REMARKS ON THE PHENOMENOLOGICAL ANALYSIS OF RESONANCES

(a) SHAPE AND POSITION

(b) DECAY CORRELATIONS AND EXCHANGES

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ABSTRACT

Some remarks are made concerning various aspects of the analysis of resonances produced in multi-particle final states resulting from high-energy collisions. The shape and position of a resonance are discussed by means of the connection between production cross-sections for stable and unstable particles, and a numerical example of N^* as seen in $KN \rightarrow KN^*$ at different energies is given. The relation between the present description of the resonant effects and others (Watson's final state interaction formula and the one meson exchange model of Chew and Low) is examined. Off-the-mass shell corrections to the one meson exchange model are obtained in a relatively simple way. The bearing of the decay angular correlations on the production mechanism is explored; co-ordinate axes appropriate for the peripheral model are specified ; the equivalence of the azimuthal angular distribution and the Treiman-Yang distribution is established. Angular correlations are given for the decay of boson resonances $(J = 1,2)$ resulting from definite spin and parity states in the t channel. Some remarks are made about the inclusion of non-resonant contributions when there is a mixture of pseudoscalar and vector exchange.

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29 April 1964

1. INTRODUCTION

In the quantitative study of resonant states produced in fundamental particle interactions one is interested in the detailed properties of the states themselves (mass, width, isospin, decay modes, spin, parity, etc.), and also in the production mechanisms. For a resonance which appears in elastic scattering, correlation of the data on scattering and associated reaction channels can determine the properties of the resonance in a manner familiar in low energy nuclear physics. A classic example of this type of analysis is the elucidation of the nature of the Y_0^* (1520) by Watson, Ferro-Luzzi, and Tripp [1)]. If a resonance is produced as one of the reaction products in a multi-particle final state, its mass and width can be determined from an invariant mass plot for its decay products. Additional information can be obtained from the angular correlations of decay. The complexity of the distribution [2)], and more elaborate analyses in terms of various moments [3)], can be used for spin and parity determination. For known spins the decay correlations throw light on the production mechanism [4)].

In the present paper we wish to make some rather simple remarks concerning (a) the shape and position of resonances produced in inelastic collisions, and (b) the decay angular correlations. Most of the results are undoubtedly known to many workers in the field, but apparently not to all.

The reasons for considering the shape and position of resonances are two-fold. On the experimental side, there is the problem of knowing what is the most reasonable resonant shape to employ in order to make a precision fit to a resonance, or in more specific and dramatic terms, of understanding why the N_{33}^* appears at 1212 MeV in a production reaction [5)], at 1225 MeV in elastic scattering, and at 1238 MeV in the tables of properties of fundamental particles. On the theoretical side, there is the question of comparison of various models for production with experiment. In some calculations the resonant state is treated as a stable particle with the appropriate internal quantum numbers [6),7)]; in the one-pion exchange model the resonant line shape is expressed in terms of the scattering cross-section [8)]; sometimes resonant effects are included as a final-state interaction [9),10)]. The interconnection of these different theoretical treatments needs discussion.

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The remarks concerning the angular correlations of the decay products from a resonant state are of two kinds. The first are of a purely kinematical nature concerning the choice of axes and the fact that the well-known Treiman-Yang distribution [11] is just the azimuthal angular distribution with respect to a certain set of axes in the rest frame of the resonance. The second kind of remark is on the implications of the form of the decay correlations for the production mechanism, especially in terms of t channel properties (appropriate for the peripheral model).

The purpose of the paper is to spell out in some detail various points of interest primarily to experimenters. There is nothing of real significance on the theoretical side. But it is hoped that the paper will be of some value in clarifying what can be learned from the study of resonant states that are produced in multi-particle reactions. Sections 2 to 5 are devoted to the shape and position of a resonance and the inter-relation between various theoretical descriptions, as well as the example of N^* production in $KN \to KN^*$. The final three sections (6-8) are concerned with the decay correlations and the connection with the production mechanism. Two appendices contain results on the energy dependence of resonant widths and a critique of the off-mass-shell correction factor of Ferrari and Selleri.

2. CONNECTION BETWEEN CROSS-SECTIONS FOR PRODUCTION OF STABLE AND RESONANT STATES

The question of the shape and position of a resonance in a production reaction will be phrased in terms of the connection between the cross-sections for the production of a stable particle and an unstable one. The two processes are indicated in Fig. 1.

Before discussing the connection between the two cross-sections we consider the phase space for n particles in a form appropriate for two particles in a resonant state. Starting from the standard invariant phase space,

$$dF_n = \int \frac{d^3p_1\, d^3p_2 \cdots d^3p_n}{(2\pi)^{3n}(2E_1)(2E_2)\cdots(2E_n)}\, \delta^4(p_1+p_2+\cdots+p_n-P)\,, \tag{1}$$

it is straightforward to show that (1) can be written as

$$dF_n = f \times \int \frac{d^3Q\, d^3p_3 \cdots d^3p_n}{(2\pi)^{3(n-1)}(2Q_0)(2E_3)\cdots(2E_n)}\, \delta^4(Q+p_3+\cdots+p_n-P) \tag{2}$$

where

$$f = \frac{1}{4(2\pi)^3}\, \frac{q}{\omega}\, d\Omega_{12}\, d\omega^2 \tag{3}$$

is the two-body phase space in the rest frame of particles 1 and 2. In (2) and (3), $\mathbf{Q} = \mathbf{p}_1+\mathbf{p}_2$ is the 3-momentum of the pair (1+2), ω is their invariant mass $(\omega^2 = -(p_1+p_2)^2)$, Q_0 is their energy $(Q_0^2 = \mathbf{Q}^2+\omega^2)$, while q is the 3-momentum of each member of the pair in their rest frame. The form of (2) is that of $(n-1)$ body phase space, with one "particle" having mass ω, times the internal phase space (3) for two particles.

Let the cross-section for the production of a stable particle R_0 of mass ω, summed over the spin states of R_0, be denoted by $d\sigma_s(\omega)$. This is the process indicated in the upper diagram of Fig. 1. The cross-section for production of the resonant state R, integrated over all angles of decay in the rest frame of the resonance, can be expressed in terms of $d\sigma_s(\omega)$ by modifying the calculation to include a propagator and decay vertex amplitude V for R, as shown in the lower part of Fig. 1. The result is

$$d\sigma = d\sigma_s(\omega)\left[\frac{\pi^{-1}\,\omega_0\Gamma(\omega)}{(\omega_0^2-\omega^2)^2+\omega_0^2\Gamma^2(\omega)}\right]d\omega^2 \tag{4}$$

where ω_0 is defined as the mass of the resonance, and $\Gamma(\omega)$ is the width, defined by

$$\Gamma(\omega) = \frac{1}{32\pi^3}\,\frac{1}{2J+1}\sum\int |V|^2\,\frac{q}{\omega}\,d\Omega_{12}. \tag{5}$$

In (5), J is the angular momentum of the resonant state, the summation is over the spins of the outgoing decay particles, and the integration is over the decay angles of the two-body phase space (3). Equation (4) has been given previously by Bergia, Bonsignori and Stanghellini [12], by Bordes and Jouvet [13], and even earlier by R.F. Christy [14].

Several remarks should be made about (4) :

(a) It is valid only if the resonance is produced "cleanly", as implied in Fig. 1. Interference effects in the final state are ignored.

(b) Although the assumption of two-body decay has been made in our discussion, (4) holds for more complicated decays provided $\Gamma(\omega)$ is suitably defined.

(c) If the resonance has alternative modes of decay and only one mode is being considered, the width in the numerator of (4) becomes the partial width for that mode while the width in the denominator is the total width.

(d) For a narrow resonance, integration over the line shape in (4) yields the stable particle cross-section evaluated at $\omega \simeq \omega_0$.

(e) Equation (4) cannot quite be derived by perturbation theory involving a simple propagator for the resonance. The resonance must be given a complex mass or the propagator must be calculated more accurately [13]. But justification for the denominator in (4) is hardly necessary.

The elastic scattering of the pair (1,2) via the resonant state gives rise to a scattering cross-section of the form

$$\sigma_{scatt}(\omega) = \frac{4\pi}{q^2} \frac{2J+1}{(2j_1+1)(2j_2+1)} \frac{\omega_0^2 \Gamma^2(\omega)}{(\omega_0^2-\omega^2)^2+\omega_0^2\Gamma^2(\omega)} \tag{6}$$

where (j_1, j_2) are the angular momenta of the pair (1,2). The presence of Γ^2 in the numerator of (6) and only Γ in (4) is a result of the fact that in scattering the resonance is formed and then decays, while in production only the decay occurs. In production the second Γ is replaced by the square of a coupling constant in $d\sigma_s(\omega)$, as we will see below. The elastic scattering (6) and production cross-section (4) can be expressed in terms of a resonant phase shift, $\delta(\omega)$, given by

$$\tan\delta = \frac{\omega_0 \Gamma(\omega)}{\omega_0^2 - \omega^2} \tag{7}$$

In particular, (4) can be written

$$d\sigma = d\sigma_s(\omega) \frac{\sin^2\delta}{\pi\omega_0\Gamma(\omega)} d\omega^2 \tag{8}$$

This form will be useful for comparison with the one meson exchange model of Ferrari and Selleri [8], and the final state interaction result of Watson [15].

The shape and position of a resonance is governed mainly by the line shape factor in (4), although the energy variation in the stable particle cross-section $d\sigma_s(\omega)$ can enter significantly, as we will see in an example later. If the resonance is broad the energy dependence of the width $\Gamma(\omega)$ can cause distortion of the line shape, with the peak position falling <u>below</u> ω_0 and the shape being skewed to higher energies *). If the two-body decay of the resonance proceeds via a partial wave of orbital angular momentum ℓ, the width $\Gamma(\omega)$ varies with energy roughly as

$$\Gamma(\omega) \simeq \Gamma_0 (q/q_0)^{2\ell+1} \tag{9}$$

With this energy dependence the shift in the peak position is found from (4) to be

$$\frac{\omega_0 - \omega_{peak}}{\Gamma_0} \simeq \frac{2\ell+1}{8}\left(\frac{\Gamma_0}{\omega_0}\right)\left[\frac{\omega_0^4 - (m_1^2-m_2^2)^2}{\omega_0^4 - 2(m_1^2+m_2^2)\omega_0^2 + (m_1^2-m_2^2)^2}\right], \tag{10}$$

the factor in square brackets being $\omega_0^2(dq^2/d\,\omega^2)_0/q_0^2$. This factor can become large for small energy release, varying inversely as the Q value if ω_0 is near the two-particle threshold. Consequently there can be an appreciable difference

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*) There are circumstances where the peak can fall <u>above</u> ω_0, for example in a partial cross-section where the total width in the denominator of (4) is essentially constant but the partial width in the numerator varies rapidly with energy. Such cases are discussed by Glashow [16].

between ω_{peak} and ω_o for low-lying, broad resonances. The peak of N^* (1238), for example, is estimated from (10) to occur roughly 23 MeV below the mass value $\omega_o = 1238$ MeV for $\Gamma_o \simeq 125$ MeV. The maximum of the scattering cross-section (6) is also shifted, of course, by an amount given by (10), with the factor $(2\ell+1)$ replaced by 2.

For quantitative fitting of data the energy variation of $\Gamma(\omega)$ given by (9) is perhaps too crude. In Appendix A some theoretical results for $\Gamma(\omega)$ are given for cases of interest, and also some empirical forms.

3. RESONANT FACTOR FOR MODIFIED PHASE SPACE CALCULATIONS - CONNECTION WITH FINAL STATE INTERACTION FORMULA OF WATSON

The proper expression for modifying phase space calculations to include a resonance between a pair of particles can be read off from (4). On the assumption that the $(n-1)$ particles and the stable resonance R_0 are produced according to phase space, the stable particle cross-section $d\sigma_s(\omega)$ will be proportional to the $(n-1)$ body phase space in (2), that is, the right-hand side of (2) with f omitted. When that expression is inserted in (4), it is seen that (4) differs from the n body phase space (1) by a factor,

$$\phi_R(\omega) = C\,\frac{\omega}{q_r}\,\frac{\Gamma(\omega)}{(\omega_0^2-\omega^2)^2+\omega_0^2\Gamma^2(\omega)} = C'\,\frac{\omega\sin^2\delta}{q_r\,\Gamma(\omega)} \qquad (11)$$

where C and C' are normalization constants. In the absence of any information on the production mechanism, the mass and width parameters of a resonance appearing in the invariant mass plot of two particles in an n particle final state can be determined by multiplying the predictions of n body phase space by $\phi_R(\omega)$. Note, however, that if Pauli principle effects are important a coherent superposition of amplitudes is necessary [9),10)].

$\phi_R(\omega)$ is essentially the final state enhancement factor of Watson [15)]. This can be seen immediately from (11) when we use the approximate energy variation of $\Gamma(\omega)$ given by (9). Apart from slowly varying factors, $\phi_R(\omega)$ is then proportional to the absolute square of $e^{i\delta}\sin\delta/q^{\ell+1}$, the result of Watson. It is this form that has been used by Dalitz and Miller [9)] and Bouchiat and Flammand [10)], with a Breit-Wigner shape for the resonance.

4. SHAPE AND POSITION OF N^* IN $K^+p \to K^0\pi^+p$

As an example of the appearance of a resonance in production we consider the reaction $K^+p \to K^0\pi^+p$ at 1.14 GeV/c [5] and at 3.0 GeV/c [17], in which N^* (1238) is produced. The three-body phase space is proportional to $p_{K^0}d\Omega_{K^0}$ times f, (3). Thus, in the absence of a model for the production process we expect the invariant mass plot for the pair (π^+p) to be given by the square bracket in (4) times p_{K^0}, i.e., $\frac{q}{\omega}\Phi_R(\omega)p_{K^0}$. But we have a model for this process, namely the ρ - exchange model of Stodolsky and Sakurai [18]. In the differential cross-section, in addition to some complicated Δ^2 dependence, there is an over-all factor of $p^3_{K^0}$, instead of the p_{K^0} of phase space [7]. Although for fixed production angle Δ^2 changes as the invariant mass varies, it is perhaps reasonable to employ $p^3_{K^0}$, not p_{K^0}, times the square bracket in (4) as a representation of the π^+p invariant mass spectrum. Since $p^3_{K^0}$ decreases with increasing ω^2, this factor tends to push the peak of the resonant line towards lower values of ω^2 and to make the shape narrower and more symmetrical. Figure 2 shows a comparison of N^* resonant shapes at 1.14 GeV/c and 3.0 GeV/c calculated in the above manner, using ω_0 = 1238 MeV, Γ_0 = 140 MeV, and the theoretical width formula (A.4). The peak positions and full widths at half maximum are ω_{peak} = 1210 MeV, $\Delta\omega$ = 82 MeV and ω_{peak} = 1217 MeV, $\Delta\omega$ = 96 MeV, respectively. The resonant shape in (4) is very similar to the 3 GeV/c curve, peaking at 1219 MeV, with a width of 100 MeV. Both curves in Fig. 2 are in reasonable accord with the respective data [5),17] and illustrate forcibly the differences between the observed position and width $(\omega_{peak}, \Delta\omega)$ and the theoretical parameters (ω_0, Γ_0).

For resonances that are narrow and/or well above threshold the effects of the energy dependence of the width and other dynamical factors are less than for the N^*. The appearance of the ρ meson, for example, is more or less unaltered from a symmetrical resonance curve, with the peak shifted downwards by about 4 MeV from $\omega_0 \simeq$ 750 MeV and the apparent width about 5 MeV less than $\Gamma_0 \simeq$ 100 MeV. For the ρ meson, of course, interference effects seem important ; these distort the shape far more than the energy dependence of the width.

5. CONNECTION WITH THE ONE MESON EXCHANGE MODEL OF CHEW-LOW

The structure of (8) is similar to the one meson pole contribution to the production process in that $\sin^2 \delta$ can be considered as proportional to the scattering cross-section for the pair (1,2). The presence of the scattering cross-section at the pole was, in fact, the motivation for the original discussion by Chew and Low [19]. In the subsequent work on the one meson exchange model [8] it is assumed that the basic diagram shown in Fig. 3 dominates the production process in the region of small, but physical, momentum transfers. Although not a necessary restriction on the model, in the applications it is usual that the incident particle 1' and the exchanged meson 2' (taken to be pseudoscalar) are the same type of particles as the pair (1,2). It is clear from Fig. 3 that in the limit that the exchanged meson is on the mass shell the production cross-section will be proportional to the elastic scattering cross-section $(1',2' \to 1,2)$. But in the physical region off-the-mass-shell effects enter, and we wish to indicate to what extent (8) includes such effects and to compare it with the so-called pole approximation of Ferrari and Selleri. Figure 3 represents only a special one meson exchange diagram in which the production of the pair (1,2) proceeds via a resonant state, but this is the situation in most applications. We will begin our discussion with the important examples of a vector meson (ρ, K^*) decaying into two pseudoscalars and a baryonic isobar (N^*, Y_1^*) decaying into a baryon and a pseudoscalar meson. The comparisons can be accomplished by inspection of the stable particle production cross-sections [7] and the formulae of Ferrari and Selleri, but we shall give a direct argument here.

When the upper vertex in Fig. 3 involves two pseudoscalar mesons (1',2') and a vector meson (R), the stable particle cross-section $d\sigma_s(\omega)$ is proportional to the square of the vertex amplitude $V' = g(p_{1'} - p_{2'}) \cdot \epsilon = 2g(p_{1'} \cdot \epsilon)$, where g is the coupling constant for $R \to 1',2'$. When summed over the vector meson's polarizations, $d\sigma_s(\omega)$ will involve a factor $g^2 p_R^2$, where p_R is the magnitude of the 3-momentum of the incident meson 1' in the rest frame of the resonance. The width $\Gamma(\omega)$ which appears in the denominator of (8) is proportional to g^2q^3/ω (see Appendix A). Thus the cross-section (8) will involve, as the factor from the upper vertex in Fig. 3,

$$\omega \frac{p_R^2}{q^3} \sin^2\delta$$

This can be compared with the factor from the pole approximation of Chew and Low [19] and Ferrari and Selleri [8],

$$\omega q\, \sigma_{scatt}(\omega) \propto \frac{\omega \sin^2\delta}{q}$$

The ratio of (8) to the pole approximation cross-section is thus

$$\frac{d\sigma}{d\sigma_{pole}} = \frac{p_R^2}{q^2} = \frac{\left[(\omega+m_1)^2+\Delta^2\right]\left[(\omega-m_1)^2+\Delta^2\right]}{\left[(\omega+m_1)^2-m_2^2\right]\left[(\omega-m_1)^2-m_2^2\right]}, \qquad (12)$$

where Δ^2 is the 4-momentum transfer, apart from a vertex form factor. In the limit $\Delta^2 \rightarrow -m_2^2$ the right-hand side of (12) is unity, but in the physical region of Δ^2 it is larger than one and increases with increasing momentum transfer. The correction factor (12) for vector meson production was first obtained by Selleri [20], using essentially the same argument.

If particle 1' is a nucleon and the resonance is a $(\frac{3}{2}^+)$ isobar, similar considerations show that the ratio of (8) to the pole approximation of Ferrari and Selleri is

$$\frac{d\sigma}{d\sigma_{pole}} \simeq \left[\frac{(\omega+m_1)^2+\Delta^2}{(\omega+m_1)^2-m_2^2}\right]\frac{p_R^2}{q^2} \qquad (13)$$

where (p_R^2/q^2) is given by (12). Salzman and Salzman [21] considered off-mass-shell effects for the N^* within the framework of the static model and obtained (13) with the first factor omitted.

For formation of a $(\frac{3}{2}^-)$ isobar with an incident nucleon 1', the result corresponding to (13) is

$$\frac{d\sigma}{d\sigma_{pole}} \simeq \left[\frac{(\omega+m_1)^2-m_2^2}{(\omega+m_1)^2+\Delta^2}\right]\left(\frac{p_R}{q}\right)^4 \qquad (14)$$

The presence of (p_R^4/q^4) is because of the d-wave character of the formation or decay.

Inspection of (8) and (9), as well as (12), (13), (14), leads to the inference that for higher spin resonances the dominant kinematic off-the-mass shell correction to the pole approximation will be

$$\frac{d\sigma}{d\sigma_{pole}} \simeq \left(\frac{p_R}{q}\right)^{2\ell}, \qquad (15)$$

where ℓ is the relative orbital angular momentum in the formation of the resonance *). For a resonance formed from two spinless bosons this result follows directly because the vertex amplitude V' of Fig. 3 will involve a product of

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*) This is remarked on by Selleri [20].

the tensor of rank ℓ describing the spin of that particle and a tensor of the same rank made up of the 4-momenta of 1' and 2'. In the particle's rest frame, V' will then have a dominant factor p_R^{ℓ} (at least if the interaction is assumed to be of short range), while the vertex V in the width (5) describing its decay will have a corresponding factor q^{ℓ}. For baryonic resonances the baryon spins introduce additional terms, but these are generally slowly varying for small Δ^2 [see (13) and (14)].

For N^* production off-the-mass-shell effects have been investigated with dispersion relation techniques by Iizuka and Klein [22] and Ferrari and Selleri [23] with the result that for small Δ^2 the correction factor (12) is obtained. Ferrari and Selleri give, in addition, a more elaborate result which is essentially (13) multiplied by the square of $(1+3\alpha_0)/(1+\alpha_0)^3$, where $\alpha_0 = (\Delta^2+m_2^2)/2m_1(\omega_0-m_1)$. This additional factor falls off rapidly at large Δ^2 and more than cancels the increase with Δ^2 present in (13). The validity of this additional factor is discussed in Appendix B, with the conclusion that (13) is more or less correct as it stands.

The correction factors (12) - (15) for the pole approximation can be generalized to processes in which 1' and 2' of Fig. 3 are not the same as 1 and 2 (e.g., $N\bar{K} \rightarrow Y^* \rightarrow Y\pi$). Then the pole approximation is proportional to the reaction cross-section, rather than the scattering cross-section. It is straightforward to show that the only change in (12) is that the masses m_1 and m_2 are replaced by m_1' and m_2', respectively. This clearly must be so because it is the initial state (1',2') that is off-the-mass-shell.

In actual applications the off-the-mass-shell corrections to the pole approximation are masked by additional Δ^2 dependence, not present in lowest order perturbation theory, and usually expressed in terms of empirical form factors [7),8)]. One can argue that the explicit Δ^2 dependence of (12) - (15) should be retained since it is basically kinematic in origin ; the form factors then represent dynamical modifications of the vertices and propagator. But if the empirical form factors result mainly from absorption effects in the low partial waves, the argument for explicit retention of (12) - (15) is less clear.

6. CHOICE OF DECAY AXES ; EQUIVALENCE OF AZIMUTHAL AND TREIMAN-YANG ANGLE

The angular correlations of the decay products of a resonance are most conveniently considered in the rest frame of the resonance. For two-body decays the direction of one of the decay particles defines the decay angles (θ, ϕ) relative to a suitable set of co-ordinate axes. For ω decay the normal to the three pion decay plane replaces the direction of one of the particles in the two-body decay.

The choice of axes is dictated by considerations on the probable mechanism of production. For peripheral collisions, in which the description of the production process in the t channel is assumed to be simple, there is a natural choice of z axis, namely the direction of the "incident particle" in the rest frame of the resonance. Figure 4 illustrates various co-ordinate frames. The upper diagram indicates the particles involved, while the lower three diagrams show the momenta in the centre-of-mass system, the rest frame of $\underline{a}$ (the "incident particle"), and the rest frame of $\underline{c}$ (the resonance). It will be noted that in the rest frame of $\underline{c}$ the momenta $\underline{a}$ and $\underline{e}$ are equal and opposite. Thus the choice of the direction of $\underline{a}$ as z axis is the same as choosing the momentum transfer direction, a natural axis for consideration of exchanged systems. The azimuthal angle ϕ is specified by choosing the normal to the production plane as the y axis.

Before discussing some of the angular distributions and their implications, we wish to make a remark concerning the Treiman-Yang angle [11]. This angle is defined as the angle between the plane containing the two particles from the decay of the resonance and the production plane, as seen in the rest frame of the "incident particle" $\underline{a}$. Treiman and Yang showed that isotropy in the distribution of this angle is consistent with pion (or K meson) exchange, although it does not prove it. Subsequently it has become customary that experimenters give a Treiman-Yang angular distribution, as well as other decay correlations. We assert that the azimuthal angle ϕ and the Treiman-Yang angle are the same, even though they are defined in different reference frames, and that exhibiting both angular distributions is redundant.

That something like this must be true can be seen from Fig. 4. In the rest frame of a, the exchanged system e and the resonance c have the same momenta. If e has spin zero the decay plane of c must show rotational symmetry around the common momentum. This is the essence of the Treiman-Yang test. But now consider the rest frame of c. Here the exchanged system e is incident along the z axis (the direction of a). Again, if e has spin zero we expect rotational symmetry in the decay of c. This is isotropy in the azimuthal angle ϕ .

A direct demonstration of the equality of ϕ and the Treiman-Yang angle can be seen from Fig. 5, where the co-ordinate axes in the rest frame of c are shown along with the momenta of the decay particles (1,2). If a Lorentz transformation is made along the z axis to reach the rest frame of a, the momenta of the decay particles (1,2) will remain in the shaded plane containing the z axis and their common line. This then is the decay plane of Treiman and Yang, while the x - z plane is the production plane. The angle between these planes is evidently ϕ .

There is only one point to discuss, namely the fact that in reaching the rest frame of a we made two Lorentz transformations, from the centre-of-mass system to the rest frame of c and then to the rest frame of a. Normally one would transform directly from the centre of mass to the rest frame of a in order to compute the Treiman-Yang angle. It is well known that these two ways of reaching the rest frame of a give results differing at most by a three-dimensional rotation, in this case around the normal to the production plane. But a rotation of all momenta around this normal does not change the angle between the two planes, and so the inequivalence of the Lorentz transformations is of no consequence.

7. DECAY ANGULAR CORRELATIONS AND EXCHANGED SYSTEMS

The structure of the decay correlation $W(\theta,\phi)$ is limited by the angular momentum J of the resonance and the conservation of parity in both the production and the decay process [4]. For a resonance with $J = 1$ decaying into two spinless particles it is

$$W_1(\theta,\phi) = \frac{3}{4\pi}\left\{ \rho_{00}\cos^2\theta + \rho_{11}\sin^2\theta - \rho_{1,-1}\sin^2\theta\cos 2\phi - \sqrt{2}\,\mathrm{Re}\,\rho_{10}\sin 2\theta\cos\phi \right\} \tag{16}$$

where $\rho_{MM'}$ are the elements of the spin density matrix of the resonance whose trace $(2\rho_{11}+\rho_{00})$ is normalized to unity. The angular distribution for a resonance of arbitrary spin is given by Eq. (17) of Ref. [4]. For a $J = \frac{3}{2}$ baryon resonance the general structure of $W(\theta,\phi)$ is the same as (16), although there is a rearrangement of terms going with the $J = \frac{3}{2}$ density matrix. The structure of (16) is general ; the specific production mechanism will determine the values of the density matrix elements $\rho_{MM'}$. For $J = \frac{3}{2}$ isobar production the fact that (16) is the most general form possible vitiates the observation of Stodolsky and Sakurai [18] that the azimuthal dependence of (16) results from vector-meson exchange. For higher spin resonances their remark is relevant, but for the important case of $J = \frac{3}{2}$ (independent of parity) the azimuthal dependence must be that of (16), regardless of the production mechanism.

If $W(\theta,\phi)$ is integrated over $d(\cos\theta)$, the Treiman-Yang angular distribution results. For $J = 1$ or $J = \frac{3}{2}$ this has the form $(a+b\cos 2\phi)$.

We now turn to the question of the production mechanism and its consequences for decay distributions like (16). We will restrict the discussion to a boson resonance produced peripherally by an incident pseudoscalar meson. The relevant diagram is shown in Fig. 4, where e represents an exchanged meson, or more generally, the t channel states. It should be recalled that in the rest frame of the resonance c the axes have been chosen so that the incident particle a and the exchanged system e are incident along the z axis and form the resonance, as is shown in the lower right-hand part of Fig. 4.

For a $J = 1^-$ resonance it is proved in Ref. [4] that for 0^- exchange only ρ_{00} is different from zero in (16) - a well-known result, for natural parity exchanges $(1^-, 2^+, \ldots)$ only ρ_{11} and $\rho_{1,-1}$ are different from zero, while for unnatural parity $(1^+, 2^-, \ldots)$, all elements of $\rho_{MM'}$ are different from zero in general. It should be noted especially that $\rho_{10} \neq 0$ demands the presence of unnatural parity states with $J \geqslant 1$ in the t channel.

The discussion of Ref. [4] is based on the properties of helicity amplitudes. It is perhaps worthwhile to give a more pedestrian derivation which can be phrased so as to include consideration of resonances of higher spin. Suppose that a resonant state of spin and parity J^P is formed from two systems of spin and parity, 0^- (the incident meson) and $J_e^{P_e}$ (the exchanged system), with relative orbital angular momentum ℓ, as indicated in Fig. 6. Parity conservation limits the ℓ values to being even (odd) for PP_e odd (even). Furthermore angular momentum conservation restricts the values of ℓ to the interval $|J-J_e| \leqslant \ell \leqslant J+J_e$. For $J = 1,2$ and $0 \leqslant J_e \leqslant 3$ the possible values of ℓ are shown in Table 1. For $J = 1^-$ (1^+) the natural (unnatural) parity states, with $P_e = (-1)^{J_e}$ $((-1)^{J_e+1})$, have a single ℓ value, $\ell = J_e$. This is a decisive factor in consideration of the density matrix. The density matrix is bilinear in the amplitude for formation of the resonance. The amplitude for a resonant state with $J_z = M$ will be proportional to the vector addition coefficient, $\langle J_e \ell M_e 0 | JM \rangle$. It is hardly necessary to mention that if $J > J_e$, only amplitudes with $|M| \leqslant J_e$ will occur, with a consequent limitation on the number of non-vanishing elements $\rho_{MM'}$. But consider in particular the amplitude for $M = 0$. It is proportional to $\langle J_e \ell 00 | J0 \rangle$, which is well known to vanish unless $J_e + \ell + J$ is even. For many of the combinations in Table 1 there will therefore be vanishing amplitudes for $M = 0$. For example, when $J = 1$ those sequences of exchanged systems with $J_e = \ell$, will give rise to vanishing density matrix elements ρ_{M0} while for the other exchanged states all $\rho_{MM'}$ will occur in general. We have thus derived the results of Ref. [4] quoted above.

Values of orbital angular momentum ℓ when a 0^- meson and a system $J_e^{P_e}$ form a resonance J^P

		$J_e^{P_e}$							
		0^+	0^-	1^+	1^-	2^+	2^-	3^+	3^-
J^P	1^+	1	-	1	0,2	1,3	2	3	2,4
	1^-	-	1	0,2	1	2	1,3	2,4	3
	2^+	-	2	1,3	2	1,3	0,2,4	1,3,5	2,4
	2^-	2	-	2	1,3	0,2,4	1,3	2,4	1,3,5

TABLE 1

By the same argument we see that for a $J = 2^+$ resonance (f meson) pseudoscalar exchange will give only ρ_{00} different from zero, while vector-meson exchange will give only ρ_{11} and $\rho_{1,-1}$ non-vanishing, and natural parity exchanges $(1^-,2^+,3^-,\ldots)$ will always give $\rho_{M0} = 0$.

Actually the f meson warrants some comments in view of the recent interest in an unambiguous spin determination [24]. For a $J = 2$ resonance decaying into two spinless bosons the general decay angular distribution is [4]

$$W_2(\theta,\phi) = \frac{15}{16\pi}\Big\{ 3\rho_{00}\left(\cos^2\theta - \tfrac{1}{3}\right)^2 + 4\rho_{11}\sin^2\theta\cos^2\theta + \rho_{22}\sin^4\theta$$

$$-2\cos\phi\sin 2\theta\left[\mathrm{Re}\,\rho_{21}\sin^2\theta + \sqrt{6}\,\mathrm{Re}\,\rho_{10}\left(\cos^2\theta - \tfrac{1}{3}\right)\right]$$

$$-2\cos 2\phi\sin^2\theta\left[2\rho_{1,-1}\cos^2\theta - \sqrt{6}\,\mathrm{Re}\,\rho_{20}\left(\cos^2\theta - \tfrac{1}{3}\right)\right]$$

$$+2\,\mathrm{Re}\,\rho_{2,-1}\cos 3\phi\sin^2\theta\sin 2\theta$$

$$+\rho_{2,-2}\cos 4\phi\sin^4\theta\Big\} \qquad (17)$$

When this distribution is integrated over ϕ it simplifies to

$$\int_0^{2\pi} W_2(\theta,\phi)\,d\phi = \frac{15}{8}\left[3\rho_{00}\left(\cos^2\theta - \tfrac{1}{3}\right)^2 + 4\rho_{11}\sin^2\theta\cos^2\theta + \rho_{22}\sin^4\theta\right] \qquad (18)$$

Furthermore, if one assumes that only pseudoscalar and vector exchanges are responsible, (17) reduces to

$$W_2(\theta,\phi) \rightarrow \frac{15}{16\pi}\Big\{ 3\rho_{00}\left(\cos^2\theta - \tfrac{1}{3}\right)^2 + 4\sin^2\theta\cos^2\theta\left(\rho_{11} - \rho_{1,-1}\cos 2\phi\right)\Big\} \qquad (19)$$

with an integral over azimuth that involves only the first two terms in (18).

In Ref. [24] only the ρ_{00} term was kept in (17) (although some s wave interference was included) since the tacit assumption of pion exchange was made. While there do not seem to be any vector mesons of the right quantum numbers (T = 1, G = -1) known at present, it should be kept in mind that the general θ dependence of decay of a J = 2 resonance is given by (18), and that some amount of ρ_{11} is possible even if ρ_{22} is likely to be very small.

8. INCLUSION OF NON-RESONANT INTERFERENCE TERMS IN THE DECAY CORRELATIONS

The discussion so far has been based on the assumption of production of a pure resonance, that is, a state with only one angular momentum and parity. For the ρ and K^* mesons it is known that this is not an adequate description[25),26)]. Significant interference effects, not present in (16), are observed above and below the resonance. We wish to point out that, to the extent that such states are formed in peripheral collisions involving only pseudoscalar and vector meson exchanges, the non-resonant effects can be incorporated at least approximately in the usual manner by use of the elastic scattering cross-section to describe the 0^- exchange. (For vector exchange, however, we neglect non-resonant effects.) The reason that this is possible is the lack of coherence between 0^- and 1^- exchange. The 0^- meson gives only amplitudes with $M = 0$, while the 1^- meson only populates $M = \pm 1$ states.

If the decay distribution is generalized to be a function of ω, as well as the angles, and we use the Chew-Low pole approximation, corrected by (15), we will have a generalization of the ρ_{00} term in $W(\theta, \phi)$ to the form

$$\frac{2\ell+1}{4\pi}\left|P_\ell(\cos\theta)\right|^2 \rho_{00} \longrightarrow$$

$$\frac{\rho_{00}}{(2\ell+1)4\pi^2\omega_0\Gamma_0}\left(\frac{q_0}{p_{R0}}\right)^{2\ell}\frac{q_0}{q}\left|(2\ell+1)\left(\frac{p_R}{q}\right)^{\ell} e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)\right|^2 \quad (20)$$

where the resonant factor has been normalized approximately to unity on integration over $d\omega^2$ (accurate in the limit of a narrow resonance). For ρ_{11} and $\rho_{1,-1}$ similar resonant shapes will occur. Equation (20) has been written in anticipation of the inclusion of non-resonant amplitudes. The factor in (20) whose absolute square is taken is just one term in the partial wave expansion of the scattering amplitude, corrected according to (15) for off-the-mass-shell effects. The interference effects can be incorporated by inserting the full scattering amplitude (actually q times it),

$$f(\theta) = \sum_{L=0}^{\infty} (2L+1)\left(\frac{p_R}{q}\right)^L e^{i\delta_L} \sin\delta_L \, P_L(\cos\theta) , \qquad (21)$$

into (20) in place of the single term, $L = \ell$. In writing (21) we have assumed that the off-mass-shell correction (15) holds for non-resonant, as well as resonant, partial waves.

Equation (20), with (21) inserted, is just the Chew-Low cross-section, corrected roughly for off-the-mass-shell effects. But the point is that, to the extent the vector-exchange part does not have significant interference effects, the pseudoscalar-exchange term involving ρ_{00} can be treated in a semi-rigorous manner, independently of the presence of appreciable vector exchange. Wojcicki's data [26] on $\bar{K}^*$ production in K^-p collisions at 1.5 - 1.7 GeV/c, for example, can be interpreted as due to roughly equal amounts of vector meson and pion exchange, even though there is a non-resonant s wave contribution to the pion exchange which is large enough to give a sizeable forward-backward asymmetry in θ that changes sign from below to above the resonance.

It should perhaps be stressed that these remarks on the inclusion of non-resonant effects for boson resonances apply only to a mixture of 0^- and natural parity states in the t channel. If appreciable contributions from unnatural parity states occur (as evidenced by a large $\sin 2\theta \cos\phi$ term in (16), for example), these contributions can interfere with the 0^- exchange amplitudes and cause departures from (20) and (21).

ACKNOWLEDGEMENTS

The author wishes to thank the members of the Track Chamber Division for conversations on the experimental aspects of the production of resonances, and K. Gottfried and H. Pilkuhn for discussions on the theory. He wishes to acknowledge the kind hospitality of CERN during his stay.

APPENDIX A

Energy Dependence of Resonant Widths

We collect here some results for the energy variation of $\Gamma(\omega)$ for two-body decays. The expressions will be written in the form

$$\Gamma(\omega) = \Gamma_0 \left(\frac{q}{q_0}\right)^{2\ell+1} \frac{\rho(\omega)}{\rho(\omega_0)} \tag{A.1}$$

where q, ω, and ℓ are the 3-momentum of each particle, the total energy, and relative orbital angular momentum, Γ_0 is the width at $\omega = \omega_0$, and $\rho(\omega)$ is a relatively slowly varying factor.

Theoretical widths

The following results for $\rho(\omega)$ are found from lowest order perturbation theory. There is therefore the possibility (even necessity) of the presence of a vertex form factor with uncalculable energy variation. Such a factor can be thought of as accounting for the difference between the theoretical and the empirical expressions for $\rho(\omega)$. The angular momentum and parity of the particles is denoted by J^P.

$\underline{(1^-) \rightarrow (0^-)(0^-)}$ $\quad \ell = 1, \quad \rho(\omega) = \omega^{-1}$ (A.2)

$\underline{(1^-) \rightarrow (0^-)(1^-)}$ $\quad \ell = 1, \quad \rho(\omega) = \omega$ (A.3)

$\underline{(\tfrac{3}{2}^+) \rightarrow (0^-)(\tfrac{1}{2}^+)}$ $\quad \ell = 1, \quad \rho(\omega) = \dfrac{(\omega + M)^2 - m^2}{\omega^2}$ (A.4)

$\underline{(\tfrac{3}{2}^-) \rightarrow (0^-)(\tfrac{1}{2}^+)}$ $\quad \ell = 2, \quad \rho(\omega) = \left[(\omega + M)^2 - m^2\right]^{-1}$ (A.5)

In (A.4) and (A.5), M and m are the masses of the spin $\frac{1}{2}^+$ and 0^- particles, respectively. For these spin $\frac{3}{2}$ decays there is some arbitrariness in the result because of arbitrariness in the positive energy projection operator. The description used for the spin $\frac{3}{2}$ particles is that given in Appendix B of Ref. [7].

Empirical widths

N^* (1238) : $(\frac{3}{2}^+) \rightarrow (0^-)(\frac{1}{2}^+)$

$$\rho(\omega) = \left[a\, m_\pi^2 + q^2\right]^{-1} \qquad \text{(A.6)}$$

This form of $\rho(\omega)$ is taken from nuclear reactions theory [27]. Anderson's fit [28] to the (3,3) phase shift leads, through (7), to $a \simeq 2.2$ and $\Gamma_o \simeq 123$ MeV. Gell-Mann and Watson [29] choose $a \simeq 1.3$, $\Gamma_o \simeq 116$ MeV.

ρ (750) : $(1^-) \rightarrow (0^-)(0^-)$

$$\rho(\omega) = \left[q_0^2 + q^2\right]^{-1} \qquad \text{(A.7)}$$

This choice is made by Selleri [20]. Its main virtue is that the energy dependence of the width (A.1) is uniquely determined by the resonance energy ω_o.

Baryonic resonances :

$$\rho(\omega) = \omega^{-1}\left[X^2 + q^2\right]^{-\ell} \qquad \text{(A.8)}$$

where ℓ is the orbital angular momentum of decay. This form is employed by Glashow and Rosenfeld [30] in fitting baryonic resonances in the context of unitary symmetry. They use $X = 350$ MeV.

Use of the empirical forms for $\rho(\omega)$ tends to make the width vary less rapidly with energy than (9). Consequently the shifts in peak position will not be quite as large as given by (10) for the same Γ_o.

A P P E N D I X B

Criticism of the off-the-mass-shell correction of Ferrari and Selleri

Ferrari and Selleri [23], using dispersion relation techniques, considered the modifications of the partial wave amplitudes for pion-nucleon scattering when the incident pion is off the mass shell. For the dominant (3,3) amplitude they obtained, in addition to the square root of (13), a multiplicative correction factor,

$$\frac{1+3\alpha_0}{(1+\alpha_0)^3} , \tag{B.1}$$

where $\alpha_0 = (\Delta^2+\mu^2)/2m(\omega_0-m)$. Here m, μ, ω_0 are the masses of the nucleon, the pion, and the (3,3) resonance, respectively, while $-\Delta^2$ is the square of the "mass" of the incident pion. Numerically, $\alpha_0 \simeq 1.9\ (\Delta^2+0.02)$, where Δ^2 is in $(\mathrm{GeV}/c)^2$. For $\Delta^2 = 0.5$, the factor (B.1) is approximately 0.5 ; it decreases rapidly for larger Δ^2, providing a strong damping for large Δ^2.

To be more precise, the correction factor (B.1) comes from Ferrari and Selleri's Eq. (65) for their amplitude $g(u,\Delta^2)$ (g is the off-shell (3,3) amplitude with the correction factor corresponding to (13) already extracted) :

$$g \simeq K(\Delta^2)\, e^{i\delta} \left[\frac{1+3\alpha_0}{(1+\alpha_0)^3}\frac{\sin\delta}{q^3} + \frac{\lambda}{u}\left(\frac{1+3\alpha}{(1+\alpha)^3} - \frac{1+3\alpha_0}{(1+\alpha_0)^3}\right)\cos\delta \right] \tag{B.2}$$

where $\alpha = (\Delta^2+\mu^2)/2m(\omega-m)$, $u = (\omega-m)/m$, $\lambda = 4f^2/3m\mu^2$, and δ is the (3,3) phase shift. $K(\Delta^2)$ is a vertex form factor. For simplicity we will omit it from now on. It can be included in a trivial fashion, if desired, but its presence encumbers the formulae. The first term of (B.2) dominates, giving the factor (B.1). The second term is very small near resonance because the coefficient of $\cos\delta$ vanishes at resonance and so does $\cos\delta$.

The result (B.2) arises from an approximate solution of a singular integral equation [Ferrari and Selleri's Eq. (51)] of the Muskhelishvili-Omnès type. We wish to show that an alternative method of solution leads to the conclusion that (B.1) should be replaced by unity, leaving (13) as the proper off-the-mass-shell correction for N^* production, and to indicate the probable source of error in the derivation of (B.2) by Ferrari and Selleri.

The integral equation to be solved is a static limit approximation to a fixed Δ^2 dispersion relation :

$$g(u) = B(u) + \frac{1}{\pi}\int_{\mu/m}^{\infty} du' \frac{e^{-i\delta(u')}\sin\delta(u')}{u'-u-i\epsilon}\, g(u') \tag{B.3}$$

with

$$B(u) = \lambda \frac{u(u+3\beta)}{(u+\beta)^3} \tag{B.4}$$

and $\beta = (\Delta^2+\mu^2)/2m^2$. The phase shift δ is assumed to be known. The amplitude $g(u)$ is a function of Δ^2 (as a parameter) ; on the mass shell corresponds to $\beta = 0$. The direct Muskhelishvili-Omnès solution of (B.3) is

$$g(u) = B(u) + \frac{\Omega(u)}{\pi}\int_{\mu/m}^{\infty} du' \frac{e^{i\delta(u')}\sin\delta(u')\, B(u')}{(u'-u-i\epsilon)\,\Omega(u')} \tag{B.5}$$

where

$$\mathcal{R}(u) = \exp \frac{1}{\pi}\int_{\mu/m}^{\infty} dx \frac{\delta(x)}{x-u-i\epsilon} \tag{B.6}$$

(B.5) is essentially Ferrari and Selleri's Eq. (52) ; the phase function (B.6) differs from theirs by the presence of an explicit phase factor $e^{i\delta}$.

Ferrari and Selleri dislike (B.5) as a solution because it involves the function $\mathcal{R}(u)$ and demands numerical integration to obtain $g(u)$. They therefore proceed to obtain their approximate result (B.2). But we wish to cast (B.5) into an alternative form from which a solution can be found easily and with little approximation. The alternative expression for the solution is found in the author's Edinburgh lectures [31)], in particular, Eq. (A.27) on p. 59. For the present problem this form is

$$g(u) = -\frac{\mathcal{R}(u)}{2\pi i}\int_{C_2} du' \frac{B(u')}{(u'-u-i\epsilon)\mathcal{R}(u')} \tag{B.7}$$

where C_2 is a contour (shown in Fig. 7 of Ref. [31)]) around the singularities of $B(u)$ only, rather than along the physical cut, as in (B.5). The only singularitiy of $B(u)$ is a pole of order 3 at $u = -\beta$ *). Thus (B.7) can be written

$$g(u) = \frac{\mathcal{R}(u)}{2} \left.\frac{d^2 f(u')}{du'^2}\right|_{u'=-\beta} \tag{B.8}$$

where

$$f(u') = -\lambda \frac{u'(u'+3\beta)}{(u'-u-i\epsilon)\mathcal{R}(u')} \tag{B.9}$$

---

*) Since, in the static model the important range of u is assumed much less than unity, meaningful results can be expected only for values of Δ^2 such that $\beta < 1$.

In order to evaluate the derivative in (B.8) we must know $\Omega(u')$. Since the point $u' = -\beta$ is removed from the position of the resonance, $u' = u_o$, it is permissible to use an approximation for $\Omega(u')$ which neglects the width of the resonance. Thus we use $\delta(x) = \pi\,\theta(x-u_o)$ and obtain *)

$$\Omega(u) \simeq \frac{1}{u_o - u} = \frac{e^{i\delta(u)}}{|u_o - u|} \tag{B.10}$$

With (B.10) in (B.9), the result for (B.8) is

$$g(u) \simeq \lambda\,\Omega(u)\left[1 + \frac{u(u+3\beta)}{(u+\beta)^3}(u_o - u)\right] \tag{B.11}$$

To cast this in a form for comparison with (B.2) we can use the knowledge that in the physical limit $\beta \to 0$, $g(u) \to e^{i\delta}\sin\delta/q^3$. Thus $\lambda\,\Omega(u)\frac{u_o}{u} \simeq e^{i\delta}\sin\delta/q^3$ **) and (B.11) can be written

$$g(u) \simeq \frac{e^{i\delta}\sin\delta}{q^3}\left(\frac{u}{u_o}\right)\left[1 + \frac{u(u+3\beta)}{(u+\beta)^3}(u_o - u)\right] \tag{B.12}$$

An alternative method is to observe that with the effective range approximation for the phase shift, $\lambda q^3 \cot\delta \simeq u_o - u$,

$$\frac{e^{i\delta}\sin\delta}{q^3} \simeq \frac{\lambda}{u_o - u - i\lambda q^3} \tag{B.13}$$

---

*) We have actually used a subtracted form of (B.6) to get (B.10). Since $\Omega(u)/\Omega(u')$ always enters, the presence of a multiplicative constant in $\Omega(u)$ does not matter.

**) This connection between $\Omega(u)$ and the scattering amplitude has been remarked upon elsewhere [32], in a comparison of dispersion theoretic and Watson's [15] treatments of final state interactions.

Comparison with (B.10) shows that $\lambda\Omega(u) \simeq e^{i\delta}\sin\delta/q^3$, and (B.12) again is obtained, apart from the factor $(u/u_0) \simeq 1$. The effective range formula allows (B.12) to be written finally as

$$g(u) \simeq e^{i\delta}\frac{u}{u_0}\left[\frac{\sin\delta}{q^3} + \frac{\lambda}{u}\frac{(1+3\alpha)}{(1+\alpha)^3}\cos\delta\right] \tag{B.14}$$

This can be compared directly with (B.2) and shows that we find a correction factor of essentially unity, rather than the Ferrari-Selleri result (B.1).

In order to understand the origin of the difference between (B.14) and (B.2) it is necessary to examine the derivation of (B.2) by Ferrari and Selleri. They begin with (B.5) and exploit the fact that, in the limit $\beta \longrightarrow 0$, the solution for $g(u)$ is known. Thus they write the integral in (B.5) as

$$\int = \int_{\mu/m}^{\infty} du' \frac{e^{i\delta(u')}\sin\delta(u')}{(u'-u-i\epsilon)\Omega(u')}\left[\frac{1+3\alpha'}{(1+\alpha')^3}\right]\left\{B(u')\right\}_{\beta=0} \tag{B.15}$$

Here the factor $[(1+3\alpha')/(1+\alpha')^3]$ is just $B(u')/\{B(u')\}_{\beta=0}$. Then they argue that the integrand peaks at $u' = u_0$ and factor out the square bracketed quantity, obtaining (their Eq. (63)) :

$$\int \simeq \left[\frac{1+3\alpha_0}{(1+\alpha_0)^3}\right]\int_{\mu/m}^{\infty} du' \frac{e^{i\delta(u')}\sin\delta(u')}{(u'-u-i\epsilon)\Omega(u')}\left\{B(u')\right\}_{\beta=0} \tag{B.16}$$

Since the integral remaining is on the mass shell, it can be readily related to the physical scattering amplitude and (B.2) follows.

If the step from (B.15) to (B.16) is justified then there is a unexplained discrepancy between (B.14) and (B.2). We now wish to indicate that the integrand in (B.15) does not peak around $u' = u_0$ (even though that in (B.3) does), and that the step from (B.15) to (B.16) is unwarranted. The point is that the factor $\mathcal{D}(u')$ in the denominator of (B.15) becomes very large near resonance and compensates for the increase in $\sin\delta$ there. This can be seen from the sharp resonance approximation (B.10) and (B.13). The supposedly resonant factor in (B.15) is actually in this approximation

$$\frac{e^{i\delta(u')}\sin\delta(u')}{\mathcal{D}(u')} \simeq \frac{\Gamma/2}{\sqrt{(u_0-u)^2+\frac{\Gamma^2}{4}}} \times |u_0-u| \qquad \text{(B.17)}$$

where $\Gamma = 2\lambda q^3$ is the width of the resonance. More elaborate and realistic descriptions of the resonance will not change the conclusion.

From the mathematical point of view, the solution (B.5) of (B.3) can be augmented by adding $P(u)\mathcal{D}(u)$, where $P(u)$ is an arbitrary polynomial in u. Since $\mathcal{D}(u) \simeq e^{i\delta}\sin\delta/q^3$ for a sharp resonance, it is clear that in this limit an arbitrary off-the-mass-shell correction in Δ^2 can be obtained. This can be traced to the dominance of the integral in (B.3), relative to the inhomogeneous termn $B(u)$, in the neighbourhood of the resonance. But the basic philosophy here (and in Ref. [23]) is to omit such added solutions of the homogeneous equation.

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FIGURE CAPTIONS

Figure 1 — Schematic diagram of production of stable particle R_o and of unstable particle R that decays into particles 1 and 2 at the vertex V.

Figure 2 — Calculated appearance of the N^* (1238) resonance as a function of the square of the effective mass of the $\pi^+ p$ system in the reaction $K^+p \rightarrow K^o\pi^+p$ at 1.14 GeV/c and 3 GeV/c incident K^+ momentum.

Figure 3 — One meson exchange diagram for production of the unstable particle R.

Figure 4 — Kinematical diagrams. The upper figure defines the various momenta. The three lower figures show the momenta in different Lorentz frames.

Figure 5 — Decay directions in the rest frame of particle c. The plane containing the z axis (direction of a) and the momenta of the decay products, 1 and 2, is the decay plane of Treiman-Yang, while the x-z plane is the plane of production. Φ is the Treiman-Yang angle.

Figure 6 — Mnemonic diagram for angular momentum and parity analysis of the formation of a resonance with spin J and parity P from 0^- meson and exchanged system of angular momentum J_e and parity P_e.

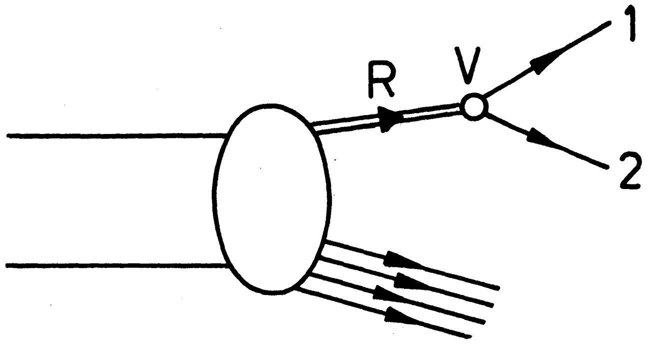

FIG. 1

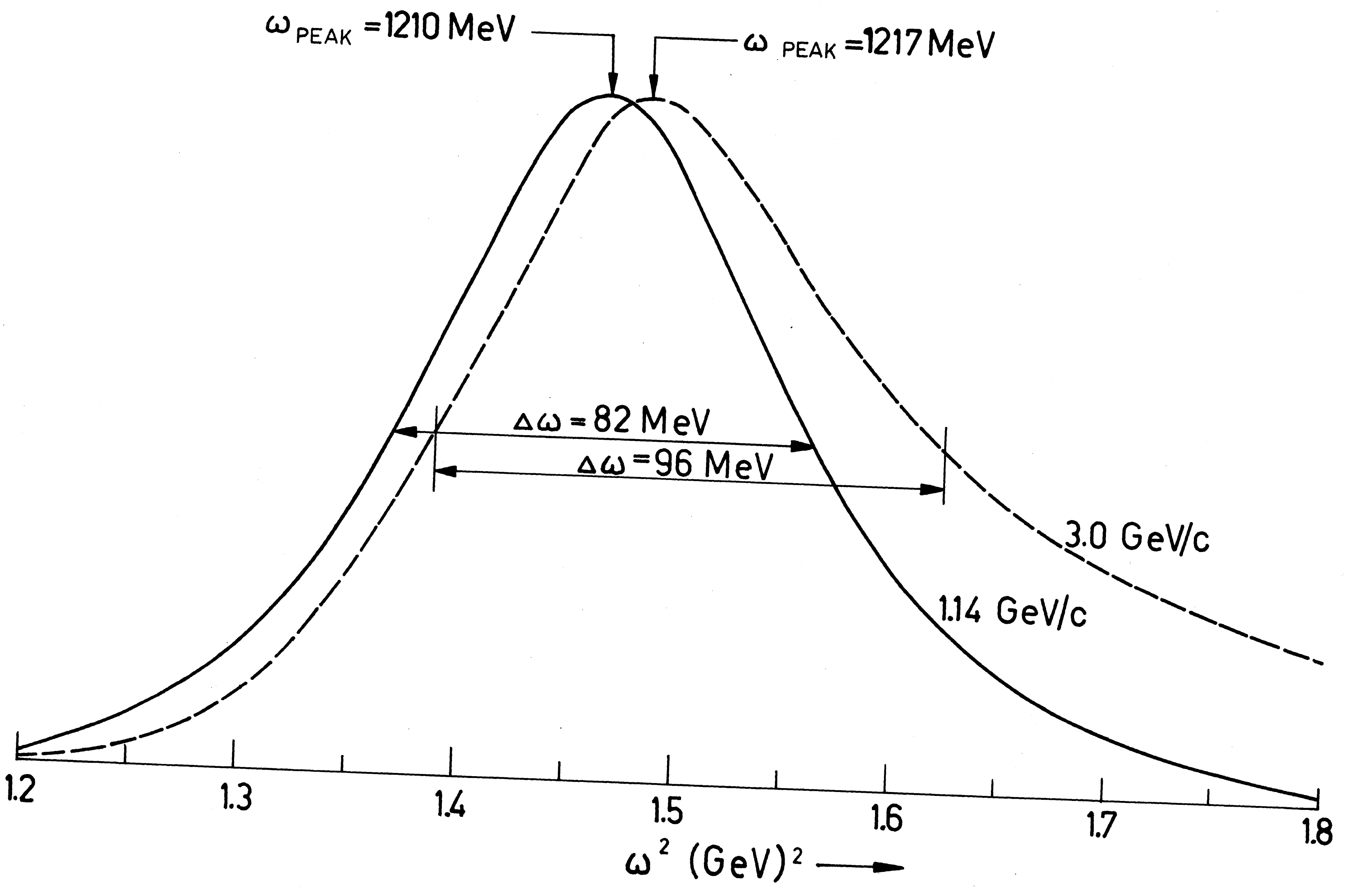
ω_{PEAK} =1210 MeV
ω_{PEAK} =1217 MeV
$\Delta\omega$ = 82 MeV
$\Delta\omega$ =96 MeV
3.0 GeV/c
1.14 GeV/c
1.2
1.3
1.4
1.5
1.6
1.7
1.8
ω^2 (GeV)^2

FIG. 2

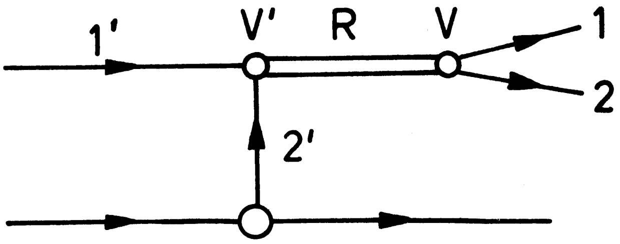
1'
V'
R
V
1
2
2'

FIG. 3

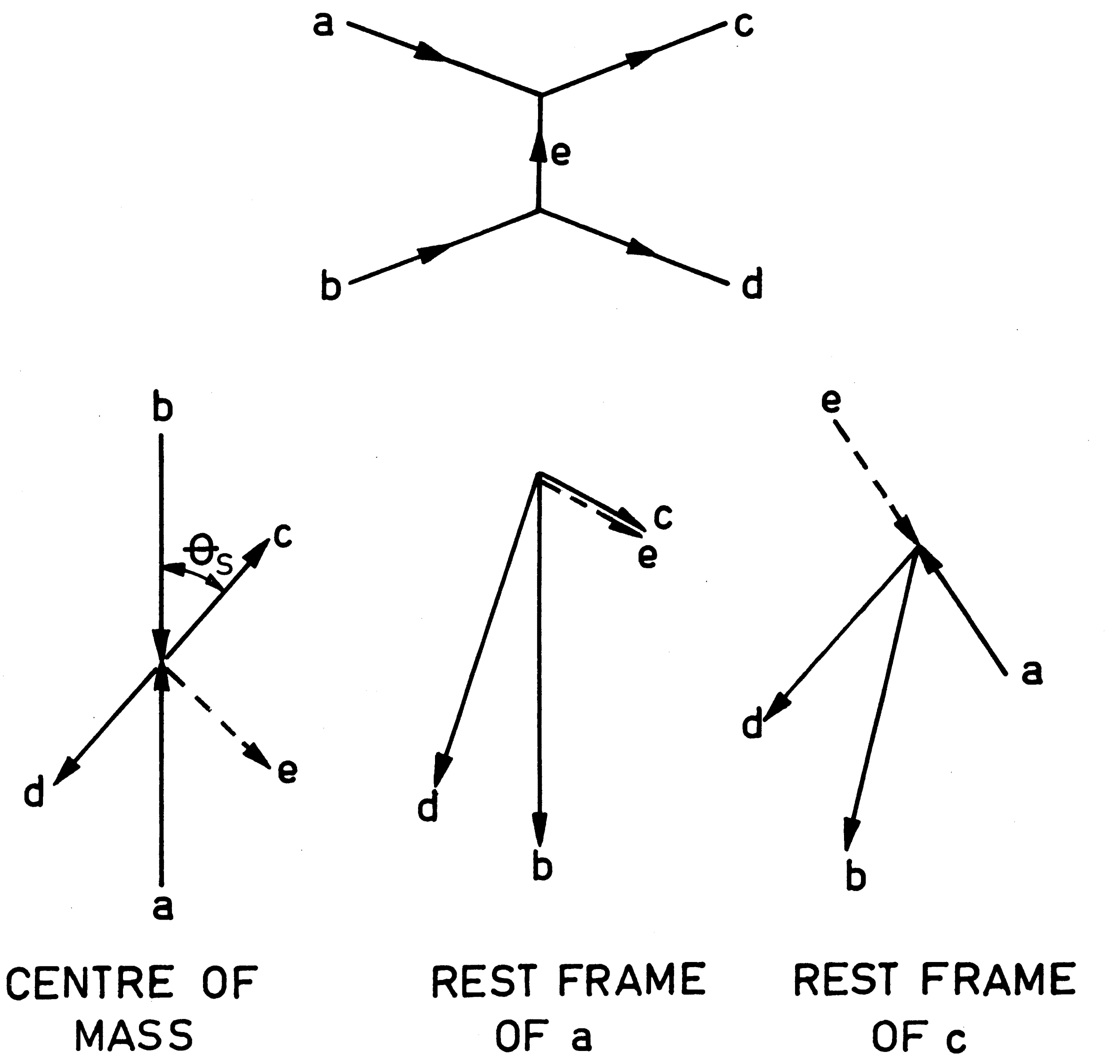
a
c
e
b
d
b
c
θS
d
e
a
c
e
d
b
e
a
d
b
CENTRE OF MASS
REST FRAME OF a
REST FRAME OF c

FIG. 4

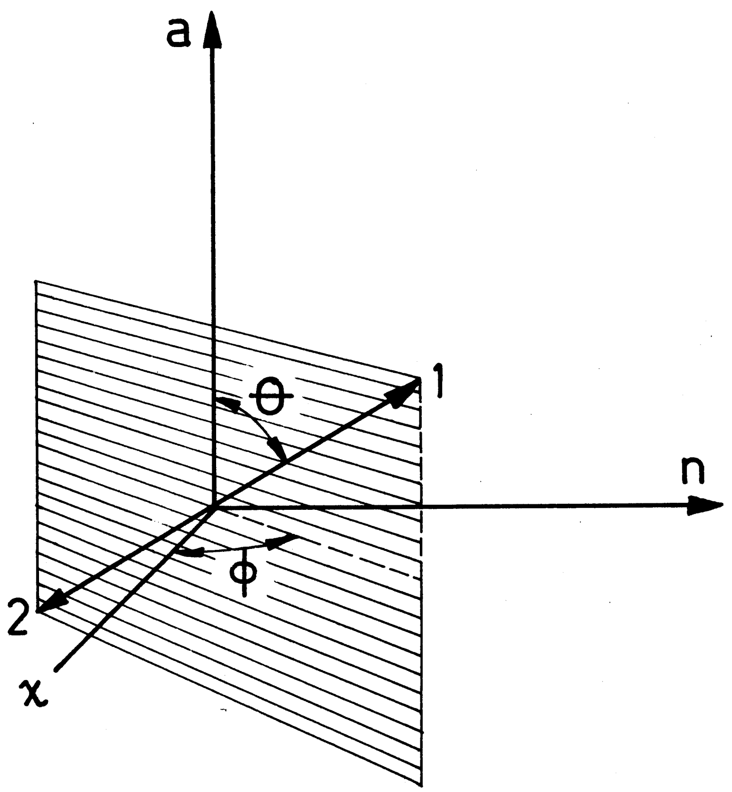

FIG. 5

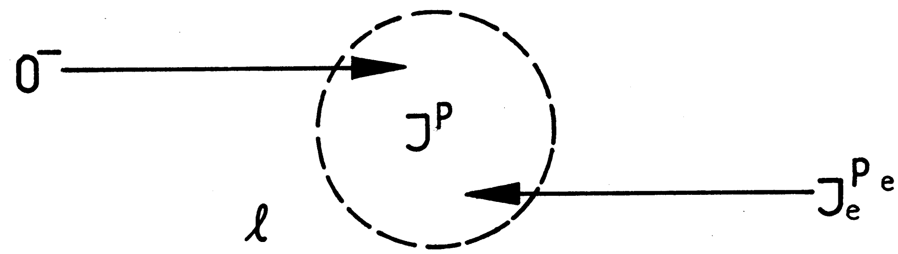

FIG. 6