CONTRIBUTION OF THE PION-PION INTERACTION TO LOW ENERGY PION-NUCLEON SCATTERING

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A B S T R A C T

The contribution of the pion-pion interaction to low energy S and P phase shifts for pion-nucleon scattering is calculated in a relativistic way by using the Cini-Fubini approximation. The relativistic effects are shown to be very important and a value of -0.95 ± 0.2 is found for the parameter C_1. The relation of these results to the non-relativistic case is also discussed.

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4903/TH.303
4 October 1962

1. INTRODUCTION

The determination of the contribution of the pion-pion interaction to the pion-nucleon scattering has been considered by several authors in these past years. A first estimate has been given by Bowcock, Cottingham and Lurié [1]. These authors, by using the Cini and Fubini one-dimensional approximation [2] to the Mandelstam representation, express the effect of the pion-pion interaction as an additional contribution to the amplitudes obtained by Chew, Goldberger, Low and Nambu [3]. They also introduce the following approximations :

a) they use, as CGLN, the non-relativistic approximation, i.e., they consider only the first term of a series of powers of $1/M$ for the expressions that occur [*];

b) they describe the pion-pion resonance in the state $T=J=1$ by a Dirac δ function.

Denoting by $X_A^{(-)}$ and $X_B^{(-)}$ the contributions of the pion-pion interaction to the usual invariant amplitudes $A^{(-)}$ and $B^{(-)}$ of the pion-nucleon scattering, we have, in the Cini and Fubini approximation :

$$X_A^{(-)} = \frac{\nu - \bar{\nu}}{\pi} \int_{4\mu^2}^{\infty} dt' \, \frac{\rho^{(-)}(t')}{t'-t} \tag{1.1a}$$

$$X_B^{(-)} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \, \frac{\sigma^{(-)}(t')}{t'-t} + C_B^{(-)} \tag{1.1b}$$

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*) Another work on the contribution of the $\pi - \pi$ interaction to the low-energy π-N scattering in the non-relativistic approximation has been done recently by Isaev and Meshcheriakov (Dubna preprint 1962) by following the approach developed by Efremov, Meshcheriakov and Shirkov. The $\pi - \pi$ interaction is shown to be important in order to fit the experimental data on S and small P waves. We thank Professor Isaev for sending us this preprint.

where :

$$\rho^{(-)}(t) = \frac{3\pi}{p}\left(\frac{M}{\sqrt{2}} \operatorname{Im} f_-^1(t) - \operatorname{Im} f_+^1(t)\right) \qquad (1.2a)$$

$$\sigma^{(-)}(t) = \frac{12\pi}{\sqrt{2}} \operatorname{Im} f_-^1(t) \qquad (1.2b)$$

Here $C_B^{(-)}$ is a constant (which plays the rôle of a correction to the Cini-Fubini approximation) to be determined by the experiment, $f_\pm^1$ are the amplitudes for the process $\pi\pi \rightarrow N\bar{N}$ in the state $J=1$ introduced by Jacob and Wick, p is the c.m. momentum of the nucleon in this process, M and μ are the nucleon and pion masses, s and t have the usual meaning.

In the approximation b) above, we have :

$$\rho^{(-)}(t) = 6\pi^2 C_2\, \delta(t_r - t) \qquad (1.3a)$$

$$\sigma^{(-)}(t) = -12\pi^2 (C_1 + 2MC_2)\, \delta(t_r - t) \qquad (1.3b)$$

where t_r is the value of t at the resonance.

BCL assumed $t_r = 22.4$ on the basis of the fits made at that time for the form factors [4].

The parameters C_1 and C_2 are related to the nucleon vector form factors. Representing the last ones with formulae of the Clementel-Villi type :

$$F_1^V = \frac{e}{2}\left(1 + \frac{\eta_1 t}{t_r - t}\right) \tag{1.4}$$

$$F_2^V = \frac{ge}{2M}\left(1 + \frac{\eta_2 t}{t_r - t}\right) \tag{1.5}$$

we have :

$$\eta_1 = -\frac{2C_1}{\gamma\sqrt{t_r}} \qquad \eta_2 = -\frac{2M}{g}\frac{C_2}{\gamma\sqrt{t_r}} \tag{1.6}$$

where γ is related to the half-width Γ of the pion-pion resonance by the relation :

$$\Gamma = \frac{\gamma q_r^3}{2\sqrt{t_r}} \tag{1.7}$$

Here q_r means the c.m. momentum of the pion in the t channel at the resonance.

From the fits on the form factors [4),5)], it follows that :

$$\eta_1 \simeq \eta_2 \tag{1.8}$$

and so C_1 and C_2 are related by :

$$C_2 = \frac{g_P - g_N}{2M} C_1 = 0.27 \mu^{-1} C_1 \qquad (1.9)$$

where g_P and g_N are the gyromagnetic ratios of the proton and neutron.

BCL, by fitting the experimental data on $\mathrm{Re}\, f_0^{(-)}$, obtain $C_1 = -1.0$ and determine also a range of possible values for the correction constant $C_B^{(-)}$. They find, moreover, that this set of parameters is consistent with the data on $\mathrm{Re}\, f_{1-}^{(-)}$. In spite of the low value used for t_r and of the non-relativistic approximation (quite crude for the S-wave) the value found for C_1 is in good agreement with the values recently observed for the half-width of the ρ meson [6].

The purpose of this note is to examine how much the relativistic corrections, that imply a contribution of the pion-pion interaction somewhat smaller than the estimate of BCL [7),8),9)], will modify the value of the parameter C_1, and then to discuss how these results are consistent with the data obtained from the form factors and the half-width of the ρ meson.

2. DETERMINATION OF THE PARAMETERS OF THE PION-PION INTERACTION

In a previous work [7], we calculated the relativistic corrections to the amplitudes for the pion-nucleon scattering given by CGLN. As our results are very sensitive to the tail of the function describing the 3/2, 3/2 resonance, we expressed the theoretical uncertainty as the difference between the results obtained by describing the resonance first by means of a fit with the experimental data [10], and then by means of an approximate solution of the integral equation for the resonance. In order to reduce this theoretical uncertainty, we give the results for the S-wave in a subtracted form as follows :

$$\mathrm{Re}\, f_0^{(-)} = g_1 + g_2 \tag{2.1}$$

Here g_1 contains the subtraction constant, but for the rest it does not depend on the pion-pion interaction, while - on the other hand - g_2 depends only on the pion-pion interaction and does not contain the subtraction constant. This has been determined by the threshold value $3\mathrm{Re}\, f_0^{(-)} = (a_1 - a_3) = 0.255$. On account of the subtraction, $\mathrm{Re}\, f_0^{(-)}$ depends very weakly on the tail of the function $\mathrm{Im}\, f_{33}$, and so we have assumed for this quantity the expression obtained by Höhler [10]. Moreover, the subtraction gives further support to the restriction, under the dispersion integral, only to the resonant amplitude $T = J = 3/2$. The D-wave resonance at 600 MeV changes the function g_1 by a few percent in the range from $q = 0$ to $q = 2$, while it becomes more important at $q = 3$ where it lowers the function g_1 by about 12 % (see Appendix). For this reason, and because we have considered only the experimental points at energies less than 270 MeV, we have neglected the contribution of the D resonance.

The explicit expression for g_1 is given in Ref. [7] [Eq. (5.11)], the one for g_2 is as follows :

$$\begin{aligned} g_2 = \frac{3}{2}\frac{C_1}{E_r}\Big\{ &-\frac{E+M}{2W}(W-M)\phi_0 - \frac{E-M}{2W}(W+M)\phi_1 + \frac{E+M}{W}(W-M) \\ &+ 0.27\Big[\frac{E+M}{2W}\Big((2E\omega+q^2)\phi_0 + q^2\phi_1 - 2(W-M)M\phi_0\Big) \\ &- \frac{E-M}{2W}\Big((2E\omega+q^2)\phi_1 + q^2\phi_2 + 2(W+M)M\phi_1\Big)\Big]\Big\} \end{aligned} \tag{2.2}$$

where

$$\phi_\alpha = \int_{-1}^{1} \frac{x^\alpha dx}{1 + \frac{2q^2}{t_r}(1-x)} \tag{2.3}$$

q is the c.m. momentum, $E = (M^2+q^2)^{\frac{1}{2}}$, $\omega = (\mu^2+q^2)^{\frac{1}{2}}$, $W = E+\omega$.

It is important to observe that, on account of the subtraction, g_2 depends only on the parameter C_1, while the parameter $C_B^{(-)}$ cancels out. So, unlike BCL, the data for the S-wave are fitted by only one parameter. The contribution of the pion-pion interaction has been again described by the equations (1.1), (1.2) and (1.3). For the mass of the ρ meson, we have assumed $t_r = 30$ and we have always put $\mu = 1$.

By comparing the dotted curves of Fig. 1 and Fig. 2, we see that the relativistic correction to the function g_1 is very important. Roughly speaking, about 2/3 of the discrepancy between the non-relativistic function g_1 and the experimental points is due to the relativistic effects, and so one can expect that the pion-pion contribution will be reduced by the relativistic calculation to about 1/3 of its non-relativistic value.

As for the best fit, we have included only the experimental points at low energy [11),12)] (by low energy we mean values of the pion laboratory kinetic energy less than $\sim$ 270 MeV). At higher energies up to now the phase shift analyses [11)-18)] are strongly ambiguous because of the very high number of the parameters involved, and sometimes they are in disagreement, so we think they have, for the moment, a rather preliminary role. For example, a very recent measurement by Rugge and Vik [18)] gives at 310 MeV for α_1 a negative value, while the value consistent with the sets of Pontecorvo [11)] and Walker [17)] would be of about 25°. Moreover, it should be remarked that the results obtained by using the Cini and Fubini approximation may become rather unreliable for energies above 300 MeV.

In this way we obtain :

$$C_1 = -0.95 \pm 0.2 \qquad (2.4)$$

and the behaviour of $3\mathrm{Re}\, f_0^{(-)}$ is shown in Fig. 1. We remark also that $|C_1|$ increases with t_r (almost linearly if we consider only values of q^2 less than $t_r/4$). Moreover, the value of C_1 here obtained is not very sensitive to the tail of the function describing the 3/2, 3/2 resonance. If we use, for example, the function (4.12) of Ref. [7], the change of C_1 remains within the error. A change of the same order of magnitude is obtained by considering also the D resonance at 600 MeV [*].

Our result (2.4) is, at first sight, rather surprising because it is very near the non-relativistic value, while, on the other hand, we have shown that the relativistic corrections are important. But the closeness of our value to the non-relativistic one is only apparent. The low value for $|C_1|$ obtained by BCL is essentially due to two reasons :

a) these authors fit also the experimental values up to 600 MeV by assuming the phase shifts of Pontecorvo and Walker. These phase shifts strongly decrease the value of $|C_1|$;

b) they assume $t_r = 22.4$ instead of $t_r = 30$.

If a best fit is made with the non-relativistic formulae, and including only low energy experimental points ($T < 270$ MeV), we find $C_1 = -3.0$, and so this is the real non-relativistic value to be compared with the relativistic one of -0.95.

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*) An improvement recently attained by Woolcock in the determination of the S-wave scattering lengths with respect to the values given in Ref. [11] (p. 101), would increase our value of $|C_1|$ of about 10 % (again within our error). We thank Professor J. Hamilton for having communicated to us Woolcock's results.

On the other hand, if we make the best fit including also the higher energy points (and assuming always $t_r = 30$) we find $C_1 = -0.23$, in complete disagreement with the width of the ρ meson. The corresponding non-relativistic value turns out to be $C_1 = -1.7$ (somewhat larger in absolute value than the BCL result, owing to the higher value of t_r).

An evaluation of the contribution of the pion-pion interaction to low energy pion-nucleon scattering has been recently done by Hamilton et al.[8], by analyzing the singularities of the S- and P-waves in the complex W-plane. These authors find for C_1 a value between -0.9 and -1.05 in agreement with our results.

Another calculation on the same subject has been recently done by Dietz[9], using a method proposed by Mandelstam and Frazer[19]. We remark that the behaviour obtained by Dietz for $\mathrm{Re}\, f_0^{(-)}$ is very similar to that obtained by us, with the value (2.4) for C_1, in the range of q from zero to 2. This shows a posteriori the agreement between the treatments of the problem by the Cini and Fubini method and the Mandelstam and Frazer approach.

Finally, we shall discuss briefly the situation for the P-wave $\mathrm{Re}\, f_{1-}^{(-)}$. The term which is independent from the pion-pion interaction has a theoretical uncertainty for the P-wave which is less than for the S-wave[7]. Therefore we have started from the unsubtracted expression and have used the Höhler formula for $\mathrm{Im}\, f_{33}$ under the dispersion integral. The expression for $\mathrm{Re}\, f_{1-}^{(-)}$ is as follows :

$$\mathrm{Re}\, f_{1-}^{(-)} = h_1 + h_2 \qquad (2.5)$$

where h_1 is independent from the pion-pion interaction and is given in Ref. [7) /Eqs. (2.11) and (2.14)/. The explicit expression for h_2 is the following :

$$h_2 \doteq \frac{3}{2}\frac{C_1}{t_\pi}\Big\{-\frac{E+M}{2W}(W-M)\phi_1 - \frac{E-M}{2W}(W+M)\phi_0 + 0.27\Big[\frac{E+M}{2W}\Big((2E\omega+q^2)\phi_1+q^2\phi_2-2(W-M)M\phi_1\Big) - \frac{E-M}{2W}\Big((2E\omega+q^2)\phi_0+q^2\phi_1+2(W+M)M\phi_0\Big)\Big]\Big\}+\frac{E-M}{2W}\frac{W+M}{4\pi}C_B^{(-)} \qquad (2.6)$$

Therefore the pion-pion part contains now also the constant $C_B^{(-)}$. We show in Fig. 3 the behaviour of $3\mathrm{Re}\, f_{1-}^{(-)}$ for $C_B^{(-)} = 0$, $C_B^{(-)} = -1$ and $C_B^{(-)} = -2$. The experimental situation up to now seems not to allow an unambiguous determination of this parameter.

Let us now sum up briefly our conclusions.

From the analysis made here it follows that the values $C_1 = -0.95 \pm 0.2$ and $-2 \leq C_B^{(-)} \leq 0$ seem to be consistent with the present experimental situation of the low energy S and P pion-nucleon phase shifts and with the width of the ρ meson. At higher energies the experimental situation for the pion-nucleon scattering is rather ambiguous and also a satisfactory theoretical description is lacking up to now. It has to be remarked, however, that in the low-energy region the experimental results available up to now for the best fits are not very numerous and so our parameters can be in some degree sensitive to a new experimental result, as well as to a further improvement of the knowledge of the low-energy parameters of pion-nucleon scattering.

ACKNOWLEDGEMENTS

We wish to thank Professors D. Amati, M. Cini and S. Fubini for their interest in this work and helpful discussions. One of the authors (G.P.) wishes to express his gratitude to Professor L. Van Hove for the kind hospitality granted at the Theoretical Study Division of CERN. We are also indebted to the "Centro di Calcolo Numerico dell'Università di Genova" for the numerical calculations.

APPENDIX

THE CONTRIBUTION OF THE D-WAVE RESONANCE

The D resonance at 600 MeV contributes to Re $f_0^{(-)}$ with the term :

$$-\frac{1}{\pi}\int_{M+\mu}^{\infty} K_D(W,W')\,\frac{\operatorname{Im} f_{2-}^{(1/2)}}{q'^2}\,dW' \tag{A.1}$$

where

$$\begin{aligned}
K_D(W,W') = {} & \frac{W'}{W}\left\{\frac{1}{3}\,\frac{1}{W'-W}\left[\frac{E+M}{E'+M}q'^2-\frac{E-M}{E'-M}q^2\right]+\frac{1}{W'+W}\,\frac{E+M}{E'-M}\left(q'^2-q^2\right)\right\} \\
& +\frac{W'}{W}\,\frac{E+M}{2}\left\{\left[f_D+(W-M)g_D\right]F_0-\frac{W'+W-2M}{E'-M}q^2F_1\right\} \\
& -\frac{W'}{W}\,\frac{E-M}{2}\left\{\left[f_D-(W+M)g_D\right]F_1-\frac{W'-W-2M}{E'-M}q^2F_2\right\}
\end{aligned} \tag{A.2}$$

$$f_D=-\left(\frac{1}{3}\frac{W'+M}{E'+M}+\frac{W'-M}{E'-M}\right)q'^2+\frac{W'-M}{E'-M}q^2\,; \qquad g_D=\left(\frac{1}{3}\frac{1}{E'+M}-\frac{1}{E'-M}\right)q'^2+\frac{1}{E'-M}q^2$$

[the quantities F_α are defined in Ref. [7]] as obtained by using the relations (2.1), (2.2) of Ref. [7] and including also the D resonance in the development of the amplitudes Im $A^{(-)}$ and Im $B^{(-)}$ in partial waves.

When the subtraction for the S-wave is performed, the expression (5.10) of Ref. [7] for $\mathrm{Re}\, f_0^{(-)}$ is modified as follows :

$$\mathrm{Re}\, f_0^{(-)} = g_1 + g_1^{(D)} + g_2 \tag{A.3}$$

where g_1 and g_2 are the same as in Ref. [7] and $g_1^{(D)}$ has the expression - obtained in the same way as in Ref. [7], Section 5 - :

$$g_1^{(D)} = \tilde{D}_D(W) - \tilde{D}_D(W_0)\,\frac{f(W)}{f(W_0)} \tag{A.4}$$

The function $f(W)$ is defined in Ref. [7], and

$$\tilde{D}_D(W) = -\frac{1}{\pi}\int_{M+\mu}^{\infty} \tilde{K}_D(W,W')\,\frac{\mathrm{Im}\, f_{2-}^{(1/2)}}{q'^2}\,dW' \tag{A.5}$$

$$\tilde{K}_D(W,W') = K_D'(W,W') - \frac{f(W)}{f(W_0)}\,K_D'(W_0,W') + K_D''(W,W') \tag{A.6}$$

where K_D' and K_D'' are the parts of K_D of order zero and of higher order respectively in $1/W'$.

For $\mathrm{Im}\, f_{2-}^{(\frac{1}{2})}$ we have used a δ function normalized to the area enclosed by the curve given by Layson [20] :

$$\mathrm{Im}\, f_{2-}^{(1/2)} = \frac{M}{W_r}\,\frac{\pi a}{q_r}\,\delta(W - W_r) \tag{A.7}$$

where $a = 0.2$, $W_r = 10.86$ and $q_r = 3.25$. In this way the quantity $g_1^{(D)}$ becomes :

$$g_1^{(D)} = \frac{a}{q_r^3}\,\frac{M}{W_r}\left[\tilde{K}_D(W_0, W_r)\,\frac{f(W)}{f(W_0)} - \tilde{K}_D(W, W_r)\right] \tag{A.8}$$

This quantity has been evaluated numerically with an IBM 1620 computer and we have obtained the results shown in the table :

TABLE A.1

Values of the function $g_1^{(D)}$.
The last column is the percent correction to the function g_1.

q	$g_1^{(D)}$	$\Delta\%$
0.50	-0.24×10^{-3}	-0.27
1.00	-0.12×10^{-2}	-1.16
2.00	-0.63×10^{-2}	-5.00
3.00	-1.71×10^{-2}	-12.3

FIGURE CAPTIONS

Fig. 1 $3\mathrm{Re}\, f_0^{(-)} = \dfrac{\sin 2\alpha_1 - \sin 2\alpha_3}{2q}$ in the relativistic case for the values $C_1 = -0.95$ and $C_1 = -0.23$. The dotted curve represents the same amplitude without the contribution of the pion-pion interaction (function g_1 in the text). Experimental points :

- from a fit by Hamilton and Woolcock [see Ref. 11) p. 101];
- from Ref. 17);
- from Ref. 11);
- from Ref. 18);
- from Ref. 12).

Fig. 2 $3\mathrm{Re}\, f_0^{(-)}$ in the non-relativistic case for the values $C_1 = -3.0$ and $C_1 = -1.7$. The dotted curve represents the same amplitude without the contribution of the pion-pion interaction (non-relativistic limit of the function g_1). Experimental points as in Fig. 1.

Fig. 3 $3\mathrm{Re}\, f_{1-}^{(-)}$ in the relativistic case for $C_1 = -0.95$, $C_B^{(-)} = 0$, $C_B^{(-)} = -1$, $C_B^{(-)} = -2$. The dotted curve represents the same amplitude without the contribution of the pion-pion interaction (function h_1 in the text), and the -.-.-.-.-.-.- curve represents its non-relativistic limit. Experimental points as in Fig. 1.

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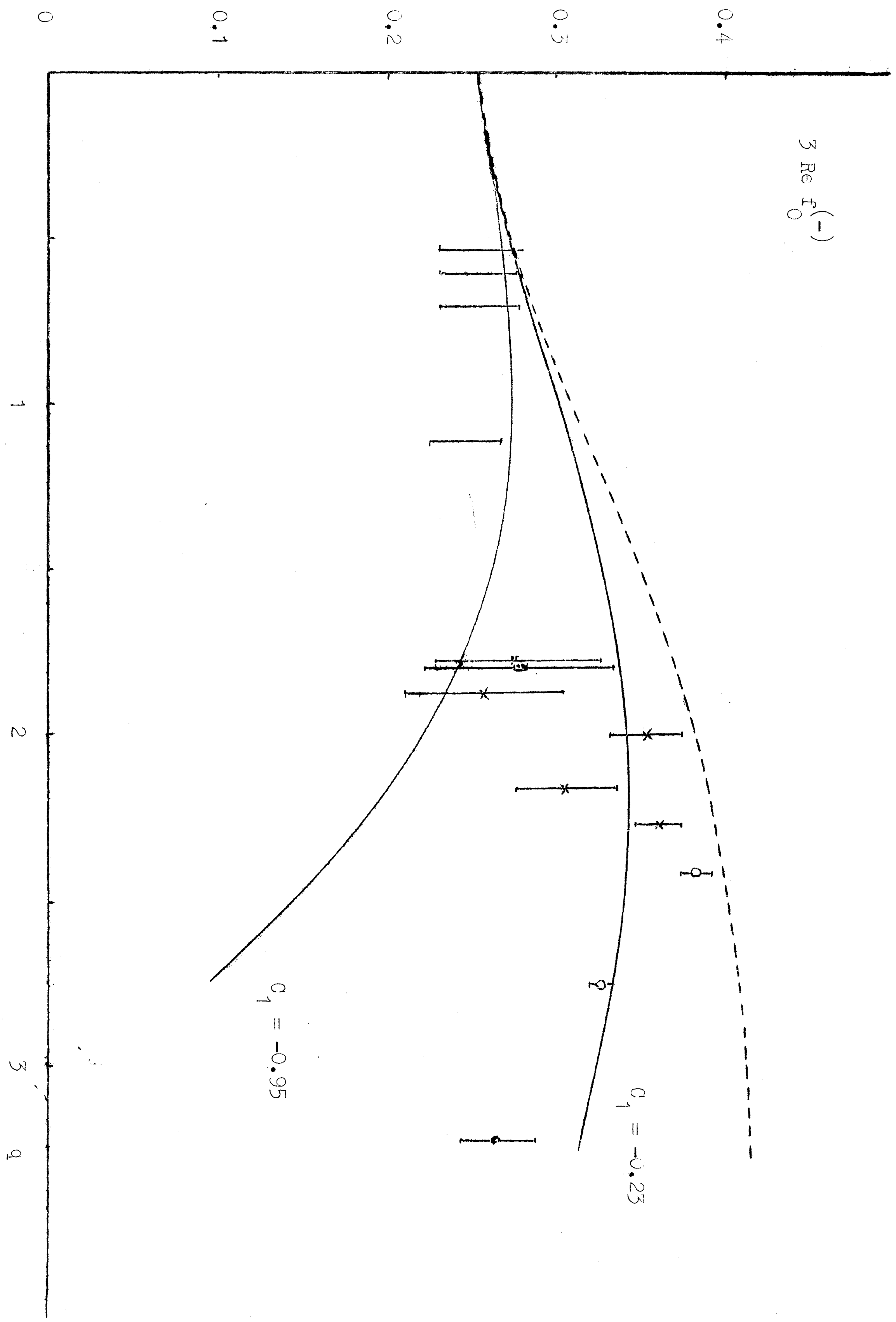

FIG. 1

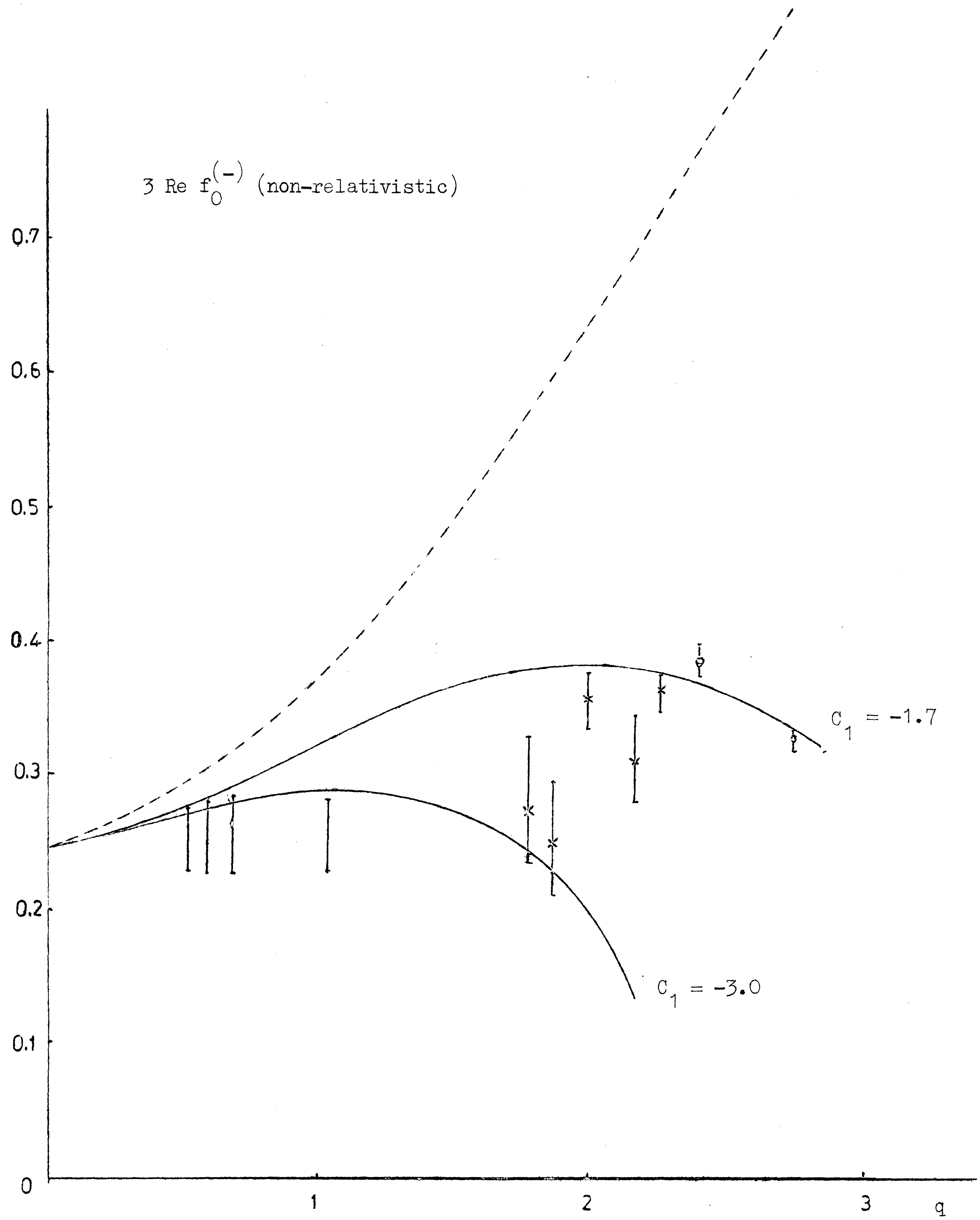

FIG. 2

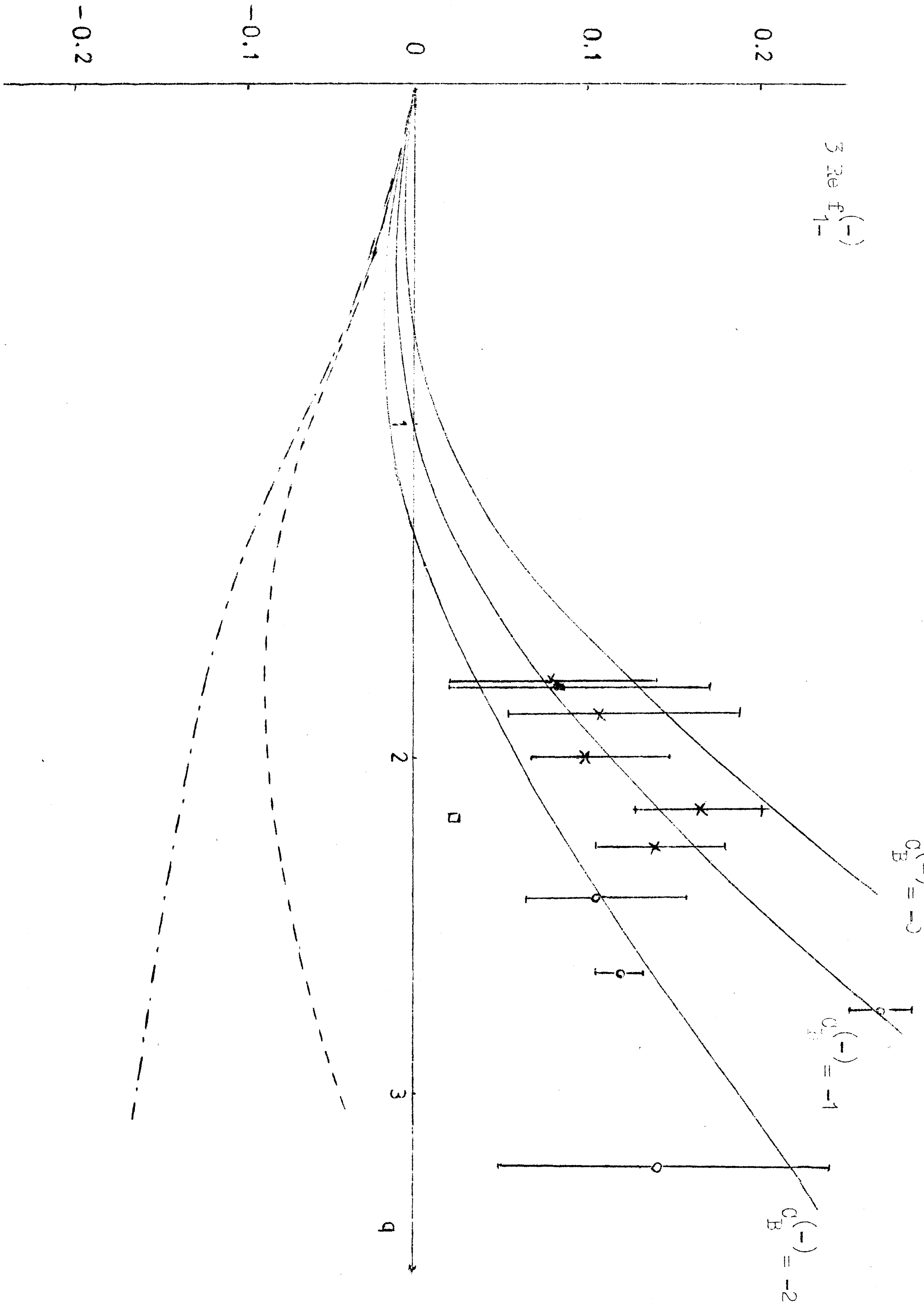

FIG. 3

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