac{v_d}{v_b} \right)^{-1}, \quad 0 \leq \psi_d < \pi, \tag{10}$$

where $M_c$, $E_c$ and $v_c$ denote the mass and the cms energy and velocity of particle c, etc. Note that the quantum numbers m and n in (8) are not helicities, except for x = 1.

When both incident particles are unpolarized, the joint spin density matrix of the outgoing particles is

$$\rho^{mm'}_{nn'} =$$
$$= \sum_{\lambda_a \lambda_b} \langle mn | T(x) | \lambda_a \lambda_b \rangle \langle m'n' | T(x) | \lambda_a \lambda_b \rangle^* . \tag{11}$$

From this, the separate spin density matrix for c is obtained by summing over the magnetic substates of d:

$$\rho^{mm'} = \sum_n \rho^{mm'}_{nn} , \tag{12}$$

and analogously for the spin density matrix of d alone:

$$\rho_{nn'} = \sum_m \rho^{mm}_{nn'} . \tag{13}$$

We choose the normalization such that

$$\sum_{mn} \rho^{mm}_{nn} = \sum_m \rho^{mm} = \sum_n \rho_{nn} = 1 . \tag{14}$$

Note that this implies $\sum |T(x)|^2 = 1$. This unconventional normalization of $T(x)$ is used in order to avoid an extra normalization factor in (11) and in some expressions in part c below.

Because $\rho$ is a Hermitian matrix we have

$$\rho_{n\,n'}^{m\,m'\,*} = \rho_{n'\,n}^{m'\,m} \quad . \tag{15}$$

Also from Eq. (7) one proves

$$\rho_{-n,-n'}^{-m,-m'} = (-1)^{m-m'+n-n'} \rho_{n\,n'}^{m\,m'} \quad . \tag{16}$$

This means especially

$$\rho_{n\,n}^{m\,m} \quad \text{real}\,, \tag{17}$$

$$\rho_{n,-n}^{m,-m} \begin{cases} \text{real if } 2(n-m) \text{ is even} \\ \text{purely imaginary if } 2(n-m) \text{ is odd} \end{cases} \tag{18}$$

b) Decay matrix elements

Up to now our discussion has been completely general; now we have to be more specific. We assume that in the decays of particles c and d, parity is conserved, and that each decay is described, in its own rest frame, by a decay direction. For the isobar $N^*$, (particle d) this is the direction of the decay proton, which we denote by $\hat{\beta}$. At the same time, $\lambda_\beta$ will denote the proton's helicity. For particle c, the decay direction is denoted by $\hat{\alpha}$. The meaning of this direction depends upon which particle is represented by c. For the anti-isobar [reaction (3)], $\hat{\alpha}$ is the direction of the antiproton and $\lambda_\alpha$ will denote its helicity. For the decay of a vector particle like K* or the $\rho$ meson, $\hat{\alpha}$ is the momentum direction of either of the two decay mesons, whereas for the three-pion decay of the $\omega$ meson, $\hat{\alpha}$ will denote the normal to the decay plane. In these latter cases, the index $\lambda_\alpha$ is of course unnecessary.

The angles of $\hat{\alpha}$ ($\hat{\beta}$) in the c(d) rest frames are denoted by $\theta_\alpha$, $\varphi_\alpha$ ($\theta_\beta$, $\varphi_\beta$). They are defined as the polar and azimuthal angles in the c(d) reference system discussed in part a) of this section. The azimuthal angles $\varphi_\alpha$ and $\varphi_\beta$ are equivalent to the Treiman-Yang angles. For the matrix elements $M_m^{s_c}(\alpha;\lambda_\alpha)$ of the decay of particle c we then have [4)]

$$M_m^{s_c}(\hat{\alpha};\lambda_\alpha)=\sqrt{\frac{2s_c+1}{4\pi}}\,\mathcal{D}_{m\lambda_\alpha}^{s_c}(\hat{\alpha})^*\;M^{s_c}(\lambda_\alpha), \tag{19}$$

where it is convenient to normalize

$$\sum_{\lambda_\alpha}|M^{s_c}(\lambda_\alpha)|^2=1 \tag{20}$$

For parity conserving decays one has [4)]

$$M^{s_c}(-\lambda_\alpha)=\eta'\,M^{s_c}(\lambda_\alpha), \tag{21}$$

where $\eta' = \pm 1$ independent of the helicities. Let us further introduce the quantities

$$M^{s_c}_{mm'}(\hat{\alpha}) = \sum_{\lambda_\alpha} M^{s_c}_{m}(\hat{\alpha};\lambda_\alpha)\, M^{s_c}_{m'}(\hat{\alpha};\lambda_\alpha)^{*} . \tag{22.a}$$

These are elements of a Hermitian matrix, which for parity conserving decays have the property

$$M^{s_c}_{-m,-m'}(\hat{\alpha}) = (-1)^{m-m'}\, M^{s_c}_{mm'}(\hat{\alpha})^{*} . \tag{23}$$

We also note the following properties, valid for both parity conserving and parity violating decays, which can easily be proved using Eq. (20) and the properties of the rotation functions,

$$\sum_{m} M^{s_c}_{mm}(\hat{\alpha}) = \frac{2 s_c + 1}{4\pi} , \tag{24.a}$$

$$\int d\Omega_{\hat{\alpha}}\; M^{s_c}_{mm'}(\hat{\alpha}) = \delta_{mm'} . \tag{24.b}$$

In exactly the same way we introduce the matrix elements $M^{s_d}_{n}(\hat{\beta};\lambda_\beta)$ for the decay of particle d; these elements obey formulae analogous to Eqs. (19)-(24) above. In particular, we define

$$M^{s_d}_{nn'}(\hat{\beta}) = \sum_{\lambda_\beta} M^{s_d}_{n}(\hat{\beta};\lambda_\beta)\, M^{s_d}_{n'}(\hat{\beta};\lambda_\beta)^{*} . \tag{22.b}$$

c) The joint decay distribution

The matrix elements between the incident particles and the decay products are now given by

$$\langle \lambda_\alpha \lambda_\beta | T(x;\hat{\alpha},\hat{\beta}) | \lambda_a \lambda_b \rangle =$$

$$= \sum_{m,n} M^{s_c}_{m}(\hat{\alpha};\lambda_\alpha)\, M^{s_d}_{n}(\hat{\beta};\lambda_\beta) \langle m\, n | T(x) | \lambda_a \lambda_b \rangle . \tag{25}$$

When both incident particles are unpolarized and the polarizations of the decay particles remain unobserved, the cross-section is

$$\frac{d^3\sigma(x;\hat{\alpha},\hat{\beta})}{dx\, d\Omega_{\hat{\alpha}}\, d\Omega_{\hat{\beta}}} = \frac{d\sigma}{dx}\, \Gamma_c\, \Gamma_d\, W(x;\hat{\alpha},\hat{\beta}), \tag{26}$$

where $\frac{d\sigma}{dx}$ is the differential production cross-section for reaction (5) irrespective of the decays, $\Gamma_c$ and $\Gamma_d$ are the partial relative widths for the observed decay modes of c and d, and

$$W(x;\hat{\alpha},\hat{\beta}) = \sum_{\substack{\lambda_\alpha \lambda_\beta \\ \lambda_a \lambda_b}} | \langle \lambda_\alpha \lambda_\beta | T(x;\hat{\alpha},\hat{\beta}) | \lambda_a \lambda_b \rangle |^2 =$$

$$= \sum_{\substack{m m' \\ n n'}} M^{s_c}_{m m'}(\hat{\alpha})\, M^{s_d}_{n n'}(\hat{\beta})\, \rho^{m m'}_{n n'}(x) \tag{27}$$

gives the joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ in the c and d reference systems defined above.

By integrating over $\hat{\alpha}$, using Eq. (24.b) and the definition (13) we obtain the separate decay distribution for the decay of particle d [4])

$$W(x;\hat{\beta}) = \sum_{nn'} M^{s_d}_{nn'}(\hat{\beta})\, \rho_{nn'}(x) \quad . \tag{28.a}$$

In the same way for the decay of c:

$$W(x;\hat{\alpha}) = \sum_{mm'} M^{s_c}_{mm'}(\hat{\alpha})\, \rho^{mm'}(x) . \tag{28.b}$$

Further, by integrating (27) over both $\hat{\alpha}$ and $\hat{\beta}$ we obtain

$$\iint d\Omega_{\hat{\alpha}}\, d\Omega_{\hat{\beta}}\; W(x;\hat{\alpha}\hat{\beta}) = 1 , \tag{29}$$

consistent with having split off the factors $\frac{d\sigma}{dx}$, $\Gamma_c$ and $\Gamma_d$ in (26).

For the rest of this paragraph, we combine the indices m and n into one index i and m' and n' into another index k. We also suppress the arguments x, $\hat{\alpha}$ and $\hat{\beta}$. Then Eq. (27) reads

$$W = \sum_{ik} M_{ik}\, \rho_{ik} \; ; \quad i \equiv (m\,n) , \quad k \equiv (m'n') , \tag{30}$$

$$M_{ik} = M^{s_c}_{mm'}(\hat{\alpha})\, M^{s_d}_{nn'}(\hat{\beta})\,. \qquad (30.a)$$

[The notation $\rho_{ik}$, which will be used in this section only, should not be confused with the symbol $\rho_{nn'}$ introduced in (13)]. The indices i and k assume $(2s_c+1)(2s_d+1)$ values each. The sum in (30) is, however, reduced when care is taken of hermiticity and parity conservation. Let us write

$$W = A + B\,; \quad A = \sum_i M_{ii}\,\rho_{ii}\,, \quad B = \sum_{i \neq k} M_{ik}\,\rho_{ik}\,. \qquad (31)$$

The hermiticity relations (15) together with $M_{ik} = M^*_{ki}$, reduce B to the form

$$B = 2\, \mathrm{Re}\Big[\sum_{i \neq k}{}' M_{ik}\, \rho_{ik}\Big]\,, \qquad (32)$$

where the dash indicates that only one member of the index pairs (i,k) and (k,i) is contained in the sum. Moreover, from the relations (16) and (23), we obtain

$$M_{-i,-k}\,\rho_{-i,-k} = M^*_{ik}\,\rho_{ik}\,, \qquad (33)$$

and thus the sums A and B can be simplified further to read

$$A = 2 \sum_{|i|} M_{ii} \rho_{ii} \; ; \quad B = 4 \sum_{|i| \neq |k|}' Re[M_{ik}] \, Re[\rho_{ik}] \tag{34}$$

Here the symbol $|i|$ in the summation means that only one member of each index pair $(i,k)$ and $(-i,-k)$ occurs in the sums.

The formula (34) gives the desired decay correlation. From this, using the definitions (19) and (22.a,b), together with the known rotation functions, the joint decay distribution can be written down for any two particle production process which is followed by the decay of both particles through parity conserving interactions. We now turn to this problem for the experimentally interesting reactions (2) and (3).

## 3. JOINT DECAY DISTRIBUTION IN REACTIONS LIKE $K^+p \to K^*N^*$

Consider first the matrix elements (19) for the case of vector meson decay. Because of Eq. (20), the matrix elements are simply the spherical harmonics for a p wave :

$$M^1_m(\hat{\alpha}) = Y^1_m(\hat{\alpha}) . \tag{35}$$

The matrix elements of (22.a) are thus products of these spherical harmonics.

For the decay of the baryon isobar into a baryon and a pseudoscalar particle we obtain for the matrix elements (22.b),

$$M_{33} = \frac{3}{8\pi} \sin^2\theta_\beta \quad , \tag{36.a}$$

$$M_{11} = \frac{3}{8\pi}\left(\frac{1}{3} + \cos^2\theta_\beta\right) \quad , \tag{36.b}$$

$$M_{31} = -\frac{\sqrt{3}}{8\pi} \sin 2\theta_\beta \, e^{i\varphi_\beta} \quad , \tag{36.c}$$

$$M_{3,-1} = -\frac{\sqrt{3}}{8\pi} \sin^2\theta_\beta \, e^{2i\varphi_\beta} . \tag{36.d}$$

Here we have introduced the short-hand notation of writing $2n$, $2n'$ instead of $n$ and $n'$ for the magnetic quantum numbers referring to the isobar. The other elements of $M_{n,n'}$ are obtained by application of (23) and the fact that $M_{n,n'}(\hat{\beta})$ is Hermitian.

We note especially that

$$M_{n,-n} = 0 \, . \tag{36.e}$$

Inserting (35) and (36) into Eq. (34), we obtain the following explicit expression for the joint decay distribution in reaction (2) :

$$\begin{aligned} W(x;\hat{\alpha},\hat{\beta})\cdot 16\pi^2 = \\ = 1 + A(\theta_\alpha) + A(\theta_\beta) + A(\theta_\alpha,\theta_\beta) + B(\theta_\alpha,\varphi_\alpha) + B(\theta_\beta,\varphi_\beta) + \\ + B(\theta_\alpha,\theta_\beta,\varphi_\alpha) + B(\theta_\alpha,\theta_\beta,\varphi_\beta) + B(\theta_\alpha,\theta_\beta,\varphi_\alpha,\varphi_\beta) \end{aligned} \tag{37}$$

$$A(\theta_\alpha) = (1-3\cos^2\theta_\alpha)(\rho^{11}-\rho^{00}) \tag{37.a}$$

$$A(\theta_\beta) = (1-3\cos^2\theta_\beta)(\rho_{33}-\rho_{11}) \tag{37.b}$$

$$A(\theta_\alpha,\theta_\beta) = \frac{1}{2}(1-3\cos^2\theta_\alpha)(1-3\cos^2\theta_\beta)(\rho_{\overline{33}}-\rho_{\overline{11}}) \tag{37.c}$$

$$B(\theta_\alpha,\varphi_\alpha) = -3\left[\sqrt{2}\sin 2\theta_\alpha \cos\varphi_\alpha \, Re(\rho^{10}) + \sin^2\theta_\alpha \cos 2\varphi_\alpha \, \rho^{1,-1}\right] \tag{37.d}$$

$$B(\theta_\beta,\varphi_\beta) = -2\sqrt{3}\left[\sin 2\theta_\beta \cos\varphi_\beta \cdot Re(\rho_{31}) + \sin^2\theta_\beta \cos 2\varphi_\beta \, Re(\rho_{3,-1})\right] \tag{37.e}$$

$$B(\theta_\alpha, \theta_\beta, \varphi_\alpha) = -\frac{3}{\sqrt{2}}\left(1-3\cos^2\theta_\beta\right) \times$$

$$\times \left[\sin 2\theta_\alpha \cos\varphi_\alpha \, \mathrm{Re}\left(\rho^{\underline{10}}\right) + \frac{1}{\sqrt{2}} \sin^2\theta_\alpha \cos 2\varphi_\alpha \cdot \rho^{\underline{1,-1}}\right] \tag{37.f}$$

$$B(\theta_\alpha, \theta_\beta, \varphi_\beta) = -\sqrt{3}\left(1-3\cos^2\theta_\alpha\right) \times$$

$$\times \left[\sin 2\theta_\beta \cos\varphi_\beta \, \mathrm{Re}\left(\rho_{\overline{31}}\right) + \sin^2\theta_\beta \cos 2\varphi_\beta \, \mathrm{Re}\left(\rho_{\overline{3,-1}}\right)\right] \tag{37.g}$$

$$B(\theta_\alpha, \theta_\beta, \varphi_\alpha, \varphi_\beta) =$$

$$= 3\sqrt{3}\Big\{\sin 2\theta_\alpha \sin 2\theta_\beta \left[\cos(\varphi_\alpha+\varphi_\beta)\,\mathrm{Re}\left(\rho^{10}_{31} - \rho^{0,-1}_{31}\right) + \cos(\varphi_\alpha-\varphi_\beta)\,\mathrm{Re}\left(\rho^{01}_{31} - \rho^{-10}_{31}\right)\right]$$

$$+ \sin^2\theta_\alpha \sin 2\theta_\beta \left[\cos(2\varphi_\alpha+\varphi_\beta)\,\mathrm{Re}\left(\rho^{1,-1}_{31}\right) + \cos(2\varphi_\alpha-\varphi_\beta)\,\mathrm{Re}\left(\rho^{-11}_{31}\right)\right]$$

$$+ \sin 2\theta_\alpha \sin^2\theta_\beta \left[\cos(\varphi_\alpha+2\varphi_\beta)\,\mathrm{Re}\left(\rho^{10}_{3,-1} - \rho^{0,-1}_{3,-1}\right) + \cos(\varphi_\alpha-2\varphi_\beta)\,\mathrm{Re}\left(\rho^{01}_{3,-1} - \rho^{-10}_{3,-1}\right)\right]$$

$$+ \sin^2\theta_\alpha \sin^2\theta_\beta \left[\cos 2(\varphi_\alpha+\varphi_\beta)\,\mathrm{Re}\left(\rho^{1,-1}_{3,-1}\right) + \cos 2(\varphi_\alpha-\varphi_\beta)\,\mathrm{Re}\left(\rho^{-11}_{3,-1}\right)\right]\Big\}. \tag{37.h}$$

When applicable, we have used the reality properties (17) and (18). We have also used, besides the definitions (12) and (13) and the normalisation (14), the abbreviations

$$\rho_{nn'}^{-} = \rho_{nn'}^{11} + \rho_{nn'}^{-1,-1} - 2\,\rho_{nn'}^{00} , \tag{38.a}$$

$$\rho_{-}^{mm'} = \rho_{33}^{mm'} + \rho_{-3,-3}^{mm'} - \rho_{11}^{mm'} - \rho_{-1,-1}^{mm'} . \tag{38.b}$$

The splitting of A and B into different pieces has been done in order to exhibit explicitly the angles which appear in each piece. The angular functions in (37.a-h), including the number 1, are orthogonal but not normalized. The total number of these functions is 20. The pieces $A(\theta_\alpha)$, $A(\theta_\beta)$, $B(\theta_\alpha, \varphi_\alpha)$ and $B(\theta_\beta, \varphi_\beta)$ are the ones given by Gottfried and Jackson[4].

## 4. JOINT DECAY DISTRIBUTION IN REACTIONS LIKE $\bar{p}p \rightarrow \bar{N}^*N^*$

The matrix elements of the decays of the isobar and anti-isobar are given by Eqs. (36). Again we introduce the short-hand notation of writing $2n, 2n'$ instead of $n, n'$ for the magnetic quantum numbers referring to the isobar, and correspondingly for the anti-isobar; we also use the abbreviations [cf. (38)]

$$\rho^{mm'}_{\underline{\quad}} = \rho^{mm'}_{33} + \rho^{mm'}_{-3,-3} - \rho^{mm'}_{11} - \rho^{mm'}_{-1,-1} , \tag{39.a}$$

$$\rho^{\overline{\quad}}_{nn'} = \rho^{33}_{nn'} + \rho^{-3,-3}_{nn'} - \rho^{11}_{nn'} - \rho^{-1,-1}_{nn'} . \tag{39.b}$$

From Eq. (34) we then get for the joint decay distribution :

$$\begin{aligned} W(x;\hat{\alpha},\hat{\beta})\,16\pi^2 = \\ = 1 + A(\Theta_\alpha) + A(\Theta_\beta) + A(\Theta_\alpha,\Theta_\beta) + B(\Theta_\alpha,\varphi_\alpha) + \\ + B(\Theta_\beta,\varphi_\beta) + B(\Theta_\alpha,\Theta_\beta,\varphi_\alpha) + B(\Theta_\alpha,\Theta_\beta,\varphi_\beta) + B(\Theta_\alpha,\Theta_\beta,\varphi_\alpha\varphi_\beta) , \end{aligned} \tag{40}$$

$$A(\Theta_\alpha) = (1 - 3\cos^2\Theta_\alpha)(\rho^{33} - \rho^{11}) , \tag{40.a}$$

$$A(\Theta_\beta) = (1 - 3\cos^2\Theta_\beta)(\rho_{33} - \rho_{11}) \tag{40.b}$$

$$A(\Theta_\alpha,\Theta_\beta) = \tfrac{1}{9}(1 - 3\cos^2\Theta_\alpha)(1 - 3\cos^2\Theta_\beta)(\rho^{33}_{\underline{\quad}} - \rho^{11}_{\underline{\quad}}) \tag{40.c}$$

$$B(\Theta_\alpha, \varphi_\alpha) =$$
$$= -2\sqrt{3}\left[\sin 2\Theta_\alpha \cos\varphi_\alpha \, Re(\rho^{31}) + \sin^2\Theta_\alpha \cos 2\varphi_\alpha \, Re(\rho^{3,-1})\right], \qquad (40.d)$$

$$B(\Theta_\beta, \varphi_\beta) =$$
$$= 1 - 2\sqrt{3}\left[\sin 2\Theta_\beta \cos\varphi_\beta \, Re(\rho_{31}) + \sin^2\Theta_\beta \cos 2\varphi_\beta \, Re(\rho_{3,-1})\right], \qquad (40.e)$$

$$B(\Theta_\alpha, \Theta_\beta, \varphi_\alpha) = \qquad (40.f)$$
$$= -\sqrt{3}\,(1 - 3\cos^2\Theta_\beta)\left[\sin 2\Theta_\alpha \cos\varphi_\alpha \, Re(\rho^{31}_{\text{—}}) + \right.$$
$$\left. + \sin^2\Theta_\alpha \cos 2\varphi_\alpha \, Re(\rho^{3,-1}_{\text{—}})\right],$$

$$B(\Theta_\alpha, \Theta_\beta, \varphi_\beta) = \qquad (40.g)$$
$$= -\sqrt{3}\,(1 - 3\cos^2\Theta_\alpha)\left[\sin 2\Theta_\beta \cos\varphi_\beta \, Re(\rho_{\overline{31}}) + \right.$$
$$\left. + \sin^2\Theta_\beta \cos 2\varphi_\beta \, Re(\rho_{\overline{3,-1}})\right],$$

$$B(\Theta_\alpha, \Theta_\beta, \varphi_\alpha, \varphi_\beta) =$$
$$= 3\Big\{\sin 2\Theta_\alpha \sin 2\Theta_\beta\left[\cos(\varphi_\alpha + \varphi_\beta) \, Re(\rho^{31}_{31} - \rho^{-1,-3}_{31}) + \cos(\varphi_\alpha - \varphi_\beta) \, Re(\rho^{31}_{13} - \rho^{-1,-3}_{13})\right] +$$
$$+ \sin^2\Theta_\alpha \sin 2\Theta_\beta\left[\cos(2\varphi_\alpha + \varphi_\beta) \, Re(\rho^{3,-1}_{31} + \rho^{1,-3}_{31}) + \cos(2\varphi_\alpha - \varphi_\beta) \, Re(\rho^{3,-1}_{13} + \rho^{1,-3}_{13})\right] + \qquad (40.h)$$
$$+ \sin 2\Theta_\alpha \sin^2\Theta_\beta\left[\cos(\varphi_\alpha + 2\varphi_\beta) \, Re(\rho^{31}_{3,-1} - \rho^{-1,-3}_{3,-1}) + \cos(\varphi_\alpha - 2\varphi_\beta) \, Re(\rho^{31}_{-1,3} - \rho^{-1,-3}_{-1,3})\right] +$$
$$+ \sin^2\Theta_\alpha \sin^2\Theta_\beta\left[\cos 2(\varphi_\alpha + \varphi_\beta) \, Re(\rho^{3,-1}_{3,-1} + \rho^{1,-3}_{3,-1}) + \cos 2(\varphi_\alpha - \varphi_\beta) \, Re(\rho^{3,-1}_{-1,3} + \rho^{1,-3}_{-1,3})\right]\Big\}.$$

The distribution is as before written as a sum of several terms to exhibit explicitly the angular dependence; the angular functions in (40.a-h), the number of which is again 20, are orthogonal but not normalized. We note that the separate decay distribution for the isobar (or the anti-isobar) is the same as in Eq. (37) and already given by Gottfried and Jackson [4).]

Up to now, the results of this paragraph are applicable not only to the specific reaction (3), but to all reactions with the same spin properties, e.g., $p+p \to N^*+N^*$ and $\bar{\Lambda}+p \to \bar{Y}_1^*+N^*$. We make this point, because reaction (3), in which both the initial and the final state contain a particle and its antiparticle has an additional symmetry, as a consequence of charge conjugation invariance. For the helicity amplitudes (6) this symmetry is

$$\langle \lambda_d \lambda_c | T(x) | \lambda_b \lambda_a \rangle = \gamma \, (-1)^{\lambda_c - \lambda_d - \lambda_a + \lambda_b} \langle \lambda_c \lambda_d | T(x) | \lambda_a \lambda_b \rangle , \tag{41}$$

where $\gamma = \pm 1$ independent of the helicities. This equation follows from a generalization of (JW.45). Using the fact that the rotation angles of Eq. (8) obey $\psi_c = \psi_d$, we then obtain the following symmetry relation for the joint density matrix elements (11)

$$\rho^{mm'}_{nn'} = \rho^{nn'}_{mm'} \quad \text{if} \quad a = \bar{b}, \; c = \bar{d}, \tag{42}$$

which from the Eqs. (12) and (13) means

$$\rho^{mm'} = \rho_{mm'} \quad \text{if} \quad a = \bar{b}, \; c = \bar{d}. \tag{43}$$

## 5. PERIPHERAL MODEL

We now give a brief discussion of what can be learned on the production mechanism from the joint decay distribution. We do this with special reference to the peripheral model [8], which is frequently used in the analysis of production processes.

For reaction (2), there are in total 12 independent production amplitudes, not counting possible isospin. In order to determine all 12 (complex) amplitudes, one would need to measure 23 (real) parameters (disregarding a common phase of the amplitudes). The joint decay distribution (37), together with the determination of $\frac{d\sigma}{dx}$, determines only 20 parameters. In the peripheral model, however, both in the form factor version and in the absorption version, all the amplitudes are real [9]. From the 20 real parameters, one should thus be able to check 8 (non-linear) relations between these amplitudes. Unfortunately, these relations seem to be rather complicated. In the form factor version of the peripheral model with only pseudoscalar meson exchange, one obtains $\rho_{11} = \frac{1}{2}$, $\rho^{00} = 1$, (and thus, by virtue of (14), $\rho_{33} = \rho^{11} = 0$), while all other decay parameters of (37) vanish; the joint decay distribution thus provides a severe test of this simple model.

For reaction (3), again omitting the isospin, the total number of independent production amplitudes is 32. If charge conjugation invariance is invoked, this number is reduced to 24. The joint decay distribution (40), together with $\frac{d\sigma}{dx}$, determines in this case 20, with charge conjugation 13, real parameters, giving only an incomplete determination of the amplitudes, even if these were real. The prediction of the form factor version of the peripheral model with pure pseudoscalar meson exchange is $\rho_{11} = \rho^{11} = \frac{1}{2}$, (and thus, by virtue of (14), $\rho_{33} = \rho^{33} = 0$), while all the other decay parameters of (40) vanish. The joint decay distribution thus provides a severe test of this simple model.

For a general test of the peripheral model it would of course be much easier if one could see directly if the imaginary parts of the production amplitudes vanish or not. Unfortunately, the imaginary part of the density matrices do not appear in the joint decay distributions (37) and (40). This is due to parity conservation in the decays, Eq. (23).

On the other hand, if one of the unstable particles decays through a parity violating interaction, the imaginary parts of the density matrix elements do occur in the decay distribution, and could thus be measured experimentally. Take for instance reaction (1). In the notation of Byers and Yang [1], the peripheral model requires

$$F_{04} = F_{34} = F_{35} = F_{16} = 0 \tag{44}$$

Another case where $\mathrm{Im}\,\rho^{mm'}_{nn'}$ enters in the decay distribution is when one (or both) of the incident particles are polarized. This would be the case in reactions with polarized targets, or in photoproduction with polarized photons.

In a way, the phase determination constitutes the most general (necessary but not sufficient) test of the peripheral model, since it makes no assumptions on the spin, coupling constants, mass, etc., of the exchanged mesons. It would be of great value to perform this test, because one knows that the production amplitudes cannot be strictly real, their real and imaginary parts being connected by the unitarity condition. This illustrates the fact that the usual peripheral model is not unitary. On the other hand, in the K-matrix model of peripheral interactions [10], which does lead to a unitary S matrix, the production amplitudes, and consequently the elements of the density matrix in the final state, are no longer purely real.

Concluding then the analysis of the joint decay distribution in double resonance production gives, presently, the maximal amount of information on the production mechanism. In order to get complete information on these reactions, it will be necessary to undertake polarization measurements.

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