Effect of a Pion-pion Scattering Resonance on Low Energy Pion-Nucleon Scattering.

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ABSTRACT

From the analytic properties of the pion-nucleon scattering amplitude postulated by Mandelstam, we derive a fixed momentum transfer dispersion relation in the energy variable which should be valid at low energies. This formulation allows us to evaluate the effect of a two-pion scattering resonance on pion-nucleon scattering. By a suitable choice of the pion-pion resonance parameters we are able to fit both the experimental pion-nucleon phase shifts and the nucleon electromagnetic form factors.

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1. Introduction

Experimental evidence has recently appeared which would seem to point to the existence of a strong pion-pion interaction. Evidence for such an interaction has been found, for example, in the analysis of single pion production data in π-p collisions by Bonsignori and Selleri[1)] and by Derado[2)]. In addition, Frazer and Fulco's work on the electromagnetic structure of the nucleon[3),4)] has shown that the experimental data on the isotopic vector parts of the nucleon form factors may be satisfactorily fitted if one assumes a $\pi - \pi$ scattering resonance in the J=1, T=1 state.

In this paper, we shall investigate, by means of the Cini-Fubini approximate version of the Mandelstam representation [5)], the effect of such a resonance on low-energy pion-nucleon scattering. The use of this representation has the effect of adding to the Chew, Goldberger, Low and Nambu fixed momentum-transfer dispersion relations[6)] an extra turn involving an integral over the absorptive parts of the $\pi + \pi \rightarrow N+\bar{N}$ amplitudes. The addition of this term allows in a natural way for the introduction of a $\pi - \pi$ resonance whose width and position may be estimated from available experimental data on the nucleon form factors and the s-wave π-N scattering phase shifts. Although, only the effect of a resonance in the J=1, T=1 state is considered, the formalism would allow other states as well to be taken into account.

Section II is devoted to the kinematics and symmetry properties of the amplitudes under consideration. In Section III, the Mandelstam representation which the scattering amplitude is assumed to satisfy is written down. From there we pass directly to the Cini-Fubini one dimensional form from which the integral equations for the $\pi + N \rightarrow \pi + N$ and $\pi + \pi \rightarrow N+\bar{N}$ amplitudes may be derived. The latter is identical with that derived recently by Frazer and Fulco. The expressions derived for the J=1, T=1 $\pi + \pi \rightarrow N+\bar{N}$ amplitudes involve divergent integrals however, which arise from the incorrect treatment of the high momentum transfer behaviour inherent in the effective range approach of the theory.

Our procedure at this point is to express these amplitudes in terms of four arbitrary parameters, three of which may be estimated from the experimental data on the nucleon electromagnetic form factors. This will be done in Section 4 and the expressions for the $\pi+\pi \rightarrow N+\bar{N}$ amplitudes which now involve only one arbitrary parameter will be inserted into the equations for the S, P and D partial waves π-N amplitudes to be derived in Section V.

A rough determination of the final additional parameter, namely the width of the $\pi-\pi$ resonance is given in Section VI, by fitting the experimental s-wave phase shifts. A more thorough comparison of the theory with experimental angular distribution data is at present under way.

2. Kinematics and symmetry properties.

a) Kinematics

We define first the kinematical variables for the scattering process $\pi + N \rightarrow \pi + N$ (channel I) denoting the four-momenta of the pions by q_1 and q_2 and the four-momenta of the nucleons by p_1 and p_2 (Fig.1).

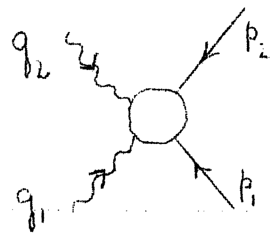

Fig. 1

In the centre-of-momentum system take*)

$$\begin{aligned} s &= -\,(q_1+p_1)^2 = (E_p + \omega_q)^2 \\ t &= -\,(q_1+q_2)^2 = -2q^2\,(1-\cos\theta) \\ \bar{s} &= -\,(q_1+p_2)^2 = (E_p - \omega_q)^2 - 2q^2(1+\cos\theta) \end{aligned} \qquad (2.1)$$

where E_p is the nucleon energy

ω_q is the meson energy

q^2 is the square of the magnitude of the meson momentum

θ is the scattering angle between pion and nucleon where the initial momenta are q_1 and p_1.

The use of our dispersion relations forces us also to consider the reaction $\pi + \pi \rightarrow N + \bar{N}$ (Channel II). For this channel denote the angle between incoming pion and final nucleon by ϕ.

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*) We use the metric such that $a.b = \mathbf{a}.\mathbf{b} - a_0 b_0$

Then

$$s = -p^2-q^2+2pq\cos\phi$$
$$t = 4(q^2+\mu^2) = 4(p^2+m^2) \qquad (2.2)$$
$$\bar{s} = -p^2-q^2-2pq\cos\phi$$

where p^2 is the square of the nucleon three-momentum.

In (2.1), (2.2) there are of course only two independent variables since we have the usual relation

$$s + \bar{s} + t = 2\,(M^2+\mu^2) \qquad (2.3)$$

b) Symmetry properties of the T matrix

(i) If the T matrix is defined by

$$S_{fi} = 1-(2\pi)^4\, i\, \delta^4\,(p_1+q_1+p_2+q_2)\frac{m}{\sqrt{4E_1E_2\omega_1\omega_2}}\; T_{fi}$$

then it has been shown[6)] that if T is to be invariant under Lorentz transformations it must be of the form

$$T = -A + \frac{i\gamma}{2}\,.(q_1-q_2)\,B \qquad (2.4)$$

A and B being scalar functions of q^2 and $\cos\Theta$.

(ii) By also requiring that the meson-nucleon interaction should be charge independent we have that

$$A_{\beta\alpha} = \delta_{\beta\alpha}A^{(+)} + \tfrac{1}{2}\,[\tau_\beta, \tau_\alpha]\,A^{(-)}$$
$$B_{\beta\alpha} = \delta_{\beta\alpha}B^{(+)} + \tfrac{1}{2}\,[\tau_\beta, \tau_\alpha]\,B^{(-)} \qquad (2.5)$$

where α, β are the isotopic spin indices of the mesons 1 and 2.

Then for channel I we find

$$A^{(+)} = \frac{1}{3} \left[A^{(\frac{1}{2})} + 2\, A^{(\frac{3}{2})} \right]$$

$$A^{(-)} = \frac{1}{3} \left[A^{(\frac{1}{2})} - A^{(\frac{3}{2})} \right] \qquad (2.6)$$

the indices $(\frac{1}{2})$, $(\frac{3}{2})$ referring to states of total isotopic spin $\frac{1}{2}$, $\frac{3}{2}$.

In channel II, however, the two allowed isotopic spins are 0,1 and we have

$$A^{(+)} = \frac{1}{\sqrt{6}}\, A^{(0)}$$

$$A^{(-)} = \frac{1}{2}\, A^{(1)} \qquad (2.7)$$

(iii) The third property of T that we use is that it should satisfy crossing symmetry, i.e.

$$A^{(\pm)}(s,\bar{s},t) = \pm\, A^{(\pm)}(\bar{s},s,t)$$

$$B^{(\pm)}(s,\bar{s},t) = \mp\, B^{(\pm)}(\bar{s},s,t) \qquad (2.8)$$

c) Partial-wave decompositions

We give here the way in which the amplitudes in channel I and channel II are decomposed into partial waves.

For channel I we follow the procedure of Ref.6) and define

$$f_1^{(\pm)} = \frac{E+m}{2W} \left\{ \frac{A^{(\pm)}+(W-M)B^{(\pm)}}{4\pi} \right\}$$

$$f_2^{(\pm)} = \frac{E-m}{2W} \left\{ \frac{-A^{(\pm)}+(W+M)B^{(\pm)}}{4\pi} \right\} \qquad (2.9)$$

W being the total energy for channel I.

Then

$$\frac{d\sigma}{d\Omega} = \sum \left| \langle f \,|\, f_1 - \frac{\vec{\sigma}\cdot\vec{q}_2\,\vec{\sigma}\cdot\vec{q}_1}{q_1 q_2}\, f_2 \,|\, i \rangle \right|^2 \tag{2.10}$$

The decomposition of $f_1^{(\pm)}$ and $f_2^{(\pm)}$ into states of definite angular momentum is given by

$$f_1^{(\pm)} = \sum_{\ell=0}^{\infty} f_{\ell+}^{(\pm)}\, P'_{\ell+1}(\cos\theta) - \sum_{\ell=2}^{\infty} f_{\ell-}^{(\pm)}\, P'_{\ell-1}(\cos\theta)$$

$$f_2^{(\pm)} = \sum_{\ell=1}^{\infty} \left(f_{\ell-}^{(\pm)} - f_{\ell+}^{(\pm)}\right) P'_{\ell}(\cos\theta) \tag{2.11}$$

where $f_{\ell\pm}$ is the scattering amplitude in a state of parity $-(-1)^{\ell}$ and total angular momentum $j = l\pm\frac{1}{2}$. Thus

$$f_{\ell\pm} = \frac{e^{i\delta_{\ell\pm}}\sin\delta_{\ell\pm}}{q} \tag{2.12}$$

It is useful to have a closed expression for $f_{\ell\pm}$ in terms of f_1 and f_2. This is given by

$$f_{\ell\pm} = \frac{1}{2}\int_{-1}^{1} \left(f_1 P_{\ell}(\cos\theta) + f_2 P_{\ell\pm1}(\cos\theta)\right) d\cos\theta \tag{2.13}$$

For channel II on the other hand it is convenient to decompose $A^{(\pm)}$ and $B^{(\pm)}$ into states of definite angular momentum and definite helicity as done by Frazer and Fulco[7]. Recalling from (2.2) that in this channel t is the total energy and ϕ the scattering angle we write

$$A^{(\pm)}(t,\cos\theta) = \frac{8\pi}{p^2} \sum_J (J+\tfrac{1}{2})(pq)^J \left\{ m f_-^{(\pm)J}(t)\, \frac{1}{\sqrt{J(J+1)}}\cos\phi\, P'_J(\cos\phi) - f_+^{(\pm)J}(t)\, P_J(\cos\phi) \right\} \tag{2.14}$$

$$B^{(\pm)}(t,\cos\phi) = 8\pi \sum_J \frac{(J+\frac{1}{2})}{\sqrt{J(J+1)}}(pq)^{J-1} f_-^{(\pm)J}(t)\, P'_J(\cos\phi)$$

For the amplitudes $f_{\pm}^{(\pm)J}(t)$, J refers to the total angular momentum of the state and the subscripts refer to definite helicities of the nucleon anti-nucleon pair; + meaning both particles have the same helicities, - meaning they have opposite helicities.

The inverse of Eq.(2.14) is given by

$$f_{+}^{(\pm)J}(t) = \frac{1}{8\pi}\left[-\frac{p^2}{(pq)^J} A_J^{(\pm)} + \frac{m}{(2J+1)(pq)^{J-1}}\left\{(J+1)B_{J+1}^{(\pm)} + JB_{J-1}^{(\pm)}\right\}\right]$$

$$f_{-}^{(\pm)J}(t) = \frac{1}{8\pi}\,\frac{\sqrt{J(J+1)}}{2J+1}\,\frac{1}{(pq)^{J-1}}\left\{B_{J-1}^{(\pm)} - B_{J+1}^{(\pm)}\right\} \tag{2.15}$$

where

$$\left[A_J^{(\pm)}(t);\ B_J^{(\pm)}(t)\right] = \int_{-1}^{1} dx\ P_J(x)\left[A^{(\pm)};\ B^{(\pm)}\right] \tag{2.16}$$

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3. Mandelstam representation and basic assumptions

We shall make here an approximation to the Mandelstam representation which we hope will give the essential structure of the invariant amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ in the region of low energy and low momentum transfer.

The approximation technique was introduced by Cini and Fubini[5] and is based on an analysis of the perturbation graphs for the amplitudes. Consider the analytic properties of the invariant amplitudes as given by the Mandelstam representation[8] and let us consider explicitly $A^{(+)}$. Then

$$A^{(+)}(s,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \, \alpha^{(+)}(s',t)\left\{\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right\} \tag{3.1}$$

This representation automatically satisfies crossing symmetry. For s and t in the physical region of channel I $\alpha^{(+)}(st) = \mathrm{Im}\, A^{+}(st)$ and from unitarity this can be expressed as the sum of a contribution from intermediate states involving pion-nucleon scattering and an inelastic contribution where additional mesons are produced in the intermediate state. We therefore write

$$\alpha^{(+)}(s',t) = \alpha^{(+)}_{el}(s',t) + \alpha^{(+)}_{inel}(s',t)$$

where $\quad \alpha^{(+)}_{inel}(s',t) = 0$ for $s' < (m+2\mu)^2$

The Mandelstam conjecture also states that $\alpha^{(+)}(s',t)$ is of the following form

$$\alpha^{(+)}(s',t) = \frac{1}{\pi}\int_{L(s')}^{\infty} \frac{u^{(+)}(s',t')\, dt'}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{M(s')} \frac{v^{(+)}(s',t')\, dt'}{t'-t}$$

Fourth order perturbation theory[9] indicates that both α^{+}_{el} and α^{+}_{inel} satisfy this representation with

$$M(s') < -4\, m\mu$$

$$L(s') \geq 4\mu^2 \text{ for } \alpha^{(+)}_{inel}$$

$$L(s') \geq 16\mu^2 \text{ for } \alpha^{(+)}_{el}$$

We will take this to be so.

Let us write Eq.(3.1) as

$$A^{(+)}(s,t) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, \alpha^{(+)}_{el}(s',t)\left\{\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right\} +$$

$$+\frac{1}{\pi}\int_{(m+2\mu)^2}^{\infty} ds'\, \alpha^{(+)}_{inel}\left\{\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right\} \tag{3.2}$$

In the first integral of (3.2) the nearest cut in the t variable begins at $t = 16\mu^2$. For this reason we shall assume the validity of keeping for this integral only the first few terms of a power series expansion in t. For the second integral in (3.2), however, the nearest cut in t begins at $4\mu^2$ whereas the cuts in s and $\bar{s}$ now begin at the inelastic threshold $(m+2\mu)^2$. Accordingly, we shall expand this integral in power series in s and $\bar{s}$ (preserving crossing symmetry), keeping again only the first few terms. Thus the second integral of (3.2) is of the form

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\, \frac{a^{(+)}(t',s,\bar{s})}{t'-t} + \frac{1}{\pi}\int_{(m+2\mu)^2}^{\infty} ds' \int_{-\infty}^{M(s')} dt'\, \frac{v_{inel}(s',t')}{(t'-t)}\left\{\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right\} \tag{3.3}$$

where $$a^{(+)}(t',s,\bar{s}) = \frac{1}{\pi}\int_{(m+2\mu)^2}^{\infty} ds'\, u^{(+)}_{inel}(s',t')\left\{\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right\} \tag{3.4}$$

The second term in (3.3) should have only a weak dependence on all three variables since the cuts are all distant. We shall replace terms of this form by real constants.

In this manner we obtain a representation of $A^{(+)}$:

$$A^{(+)}(s,t) = \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \alpha_{el}^{(+)}(s',t) \left\{ \frac{1}{s'-s} + \frac{1}{s'-\bar{s}} \right\} ds' + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{a^{(+)}(t',s,\bar{s})}{t'-t} dt' + C_A^{(+)} \tag{3.5a}$$

where $\alpha^{(+)}(s',t)$ is a real polynomial of low degree in t and $a^{(+)}(t',s,\bar{s})$ is a sum of a real polynomial of low degree in s and an identical polynomial in $\bar{s}$.

$A^{(-)}$ has a similar representation except that from crossing symmetry it must be odd under interchange of s and $\bar{s}$, so that the arbitrary constant must be zero in this case. Thus

$$A^{(-)}(s,t) = \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \alpha_{el}^{(-)}(s',t) \left\{ \frac{1}{s'-s} \quad \frac{1}{s'-\bar{s}} \right\} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{a^{(-)}(t',s,s)}{t'-t} dt' \tag{3.5b}$$

Representations of identical form (satisfying the symmetry properties of Eq.(2.8) hold for $B^{(+)}$ and $B^{(-)}$ except that to these must be added the single nucleon pole term, i.e.

$$B^{(\pm)} = g_r^2 \left(\frac{1}{m^2-s} \mp \frac{1}{m^2-\bar{s}} \right) + \text{terms similar to (3.5)}$$

Since the weight functions $\alpha^{(+)}$, etc. are real we can make the association

$$\text{Im}A^{(+)}(s,t) = \alpha_{el}^{(+)}(s,t) \tag{3.6}$$

at least for s and t in the physical region of channel I.

From (2.9) and (2.11) $\mathrm{Im}A^{(+)}(s,t)$ in this region is expandable as a sum over partial wave pion nucleon scattering amplitudes. We will take this sum to be saturated by the $P_{3/2\;3/2}$ resonance which we take from experiment. From (3.6) this gives that $\alpha^{(+)}(s,t)$ is a first order polynomial in t, the behaviour of which as a function of s is given by the $P_{3/2\;3/2}$ partial wave amplitude. As t is not necessarily a physical momentum transfer corresponding to the energy s' in the region of integration of (3.5) an analytic continuation in t of $\mathrm{Im}\,A(s',t)$ from the physical region of its arguments is implied.

It is clear that the first terms in Eq. (3.5a and b), and in the corresponding equations for $B^{(\pm)}$, are identical with the representation used by Chew, Low, Goldberger and Nambu[6] since both in their work and in the present one the integrals are saturated with the (3,3) resonance. The difference lies in the addition, here, of a strongly t-dependent term representing the contribution of inelastic pion-nucleon scattering.

The weight function $a^{(+)}(t,s,\bar{s})$ is equal to $\mathrm{Im}\,A^{(+)}(t,s,\bar{s})$ in the region $t > 4\mu^2$ for small values of s and $\bar{s}$. $t > 4\mu^2$ corresponds to the energy region of pion-pion scattering although for $t < 4m^2$, t is below the threshold for $N\bar{N}$ production. What must be fed in here is the imaginary part of the $\pi + \pi \rightarrow N + \bar{N}$ production amplitude analytically continued to the relevant values of $s, \bar{s}$ and t.

The integral equations for this production amplitude can be derived directly from our representation (3.5) by means of analyticity arguments similar to those of Frazer and Fulco[7]. We shall use the amplitudes $f_{\pm}^{(\pm)\,J}(t)$ as defined in (2.14).

The analytic properties of $A_J^{(\pm)}$ and $B_J^{(\pm)}$ may be deduced from (2.16) and (3.5). For $A_J^{(\pm)}(t)$ for example, one finds a right-hand cut starting at $t=+4\mu^2$ and extending to $t=+\infty$ due to the second term in (3.5) and a left-hand cut from $-\infty$ to 0 arising from the vanishing of the denominators in the first term. The same is true of $B_2^{J(\pm)}$ except that the left-hand cut will extend from $-\infty$ to $a=4\mu^2(1-\frac{\mu^2}{4m^2})$ because of the presence of the pole terms. From Eq.(2.15) this leads to the following dispersion relations for $f_{\pm}^{J}$, identical with those of Frazer and Fulco :

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$$f_{\pm}^{J}(t) = \frac{1}{\pi} \int_{-\infty}^{a} \frac{\mathrm{Im}\, f_{\pm}^{J}(t')dt'}{t'-t-i\varepsilon} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\mathrm{Im}\, f_{\pm}^{J}(t')\, dt'}{t'-t-i\varepsilon} \tag{3.7}$$

where $\mathrm{Im}\, f_{\pm}^{(\pm)J}(t)$ on the left hand cut is given in terms of the pion nucleon $P_{3/2\,3/2}$ resonance parameters and the single nucleon pole term as given by Eq.(5.7) and (5.8) of Ref.[7].

It can be seen from a theorem by Fubini, Nambu and Wataghin[10] that below the threshold for inelastic processes the phase of the production amplitudes $f_{\pm}^{(\pm)J}$ are equal to the phase shift of the two colliding particles, i.e. the $\pi\pi$ scattering phase shift. As there is at present no experimental data on pion-pion scattering we shall make the currently favoured assumption of a resonance in the J=1, T=1 state.

Returning to Eq.(3.5), we can then replace $a^{(\pm)}(t',s,\bar{s})$ by its expansion in terms of $\mathrm{Im}\, f_{\pm}^{(\pm)J}(t')$ using Eq.(2.16). Keeping only the J=1 T=1 terms in the sums and noting that $A^{(+)}$ and $B^{(+)}$ are unaffected by a T=1 pion-pion resonance, we get the following representations for the four invariant amplitudes

$$A^{(+)} = \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \alpha_{el}^{(+)}(s',t) \left\{ \frac{1}{s'-s} + \frac{1}{s'-\bar{s}} \right\} ds' + C_A^{(+)}$$

$$A^{(-)} = \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \alpha_{el}^{(-)}(s',t)\, ds' \left\{ \frac{1}{s'-s} - \frac{1}{s'-\bar{s}} \right\} + (s-\bar{s}) \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\rho(t')}{t'-t}\, dt' \tag{3.8}$$

$$B^{(+)} = g_r^2 \left(\frac{1}{m^2-s} - \frac{1}{m^2-\bar{s}} \right) + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \beta_{el.}^{(+)}(s,t) \left\{ \frac{1}{s'-s} - \frac{1}{s'-\bar{s}} \right\} ds'$$

$$B^{(-)} = g_r^2 \left(\frac{1}{m^2-s} + \frac{1}{m^2-\bar{s}} \right) + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \beta_{el.}^{(-)}(s',t) \left\{ \frac{1}{s'-s} + \frac{1}{s'-\bar{s}} \right\} ds'$$

$$+ \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\sigma(t')}{t'-t}\, dt' + C_B^{(-)}$$

where

$$\rho(t') = \frac{3\pi}{p'^2}\left(\frac{m}{\sqrt{2}}\ \mathrm{Im}\ f_-^{(-)1}(t) - \mathrm{Im}\ f_+^{(-)1}(t)\right)$$

$$\sigma(t') = \frac{12\pi}{\sqrt{2}}\ \mathrm{Im}\ f_-^{(-)1}(t')$$

In the absence of a solution to the Chew-Mandelstam $\pi - \pi$ scattering equations[11] we shall adopt a resonance form that satisfies the requirements of unitarity, correct low energy behaviour and the existence of a cut from $t=4\mu^2$ to ∞. Such a form is

$$f_{\pi\pi} = \frac{e^{i\delta_{\pi\pi}}\ \sin\delta_{\pi\pi}}{q^3} = \frac{\gamma}{t_r - t - i\gamma q^3} \qquad (J=T=1) \tag{3.9}$$

If this were a correct solution to the Chew-Mandelstam equations, γ would be a function of t. For $f_{\pi\pi}$ to be a resonance, however, γ would have to be a slowly varying function of t, at least in the resonance region; we shall take it to be a real constant.*)

One can now directly write down an expression for $f_\pm^{(-)1}(t)$ that satisfies the analyticity requirements of (3.9) and has the desired phase in the region $4\mu^2 \leqslant t \leqslant 16\mu^2$. A solution, equivalent to that of Omnès[12], having all the required properties is

$$f_\pm^{(-)1}(t) = f_{\pi\pi}(t)\int_{-\infty}^{a}\frac{\mathrm{Im}\ f_\pm^{(-)1}(t')}{f_{\pi\pi}(t')(t'-t-i\varepsilon)}\,dt' \tag{3.10}$$

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*) Taking γ to be a constant introduces a spurious pole in $f_{\pi\pi}(t)$. However, for a sharp resonance this pole will have a small residue and be distant from the region of interest.

into which we must now substitute the expressions given by Eq.(5.7) and (5.8) of Ref.7) for Im $f_{\pm}(t')$. If we attempt, however, to evaluate these integrals keeping only the contributions from the nucleon pole term and the (3,3) resonance we find that the integrals do not converge, essentially due to the fact that we are using a power series expansion in t' for $f^J_{\pm}(t')$ over a region where the expansion no longer converges. Frazer and Fulco have attempted to estimate the integrals appearing in (3.10) by introducing a cut-off. Our procedure is to replace them by constants representing their value at $t=t_r$, this approximation being based on the assumption that the structure of $f^J_{\pm}(t)$ in the region of interest $4\mu^2 < t < \infty$ is dominated by a strongly peaked π-π resonance.

Let us write therefore

$$f_{\pm} = \frac{N_{\pm}}{t_r - t - i\gamma q^3} \qquad (3.11)$$

In the following section we shall show how one may estimate these constants by comparison with nucleon electromagnetic structure data.

4. The electromagnetic structure of the nucleon

It has been pointed out by Frazer and Fulco(3,4) that both the magnitude of the isotopic vector part of the anomalous magnetic moment of the nucleon and the radii of the charge and magnetic moment distributions can be adequately explained if the pion-pion interaction is assumed to have a resonance in the J=1, T=1 state. Earlier attempts by dispersion relation techniques to explain these properties, neglecting the pion-pion interaction, were unable to account for them all simultaneously.

We shall use the notation of Federbush, Goldberger and Treiman[13]. They consider the nucleon current density operator j_μ taken between single nucleon states. From Lorentz and gauge invariance this can be expressed in terms of two scalar functions

$$\langle p' | j_\mu | p \rangle = \left(\frac{m^2}{p'_o p_o} \right)^{\frac{1}{2}} \bar{u}(p') \left[F_1(t) i \gamma_\mu - F_2(t) i \sigma_{\mu\nu} (p'-p)^\nu \right] u(p)$$

where u(p) and u(p') are the Dirac spinors for the initial and final nucleons and $t = -(p-p')^2$.

The functions F_1 and F_2 may be subdivided into isotopic scalar and vector parts

$$F_1 = F_1^S + \tau_3 F_1^V \qquad (4.1)$$

$$F_2 = F_2^S + \tau_3 F_2^V \qquad (4.2)$$

We consider here only the isotopic vector parts F_1^V and F_2^V as these are the easiest to handle by dispersion relation techniques. For t=0 they satisfy the relations

$$F_1^V(0) = \frac{e}{2} \qquad (4.3)$$

$$F_2^V(0) = \frac{\mu_P - \mu_N}{2} = \frac{ge}{2m} \qquad (4.4)$$

μ_N and μ_P are the anomalous magnetic moments of the neutron and proton respectively (experimentally g, the gyromagnetic ratio = 1.83) e is the electron charge. The functions $F_1^V(t)$ and $F_2^V(t)$ are taken to satisfy the dispersion relations :

$$F_1^V(t) = \frac{e}{2} + \frac{t}{\pi} \int_{4\mu^2}^{\infty} \frac{\text{Im } F_1^V(t')\, dt'}{t'(t'-t)} \tag{4.5}$$

$$F_2^V(t) = \frac{ge}{2m} + \frac{t}{\pi} \int_{4\mu^2}^{\infty} \frac{\text{Im } F_2^V(t')\, dt'}{t'(t'-t)} \tag{4.6}$$

Subtractions have been performed in order to ensure better convergence of the integrals. The reader is referred to the paper by Federbush, Goldberger and Treiman[13] for a discussion of these equations and for an evaluation of the two pion contributions to Im $F_1^V(t')$ and Im $F_2^V(t')$ in terms of the pion electromagnetic form factors and the $\pi+\pi \rightarrow N+\bar{N}$ production amplitudes.

They obtain expressions for this contribution which can be written in terms of the $\pi+\pi \rightarrow N+\bar{N}$ Jacob and Wick[14] amplitudes as[4]

$$\text{Im } F_1^V(t) = - \frac{e\, F_\pi^*(t) q^3}{2E} \Gamma_i(t) \tag{4.7}$$

where $$\Gamma_2(t) = \frac{1}{2p^2} \left\{ \frac{m}{\sqrt{2}} f_-^{(-)1}(t) - f_+^{(-)1}(t) \right\}$$

$$\Gamma_1(t) = \frac{m}{p^2} \left\{ - \frac{E^2}{m\sqrt{2}} f_-^{(-)1}(t) + f_+^{(-)1}(t) \right\}$$

$F_\pi(t)$ is the pion electromagnetic form factor. By arguments similar to those used in the derivation of the integral representations for the nucleon electromagnetic form factors it can be shown that F (t) satisfies the integral representation[3]

$$F_\pi(t) = 1 + \frac{t}{\pi} \int_{4\mu^2}^{\infty} \frac{dt' \text{ Im } F_\pi(t')}{t'(t'-t)} \tag{4.8}$$

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Also, by the Fubini, Nambu, Wataghin theorem[10] $F_\pi(t)$ has the same phase as the $\pi-\pi$ scattering amplitude in the J=1, T=1 state below the inelastic threshold $t=16\mu^2$. The problem of the construction of $F_\pi(t)$ is analogous to that of $f_\pm^{(\pm)1}(t)$ except that the former has no left hand cut. A solution arrived at in the same manner, equivalent to the Omnès solution[12], is

$$F_\pi(t) = \frac{t_r+\gamma}{t_r-t-i\gamma q^3} \tag{4.9}$$

Using this expression and Eq.(4.7) we obtain

$$\text{Im } F_2^V(t) = -\frac{e}{2E} q^3 \frac{t_r+\gamma}{(t_r-t)^2+\gamma^2 q^6} \frac{1}{2p^2}\left(\frac{m}{\sqrt{2}} N_- - N_+\right) \tag{4.10}$$

$$\text{Im } F_1^V(t) = -\frac{e}{2E} q^3 \frac{t_r+\gamma}{(t_r-t)^2+\gamma^2 q^6} \frac{m}{p^2}\left(-\frac{E^2}{m\sqrt{2}} N_- + N_+\right) \tag{4.11}$$

If we now make the assumption that the resonance is narrow, we can replace

$$\frac{q^3\gamma}{(t_r-t)^2+\gamma^2 q^6} \quad \text{by } \pi\delta(t_r-t)$$

and obtain by substitution into (4.5) and (4.6)

$$F_1^V(t) = \frac{e}{2}\left(1-\frac{at}{t_r-t}\right) \tag{4.12}$$

$$F_2^V(t) = \frac{ge}{2m}\left(1-\frac{bt}{t_r-t}\right) \tag{4.13}$$

The two constants a and b are

$$a = \frac{C_1}{E_r\gamma}\,\frac{t_r+\gamma}{t_r}$$

$$b = \frac{m}{g}\,\frac{C_2}{E_r\gamma}\,\frac{t_r+\gamma}{t_r}$$

$$C_2 = \frac{1}{2p_r^2}\left(\frac{m}{\sqrt{2}} N_- - N_+\right)$$

$$C_1 = \frac{m}{p_r^2}\left(-\frac{E_r^2}{\sqrt{2}m} N_- + N_+\right)$$

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p_r and E_r are p and E evaluated at $t=t_r$.

The form factors $F_1^V(t)$ and $F_2^V(t)$ have been investigated experimentally by high energy electron scattering from protons and deuterons[15]. It appears from this experimental data that it is consistent to assume that

$$\frac{2}{e} F_1^V(t) = \frac{2m}{ge} F_2^V(t) = \frac{1}{\mu_p} F_2^P(t) = \frac{1}{\mu_N} F_2^N(t) = \frac{1}{e} F_1^P(t) \text{ for } 0 > t > -25\mu^2$$

where $F_2^P(t)$ is the magnetic moment form factor of the proton.

Taking this to be so, the equality of $\frac{2}{e} F_1^V(t)$ and $\frac{2m}{ge} F_2^V(t)$ gives the relation

$$\frac{C_2}{C_1} = \frac{g}{m} \tag{4.14}$$

In order to fit the form of $F_2^P(t)$, we note that our form factors (4.12) and (4.13) have the same form as those predicted by the Clementel and Villi model[16] for the proton charge and magnetic moment distributions. This model is known to give a good fit to the experimental data with values of the parameters a=b=1.2[17]

$$t_r = 22.4\,\mu^2 \tag{4.15}$$

and

$$\frac{C_1}{\Gamma} = -\frac{1.2E_r}{q_r^3} = -\frac{.286}{\mu^2} \tag{4.16}$$

$$\Gamma = \gamma\, q_r^3$$

corresponding to an r.m.s. charge radius of the proton and radius of the proton and neutron magnetic moment distributions of $.8\text{x}10^{-13}$ cm.

5. Projection of partial wave amplitudes.

In order to compare the theoretical and experimental phase shifts we need to evaluate expressions for the $f_{\ell\pm}$ defined in Eq.(2.12). This is done by using the relation (2.13) into which we must substitute the expressions for f_1 and f_2 from (2.9) and use the approximate A's and B's evaluated in Section 3. In what follows we shall restrict ourselves to the calculation of S, P and D waves only. Applying these operations to a) the pole terms and b) the integrals involving $\frac{1}{s'-s}$ and $\frac{1}{s'-\bar{s}}$ and keeping only the contribution from the (33) state in the absorptive parts of f_1 and f_2, we obtain, in the static limit, expressions identical to those written down by Chew, Goldberger, Low and Nambu[6]. These are reproduced here.

$$f_S = \begin{Bmatrix} f_S\,(T=\frac{1}{2}) \\ f_S\,(T=\frac{3}{2}) \end{Bmatrix} = -2\lambda^+ + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{\omega}{m}\lambda^- \qquad (5.1)$$

where

$$\lambda^+ = \frac{g^2}{2m} - \frac{4m}{3\pi}\int_1^\infty \frac{d\omega'}{q'^2}\left(1+\frac{2\omega'}{m}\right) \mathrm{Im}\, f_{33}(\omega')$$

$$\lambda^- = \frac{g^2}{2m} - \frac{4}{3}\frac{m}{\pi}\int_1^\infty \frac{d\omega'}{q'^2}\, \mathrm{Im}\, f_{33}(\omega') \qquad (5.2)$$

$$f_{11} = -\frac{8}{3}\frac{f^2q^2}{\omega} + \frac{16}{9}\frac{q^2}{\pi}\int_1^\infty \frac{d\omega'}{q'^2}\,\frac{\mathrm{Im}\, f_{33}(\omega')}{\omega'+\omega}$$

$$f_{13} = f_{31} = \frac{1}{4} f_{11} \qquad (5.3)$$

$$f_{33} = \frac{4}{3}\frac{f^2q^2}{\omega} + \frac{q^2}{\pi}\int_1^\infty \frac{d\omega'}{q'^2}\, \mathrm{Im}\, f_{33}\left(\frac{1}{\omega'-\omega}\right)$$

With the assumption of a narrow $P_{3/2,3/2}$ resonance these expressions become

$$f_{11} = -\frac{8}{3}\frac{f^2 q^2}{\omega}\frac{1}{1+\frac{\omega}{\omega_r}} \tag{5.4}$$

$$f_{13} \simeq f_{31} \simeq \frac{1}{4} f_{11}$$

where ω_r is the centre of mass pion energy corresponding to the resonant (3,3) state. For D-waves

$$\delta^{\frac{1}{2}}_{D\frac{3}{2}} = -\lambda_D \left[1+\frac{112}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2 \right]$$

$$\delta^{\frac{3}{2}}_{D\frac{3}{2}} = \lambda_D \left[2-\frac{28}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2 \right]$$

$$\delta^{\frac{1}{2}}_{D\frac{5}{2}} = \lambda_D \left[4-\frac{32}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2 \right] \tag{5.5}$$

$$\delta^{\frac{3}{2}}_{D\frac{5}{2}} = -\lambda_D \left[8+\frac{8}{9}\left(\frac{\omega}{\omega+\omega_r}\right)^2 \right]$$

with

$$\lambda_D = \frac{1}{15}\frac{f^2}{m}\frac{q^5}{\omega^2}$$

We must now evaluate the contribution to f_s of the terms in f_1 and f_2, representing the effect of the $\pi - \pi$ interaction. These terms are given by (2.9), (3.8) and (3.11), and under the assumption of a narrow $\pi\pi$ resonance, are given in the static limit by

$$\left[f_1^{(\pi\pi)}\right] = \begin{bmatrix} f_1^{(\pi\pi)}(T=\frac{1}{2}) \\ f_1^{(\pi\pi)}(T=\frac{3}{2}) \end{bmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{m}{W} \frac{2}{t_r - t} \left(\frac{3}{2} q^2 \cos\theta \, C_2 - \frac{3}{2}\omega C_1 \right)$$

$$f_2^{(\pi\pi)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{m}{W} \frac{-2}{t_r - t} \frac{3}{2} q^2 \left(C_2 + \frac{C_1}{2m} \right) \tag{5.6}$$

Here the superscript $(\pi\pi)$ denotes the contribution from the terms of (3.13) with denominator t_r-t. The factor $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ comes from having kept the contribution of the (T=1) pion-pion scattering state only.

From Eq.(2.13),

$$f_s = \frac{1}{2} \int_{-1}^{1} (f_1 + f_2 \cos\theta) \, d\cos\theta$$

we find that the terms in C_2 do not contribute in the static limit and we get

$$\left[f_s^{(\pi\pi)}\right] = -\frac{3}{2} \frac{m}{W} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{\omega}{t_r} C_1 F_0\left(\frac{2q^2}{t_r}\right) \tag{5.7}$$

where

$$F_\alpha\left(\frac{2q^2}{t_r}\right) = \int_{-1}^{1} \frac{dc \, c^\alpha}{1 + \frac{2q^2}{t_r}(1-c)}$$

$c = \cos\theta$

This result is quite general in the sense that for all higher values of ℓ the terms in C_2 do not contribute in the static limit to the non-spin flip amplitude. Indeed, the non-spin flip amplitude for scattering in an ℓ state is

$$\left[(\ell+1)\, f_{\ell+} + \ell f_{\ell-} \right] \quad P_\ell \tag{5.8}$$

and one may easily verify, using (2.13), that

$$(\ell+1)\, f_{\ell+} + \ell f_{\ell-} = \frac{2\ell+1}{2} \int_{-1}^{1} dc\ (f_1+f_2c)\ P_\ell \tag{5.9}$$

from which the result follows immediately. The spin flip amplitude

$$\left(f_{\ell-} - f_{\ell+}\right) P_\ell^1\ e^{i\phi} \tag{5.10}$$

on the other hand, depends, in the static limit, on the linear combination

$$C_2' = C_2 + \frac{C_1}{2m} \tag{5.11}$$

as can be seen from the relation

$$f_{\ell-} - f_{\ell+} = \frac{2\ell+1}{2\ell(\ell+1)} \int_{-1}^{1} dc(1-c^2)\ P_\ell'\ \ f_2 \tag{5.12}$$

and Eq.(5 6).

We recall that by the discussion of the preceding section, C_1 and C_2 are related to the vector part of the charge distribution radius and anomalous magnetic moment, respectively. C_2' corresponds therefore to the total magnetic moment :

$$\mu_{\text{anom.}} + \frac{e}{2m} .$$

It is interesting to note that, insofar as the terms depending on the resonance are concerned, the non-spin-flip amplitude is related in the static limit only to the charge distribution radius of the nucleon whereas the spin flip amplitude is connected only to its total magnetic moment.

Evaluating (2.13) for P and D waves we find that the terms to be added onto the CGLN expressions for f_{P_1}, f_{P_3}, f_{D_3}, f_{D_5} are

$$f_{P_1}^{(\pi\pi)} = -\frac{3}{2}\ \frac{m}{W}\begin{pmatrix}2\\-1\end{pmatrix}\frac{1}{t_r}\left[\omega\, C_1F_1 + q^2C_2'\,(F_0-F_2)\right]$$

$$f_{P_3}^{(\pi\pi)} = -\frac{3}{2}\ \frac{m}{W}\begin{pmatrix}2\\-1\end{pmatrix}\frac{1}{t_r}\left[\omega\, C_1F_1 - \frac{q^2}{2}\,C_2'\,(F_0-F_2)\right]$$

$$f_{D_3}^{(\pi\pi)} = -\frac{3}{4}\ \frac{m}{W}\begin{pmatrix}2\\-1\end{pmatrix}\frac{1}{t_r}\left[\omega\, C_1(3F_2-F_0) + 3q^2C_2'\,(F_1-F_3)\right] \tag{5.13}$$

$$f_{D_5}^{(\pi\pi)} = -\frac{3}{4}\ \frac{m}{W}\begin{pmatrix}2\\-1\end{pmatrix}\frac{1}{t_r}\left[\omega\, C_1(3F_2-F_0) - 2q^2C_2'\,(F_1-F_3)\right]$$

We must still evaluate the effect on f_s of the extra constants C_A^+ ... introduced in Section 2 when passing from the Mandelstam representation to the one-dimensional Cini-Fubini form. (The effect on the $f_{P_{\frac{1}{2}}}$ amplitude is only a non-static correction and constants have no effect on the other amplitudes). One easily verifies that these constants simply add to f_s a term of the form

$$\alpha + \begin{pmatrix}2\\-1\end{pmatrix}\beta\,\omega \tag{5.14}$$

This correction is of exactly the same form as the CGLN contribution (5.3).

As for the $P_{3/2\ 3/2}$ equation, our additional term is small in the region of the resonance itself and hence the resonance solution given in Ref.6) will be modified simply by the addition of a small real part.

6. Comparison with experiment and conclusions

We have postulated a simple possible model for the effect of the pion-pion interaction on the pion-nucleon scattering amplitude. Even so, a direct comparison of the phase-shift predictions of this model with experiment is not very fruitful because of the large uncertainties involved in the phase-shift analysis of the experimental data.

The best established low energy results apart from the existence of the $P_{3/2\ 3/2}$ resonance are the values of the S-wave scattering lengths. Those quantities, however, are implicitly very dependent on the core contribution to the scattering amplitude, a fact which is reflected in our having taken the high energy contributions to our dispersive integrals as arbitrary constants. These arbitrary constants give contributions to the S-wave scattering amplitude of the same form as the Chew, Goldberger, Low and Nambu terms, namely the expression (5.14). These terms by themselves give an adequate description of the scattering lengths. However, at higher energies the experimental data cannot be adequately explained by the simple dependence given in (5.14). Using our model up to an energy of about $q=1,5\mu$ enables us to fit all the S-wave data and gives us a value of the "width" Γ of our resonance. Fitting the data with $t_r=22.4\,\mu^2$ gives $C_1= -.58\pm.15$ which corresponds to a positive Γ as it must if our postulate of a resonance is correct and a value of $1.7\mu^2<\Gamma<3.0\mu^2$. One should note that Γ is not the conventional width of the resonance but that with our resonance form (3.11) the energy difference between the points where the resonance reaches half its maximum value is given by $\frac{\Gamma}{\sqrt{t_r}}$.

With this value of t_r and the mean value of $C_1=-.58$ the P and D wave threshold behaviour as calculated from (5.13) is shown in the table. In the determination of t_r from the isovector parts of the nucleon form factors, if one allows for the rather large uncertainties in F_1^N and F_2^N then t_r may very well be $15\mu^2$ instead of $22.4\mu^2$. The values of the P and D waves at threshold corresponding to $t_r=15\mu^2$ are therefore given too.

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We also give the contribution from the Chew et al. terms as calculated from (5.4) and (5.5) and compare the total with the experimental results (taken from the 1958 Annual International Conference on High Energy Physics at CERN)

TABLE OF THRESHOLD VALUES FOR P and D WAVES

	$\pi - \pi$ contribution		Chew	Total		Exp.
Term	t_r=22.4	t_r=15	term	t_r=22.4	t_r=15	-
$f^{\frac{1}{2}}_{P\frac{1}{2}}/q^2$	.046	.049	-.14	-.094	-.091	-.038$\pm$.038
$f^{\frac{1}{2}}_{P\frac{3}{2}}/q^2$	-.016	-.013	-.035	-.051	-.048	-.039$\pm$.022
$f^{\frac{3}{2}}_{P\frac{1}{2}}/q^2$	-.023	-.025	-.035	-.058	-.060	-.044$\pm$.005
$f^{\frac{3}{2}}_{P\frac{3}{2}}/q^2$	.008	.007	.213	.221	.220	.234$\pm$.019
$f^{\frac{1}{2}}_{D\frac{3}{2}}/q^4$	.0013	.0020	-.0019	-.0006	+.0001	-
$f^{\frac{3}{2}}_{D\frac{3}{2}}/q^4$	-.0007	-.0010	.0013	-.0006	-.0003	-
$f^{\frac{1}{2}}_{D\frac{5}{2}}/q^4$	-.0005	-.0007	.0029	+.0024	+.0022	-
$f^{\frac{3}{2}}_{D\frac{5}{2}}/q^4$	.0003	.0004	-.0065	-.0062	-.0061	-

While the theoretical numbers quoted in the table are not in exact agreement with the experimental numbers one must remember that the contribution from the π-π interaction may be varied by varying C_1 within its limits as given by fitting the S-waves, so that agreement with the experimental data is possible.

One feature which is not clearly brought out by just inspecting the threshold behaviour of the partial waves is the relative size of the π-π contribution and of the Chew term. It might appear from the table that in general the π-π contribution is the smaller one. However, one finds in fact that the energy variation of the π-π contribution is stronger than that of the Chew term and therefore at higher energies the π-π term becomes comparable or larger than the Chew term and determines to a large extent the variation of the phase shifts with energy.

For example, for the D wave with $J=\frac{3}{2}$ and $T=\frac{1}{2}$ at threshold the contribution from the π-π term and the Chew term are almost equal and of opposite signs. However, with increasing energy the (positive) π-π term increases much more rapidly than the (negative) Chew term so that this phase shift becomes positive, increasing quite rapidly with energy. This is just the behaviour that one would like to have since we know that this state passes through a resonance around 600 MeV.

Of course more extensive calculations must still be done and a direct comparison of theoretical and experimental phase shifts is probably not the best way to put the theory to a severe test. For in analysing an experimental cross-section into partial waves one usually arrives at phase shifts with large uncertainties on them and whose value may depend on the assumed values of other partial waves e.g. the D waves. On the other hand, the theory presented here gives simultaneously definite values for S, P and D waves. A better procedure for comparing theory and experiment is therefore to compare directly the differential cross-sections on which the experimental errors are rather small. Such a programme is at present in progress.

In conclusion we would like to remark that it is amusing that one can probably reach an approximation to low energy pion-nucleon scattering identical to the one presented here by considering a model in which both the (3,3) resonance and the π-π, J=1, T=1 resonance are replaced by isobars having the corresponding masses, spins and isospins. Of course the formulation presented here is much more general since it allows for the insertion of experimental data on π-π scattering in all spin and isospin states when such information becomes available.

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