

PREPRINT 256

**The Riemann-Cartan-Weyl quantum geometry
and supersymmetric systems.**

Diego L. Rapoport

Diciembre 1995

CERN LIBRARIES, GENEVA



SCAN-9709113

SW9739

Comité de Redacción

Corach, Gustavo

Fava, Norberto A.

Larotonda, Angel R.

Segovia, Carlos

THE RIEMANN-CARTAN-WEYL QUANTUM GEOMETRY AND SUPERSYMMETRIC SYSTEMS

DIEGO L. RAPOPORT

Instituto Argentino de Matemática, CONICET, Viamonte 1636,
Buenos Aires, ARGENTINA and Dept. Ciencias y Tecnología ,
Univ. Nacional de Quilmes; e-mail: raport@iamba.edu.ar

ABSTRACT. In this first article of a series dealing with the geometry of Quantum Mechanics, we introduce the Riemann-Cartan-Weyl (RCW) geometries of Quantum mechanics, for spin 0 systems as well as for systems of non-zero spin. The central structure is given by a family of Laplacian (or D'Alembertian) operators on forms of arbitrary degree associated to the RCW geometries. We show that they are conformally equivalent with the Laplacian operators introduced by Witten in the topological quantum field theories. We show that the Laplacian RCW operators yield a supersymmetric system, in the sense of Witten, and study the relation between the RCW geometries and the symplectic structure of loop space. The RCW family of Laplacians are the infinitesimal generators of diffusion processes on non-degenerate space-time, of systems of arbitrary spin.

INTRODUCTION

There are several concurrent problems in mathematical physics which cannot be considered as settled definitely.

One of them is the determination of the geometrical structures to account for gravitation and the non-linear gauge theories. As it has become known recently, the solutions of the monopole equations have a remarkable dependence on the sign of the scalar curvature of the metric of the four-dimensional manifold; (Witten, 1994). This might appear as rather strange, taking in account that the non-linear

non - abelian gauge theories were primordially conceived to account for "internal" degrees of freedom.

The theory of gravitation admits extensions to Cartan geometries with torsion, so that the metric structure appears as describing partially gravitation (Cartan and Einstein, 1979); moreover, the Cartan geometries appeared in the context of the Poincaré group theory of the theory of gravitation; (Sciama, 1962; Hehl, 1976; Changui & Dehnen, 1991).

The other problem is the formulation of quantum mechanics in terms of trajectories of Brownian processes and the establishment of correlations described by the quantum potential; (Bohm, Bohm and Vigier, 1952 and 1953, Holland, 1994).

It is remarkably that both problems are connected through the Cartan geometries, specifically those which have a torsion tensor which reduces to its trace, Q , further described as a trivial Weyl 1-form: $Q = d \ln \psi$, with ψ a positive scalar field defined on space-time. We called these geometries as RCW (for Riemann-Cartan-Weyl) geometries. As was showed in (Rapoport, 1991), the Weyl trace-torsion 1-form accounts for the average displacement of the most general diffusion process on space-time, while the metric describes the covariance of these processes. Both together, i.e. first and second moments of the probability density of the processes, determine all higher moments of them. Yet, what is peculiar of this description, that there exists a single geometrical laplacian operator which incorporates these two moments, and thus, determines *all* the statistical profiles of the diffusions. This operator, which is the infinitesimal generator of the diffusions, and is the most general invariant second-order elliptic operator on a smooth manifold (when one assumes conservation of probability), is the laplacian operator associated to the RCW geometry; (Rapoport 1995). Yet, this operator does not only determine the probability density of the diffusions, yet its path integral representation through the Onsager-Machlup lagrangian, and ultimately the classical smooth approximations which with maximal probability realize the diffusions (Rapoport, 1995 a, b). One finds out that these realizations are deviations of the geodesic flow due to the Weyl torsion, yet this does not conflict with the principle of equivalence, since the diffusions represents an interacting ensemble (Rapoport, 1995 b).

In this article, we shall introduce the RCW geometries and their associated

laplacian operators, not just on spin 0 systems, still defined on differential forms of arbitrary degree. It was observed by Witten, that geometrical laplacians on forms of arbitrary degree, are basic examples of supersymmetric systems, in which forms of odd (even) degree are fermions (bosons). We shall see that the RCW laplacian on forms are conformally conjugate to Witten's deformed laplacian in the topological quantum field theories (TQFT); (Witten, 1982). This is quite striking on regards of TQFT, since in this theory the field ψ is rather a functional on the infinite dimensional loop space, not just the solution of the conformal invariant equation on space-time, as it turns to be in the present theory; (Rapoport, 1995). Yet, we shall see that the role of the RCW geometry is essential to the definition of the general symplectic structure on loop space, which as is well known already in the Riemannian case, is the key to the obtention of exact representations of the trace of the Riemannian heat kernels, and ultimately leads to a direct proof of the Atiyah-Singer index theorem; (Atiyah, 1985).

We would like to close this introduction with the description of the origin of the method which yields Cartan's geometry the universality to describe classical spinning test-particle systems (Rapoport and Sternberg, 1984) as well as interacting quantum ensembles undergoing diffusions generated by the RCW laplacian.

In Cartan's approach to Classical Mechanics, as explained in (Rapoport and Sternberg, 1984) we start with a little group H embedded as a closed subgroup in a bigger group G such that the homogeneous space G/H has the same dimension as the configuration manifold M . Here, H and G are respectively the Lorentz and Poincaré group, and M is space-time, or still, we can take de Sitter group $O(1,4)$ instead of the Poincaré group and then M is a de Sitter space-time. It is the soldering form which infinitesimally identifies both spaces. Of course, Cartan was thinking in dynamical systems with classical (i.e. smooth) trajectories. The whole point of our theory seen from Cartan's conception is its applicability to quantum fluctuations with continuous but non-differentiable Brownian paths. Instead of copying classical dynamical systems on homogeneous space to space-time, the quantum counterpart of Cartan's method is the copying through a RCW connection on space-time of a *standard* Wiener process. Yet, to deal with Brownian paths, Cartan's calculus on manifolds is obviously unapplicable and instead one has to apply the Itô stochastic calculus, or still, the Stratonovich calculus which obeys the same rules for derivatives that the classical calculus in R^n , thus

establishing a stochastic calculus which sets the stochastic extension of Cartan's method. The relation between classical and quantum motions is established, as already described above, by the Onsager-Machlup lagrangian representation of the transition density of the RCW diffusion. The classical system appears thus as the most probable approximation of the quantum diffusion. This establishes the universality of Cartan's method in describing also the relation between quantum and classical motions.

This article is organized in the following way. We first introduce the Cartan soldering form and the Riemann-Cartan (RC) geometries, and particularly, the RCW geometries which we introduce from the point of view of conformal transformations. We then introduce the laplacian operators associated to the RC geometries, and see that to obtain a one-to-one correspondance (in general dimension other than 2), between geometries and laplacians, we need to restrict the theory to RCW geometries. We then generalize the laplacian to differential forms, through the introduction of Dirichlet quadratic forms of differential forms of arbitrary degree, to further introduce Witten's deformed laplacian. Later we see that the RCW laplacians yield a supersymmetric system, which is conformally conjugate to Witten's laplacian system. We finally discuss the symplectic structure of loop space and its relation with the RCW geometries. Finally, as a preparation of a forthcoming article on the stochastic extension of Cartan's classical method, we give a local description of Cartan's classical method.

I. CARTAN CONNECTIONS AND RIEMANN-CARTAN-WEYL STRUCTURES.

We shall follow here the presentation due Rapoport, 1991 and Rapoport & Sternberg.

We recall some basic facts and definitions in order to establish notation. Let G be a Lie group and M a differentiable manifold. A principal G -bundle over M is a manifold P on which G acts freely on the right, and such that the quotient of this G action is M . Thus we have a smooth map $\pi : P \rightarrow M$ and $\pi^{-1}(x)$ is a G orbit for each $x \in M$. We also assume that P is locally trivial in that about each x there is a neighborhood U , such that $\pi^{-1}(U)$ is isomorphic to $U \times G$

(with the obvious definition of isomorphism). We shall let $R_a : P \rightarrow P$ denote right multiplication by a^{-1}

$$R_a(p) = pa^{-1}, \quad p \in P, a \in G$$

so that R_a gives a left action of G on P

$$R_{ab} = R_a R_b .$$

Let F be some differentiable manifold on which G acts on the left. We can then form the quotient of the product space $P \times F$ by the G action; call it $F(P)$. Let $\pi_F : F(P) \rightarrow M$ denote the bundle projection, and $\rho : P \times F \rightarrow F(P)$ denotes the passage to the quotient. Then also $F(P)$ is also fibered over M by

$$\pi_F(\rho(p, f)) = \pi(p) .$$

$F(P)$ is called the associated bundle (to the G action on F and the principal bundle P).

Let $f : P \rightarrow F$ be a smooth function satisfying

$$f(pa) = a^{-1}f(p) \tag{1.1}$$

Then

$$\alpha(p, f(p)) = (pa^{-1}, af(p)) = (pa^{-1}, f(pa^{-1})) .$$

Hence $\rho(p, f(p))$ is independent of the choice of $p \in \pi^{-1}(x)$. In other words, defines a section, s , of $F(P)$; i.e. a map

$$\begin{aligned} s : M &\rightarrow F(P) & \pi \circ s &= \text{id} \\ s(x) &= \rho(p, f(p)) & \pi(p) &= x . \end{aligned}$$

Conversely, given a section s we may define the function f by the preceding equations and f satisfies $f(pa) = a^{-1}f(p)$. Thus:

Lemma 1.1 We have an identification of the space of sections of $F(P)$ with the space of maps $f : P \rightarrow F$ satisfying $f(pa) = a^{-1}f(p)$.

There are two especially important cases of this construction.

The first one: Suppose that $F = G/H$ where H is a closed subgroup of G . So $F(P)$ is a bundle of homogeneous spaces. Let $f : P \rightarrow F$ satisfy the identity of (1.1) and thus be equivalent to a section, s , of $F(P)$. Consider

$$f^{-1}(H) = \{p \in P / f(p) = H \in G/H\} .$$

If $p \in f^{-1}(H)$ then $f(pa) = a^{-1}H = H$ if and only if $a \in H$. Thus

$$f^{-1}(H) = P_H$$

is an H sub-bundle of P , a reduction of the principal G bundle to an H bundle. Conversely, suppose that P_H is an H sub-bundle of P . Then define $f : P \rightarrow G/H$ by $f(P_H) = H$ and if

$$p = qa, \quad q \in P_H$$

then

$$f(p) = a^{-1}H .$$

This is well defined as can easily be checked and defines a function f satisfying the condition (1.1). Thus: A section of the bundle $(G/H)(P)$ is the same as a reduction of P to an H bundle. P_H . (1.2)

A second important case is where F is a vector space and the action of G is linear. Then $F(P)$ is a vector bundle. In this case we can consider k -forms, Ω , on P with values in F . We can consider forms which are horizontal in the sense that

$$i(\xi)\Omega = 0 \text{ for any vertical tangent vector } \xi . \quad (1.3)$$

where **vertical** means tangent to the fiber, and $i(\xi)$ denotes the operator of inner product with the vector field ξ . We can also consider forms which are equivariant in the sense that (here R_a^* stands for the adjoint of the tangent extension of R_a)

$$R_a^* \Omega \equiv \Omega \circ dR_a = a\Omega \quad \text{for all } a \in G . \quad (1.4)$$

It is easy to check that (1.1) generalizes to:

Lemma 1.2 An f -valued k -form on P satisfying (1.3) and (1.4) is equivalent to an $F(P)$ -valued k -form on M . (1.5)

(The case $k = 0$ of (1.5) is then (1.1)).

We can combine the preceding two cases. Suppose that we are given a section s of $(G/H)(P)$, so we get a reduced bundle, P_H . We can consider the vector bundle associated to the adjoint action of H on $\mathfrak{g}/\mathfrak{h}$, the homogeneous space given by the Lie algebras \mathfrak{g} and \mathfrak{h} of G and H respectively. This vector bundle can be identified with the bundle of vertical tangent vectors to $(G/H)(P)$ along the section s . Indeed, we can consider $(G/H)(P)$ as the bundle associated to P_H relative to the H action on G/H . On the principal bundle P_H the section s corresponds to the identically constant function $f \equiv H$. At the point H we have an identification of $T(G/H)$ with $\mathfrak{g}/\mathfrak{h}$. This identification is consistent with the H action. Thus we may identify $(\mathfrak{g}/\mathfrak{h})(P_H)$ with the bundle of vertical tangent vectors to $(G/H)(P)$ along s . Now suppose that $\Theta : TP \rightarrow \mathfrak{g}/\mathfrak{h}$ is a 1-form which satisfies (1.3) and (1.4) relative to the group H . Then Θ can be thought as a 1-form on M with values in $(\mathfrak{g}/\mathfrak{h})(P_H)$. Thus: Let s be a section of $(G/H)(P)$ and P_H the corresponding reduced bundle. Let Θ be a 1-form on P_H with values in $\mathfrak{g}/\mathfrak{h}$ which satisfies (1.3) and (1.4) relative to the group H . Then:

Θ can be regarded as a 1-form on M with values in the bundle of vertical tangent vectors to $(G/H)(P)$ along s . (1.6)

In the particular case that $\dim G/H = \dim M$, we can further demand that the 1-form Θ on M give an isomorphism between TM and the bundle of vertical tangent vectors. This is the method conceived by Cartan. The form Θ is called a **soldering form** (or "soudage", in French).

For example, suppose that $H = O(V)$ is the orthogonal group of a vector space with a non-degenerate scalar product and $G = H \ltimes V$ (semi-direct sum). Then $G/H = V$. A soldering form Θ then gives an identification of TM with $V(P_H)$. In particular, this puts a (pseudo or) Riemannian metric on M and also allows us to identify P_H with the bundle of orthogonal frames. Similarly, if we take $H = Gl(V)$ then a soldering form Θ allows us to identify P_H with the bundle of all linear frames on M .

Conversely, let P_H denote the bundle of frames of a differentiable manifold M , where $H = Gl(R^n)$, $n = \dim M$. then P_H carries a canonical 1-form Θ values in R^n , and Θ satisfies (1.2) and (1.3). Namely, Θ is the so-called

"structure form" defined by

$$\Theta(\xi) = p(d\pi_p \xi) \quad \xi \in TP_H$$

$$p : T_{\pi(p)}M \rightarrow R^n \quad p \in P_H .$$

Of course, we can enlarge the bundle P_H to a G -bundle $P_G = G(P_H)$ and then we are back in the situation described above.

Let P be a vector bundle with structure group G . For each $\xi \in \mathfrak{g}$, the Lie algebra of G , let ξ denote the corresponding vector field on P_G given by the right action of G on P . Recall that a connection on P_G can be described as a \mathfrak{g} -valued 1-form ω_G on P_G which satisfies

$$i(\widehat{\xi})\omega = \xi, \quad \xi \in \mathfrak{g} , \quad (1.7)$$

and

$$R_a^* \omega = Ad_a \omega_G . \quad (1.8)$$

The **horizontal space** of a connection ω_G at a point $p \in P_G$ consists of all tangent vectors at p which are annihilated by ω_G , i.e., those which satisfy

$$i(\nu)\omega_G = 0 . \quad (1.9)$$

Any curve on M lifts to a unique horizontal curve on P_G (one whose tangents are everywhere horizontal) once a lift at one point has been specified. This is the notion of parallel transport along a curve. We shall give a more detailed description of the horizontal space in the last section of this article.

Now suppose we are in the following situation: we have a reduction of P_G to an H bundle, P_H , and we are given a connection, ω_G on P_G . Then the restriction of ω_G to P_H defines a \mathfrak{g} -valued 1-form on P_H which satisfies (1.7) with $\xi \in \mathfrak{h}$ and (1.8) with $a \in H$. As \mathfrak{h} is an invariant subspace of \mathfrak{g} , we can define the form

$$\Theta = \text{restriction of } \omega_G \text{ to } P_H / \mathfrak{h}$$

as a $\mathfrak{g}/\mathfrak{h}$ valued 1-form on P_H satisfying the conditions of vanishing on all vertical vector fields and of being equivariant with respect to the action of H .

If the group H is reductive -or more generally, if h has an H invariant complement, \mathfrak{n} in \mathfrak{g} - we can decompose

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{n} \\ \omega_G|_{P_H} &= \omega_H + \Theta \end{aligned}$$

where we have identified \mathfrak{n} with $\mathfrak{g}/\mathfrak{h}$. Here ω_H is an h -valued 1-form and Θ is an \mathfrak{n} -valued 1-form. The form ω_H satisfies all the conditions for a connection on P_H . Thus:

Lemma 1.3: If the group H is reductive, then the restriction of ω_G to P_H determines an \mathfrak{n} -valued 1-form Θ on P_H and a connection ω_H on P_H . The 1-form Θ is horizontal and equivariant relative to H . (1.10)

It is important to note that the horizontal subspaces for the connections ω_G and ω_H will differ, in general, at points of P_H . However, given ω_H and Θ , one can reconstruct ω_G along P_H and hence on all of P_G . Thus the data ω_H, Θ on P_H is equivalent to a connection ω_G .

In particular if $\dim G/H = \dim M$ we can consider the condition that Θ be a soldering form. If this case holds, then ω_G is known as a **Cartan connection**. If G is the affine group then a Cartan connection is called an **affine connection**.

Let F be a vector space on which G acts. We will let $A^k(F)$ denote the space of k -forms on M with values in $F(P)$. In particular, we can identify $A^0(F)$ with the space of sections of $F(P)$, a space which we also denote by $\Gamma(P)$. A connection ω on P defines a covariant derivative

$$\begin{aligned} \nabla^\omega : A^k(F) &\rightarrow A^{k+1}(F) \\ \nabla_\omega \Omega &= d\Omega - \omega \cdot \Omega \end{aligned}$$

One also defines the **curvature 2-form** of the connection ω by

$$\text{curv}(\omega) = \nabla_\omega \omega = d\omega - 1/2[\omega, \omega].$$

For the case of a reduced bundle P_H with a reductive group H , the restriction of the $\text{curv}(\omega_G)$ to P_H is given by

$$d\omega_H + d\Theta - 1/2[\omega_H + \Theta, \omega_H - \Theta]$$

$$\begin{aligned}
&= d\omega_H - 1/2[\omega_H, \omega_H] + d\Theta - \Theta \cdot \omega - 1/2[\omega_H, \omega_H] \\
&= \text{curv}(\omega_H) + \nabla_\omega \Theta - 1/2[\Theta, \Theta] .
\end{aligned}$$

In particular, for the case of an affine connection $[\eta, \eta] = 0$, so that the last term vanishes. The term

$$\nabla_\omega \Theta = d\Theta - \omega_H \cdot \Theta$$

is known as the **torsion 2-form** of the connection ω_H . In the Levi-Civita theory of connections, the torsion entered for describing the non-closure of an infinitesimal parallelogram formed by parallel transport of infinitesimal vectors; the non-commutativity of the transport enters into the Christoffel symbols. The torsion tensor, in local coordinates, is given by the skew-symmetric components of the Christoffel symbols. But in the Cartan theory the torsion enters as a (translational) component of the curvature. In condensed matter physics, it plays the role of a dislocation density; (Kleinert, 1989).

We shall assume from now on, otherwise stated, that all geometrical structures are infinitely differentiable, and that space-time M has dimension equal to 4. This second condition is unessential.

We shall be mainly interested in the sequel in G being the Poincaré group and H the Lorentz group, so that G/H as an homogeneous space can be identified with $R^{1,3}$, or still $H = O(R^4)$ and $G = H \oplus R^4$ (on what follows, we shall write undistinctly the degenerate and non-degenerate cases). Then Θ is an $R^4(P_H)$ -valued soldering 1-form on M , and ω_H is an $h(P_H)$ -valued connection 1-form on M . We then obtain a Riemannian metric on M (or Lorentzian). Then, for any $x \in M$, $\Theta(x)$ can be thought as a 1-form on M with values in $T_x M$, due to the canonical isomorphism between $R^4(P_H)$ and TM . Then, given a local coordinate system (x^α) on M , where $\alpha = 1, \dots, 4$, we get a local coordinate system $(x^\alpha, \partial/\partial x^\alpha)$ on TM . Locally, Θ takes the form

$$\Theta(x) = (\Theta_\alpha^a(x) dx^\alpha) , \tag{1.11}$$

with inverse $e_a^\alpha \partial/\partial x^\alpha$, $a = 0, 1, \dots, 3$ representing the indices of an anholonomic basis in R^4 ; thus

$$\Theta_\alpha^a e_b^a = \delta_b^\alpha, \text{ and } , \Theta_\alpha^a e_a^\beta = \xi_\alpha^\beta . \tag{1.12}$$

If (g_{ab}) denotes a metric on R^4 , say Euclidean or Minkowski, the local expression for the metric that Θ puts on M , is

$$g_{\alpha\beta} = g_{ab}\Theta_a^\alpha\Theta_b^\beta, \quad (1.13)$$

which then has the same signature of that of (g_{ab}) .

If we have a Lorentz (or, orthogonal) linear connection on M , $\omega_H = (\omega_\mu^{ab})$, then ω^H is skew-symmetric in a, b . Now, since locally any element in h takes the form $u \wedge v$, where u and v belong to $V (= R^4, \text{ or } R^{1,3})$, then ω^H defines a connection 1-form on M with values in $\wedge^2 V \simeq \wedge^2 TM$, $\tilde{\omega}^H = (\omega_\mu^{\alpha\beta}) = (\omega_\mu^\alpha e_\alpha^\beta)$, with associated TM -valued torsion 2-form

$$T = 1/2 T_{\beta\gamma}^\alpha dx^\beta \wedge dx^\gamma, \quad (1.14)$$

with coefficients given by the torsion tensor defined by

$$T_{\beta\gamma}^\alpha = (\Theta^{-1} \nabla_\omega \Theta)_{\beta\gamma}^\alpha = e_a^\alpha \nabla_\omega \Theta = e_a^\alpha \left(\partial_\gamma \Theta_{\beta}^a - \omega_{b[\gamma}^a \Theta_{\beta]}^b \right), \quad (1.15)$$

where $(\omega_{b\mu}^a)$ denotes the Christoffel coefficients of the connection form ω^H induced by the isomorphism $V \times V = V^* \times V^*$ induced by g , so that $\omega_{b\mu}^a = \omega_\mu^{ac} g_{bc}$. Thus, the Christoffel coefficients of the connection 1-form $\tilde{\omega}^H$ are given by

$$\Gamma_{\beta\mu}^\alpha = e_a^\alpha \Theta_\beta^b \omega_{b\mu}^a + e_a^\alpha \partial_\mu \Theta_\beta^a \quad (1.16)$$

$\tilde{\omega}_H$ is known in Poincare gauge theory as the **space-time linear connection**.

If we assume that ω^H is compatible with the metric on V , i.e. $\nabla_{\omega_H\mu} g_{ab} = 0$, then $\tilde{\omega}^H$ is compatible with the space-time metric, i.e.

$$\nabla_{\tilde{\omega}_H\mu} g_{\alpha\beta} = 0 \quad (1.17)$$

Thus, lengths of vector fields are preserved under parallel transport. This means that Θ has reduced the bundle of linear frames to the orthogonal bundle.

What is essential to the connection on M defined by (1.16), is its non-symmetric character, i.e. it has a non-zero torsion tensor

$$T_{\nu\mu}^\alpha = 1/2 (\Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha). \quad (1.18)$$

This geometry is called the Riemann-Cartan (RC) structure.

Let us introduce a conformal structure on the tangent space of M . For this we shall follow Einstein's last work (Einstein and Kauffman, 1955; Obukhov, 1982), which lifts Einstein's original criticism to Weyl's abelian (and historically first) gauge theory of 1918.

We define the Weyl transformation on the soldering form

$$W(\Theta_b^a) = \psi \Theta_b^a, \quad (1.19)$$

so that $W(e_a^\alpha) = (1/\psi)e_a^\alpha$, and a Weyl transformation on Γ (which by abuse of notation we denote by W as well as for the other derived transformations)

$$W(\Gamma_{b\mu}^a) = \Gamma_{b\mu}^a \quad (1.20)$$

then we can derive the following transformation on the metric on M

$$W(g_{\alpha\beta}) = \psi^2 g_{\alpha\beta}, \text{ and, } W(g^{\alpha\beta}) = \psi^{-2} g^{\alpha\beta} \quad (1.21)$$

These are the well known conformal transformations of the metric on M . In the above definitions, ψ is a function defined on M with values on R^+ .

The Riemann-Cartan structure, under the above transformations becomes

$$W(\Gamma_{\beta\mu}^\alpha) = \Gamma_{\beta\mu}^\alpha + \delta_\beta^\alpha \partial_\mu \ln \psi \quad (1.22)$$

with torsion tensor

$$T_{\beta\mu}^\alpha + 1/2 (\delta_\beta^\alpha \partial_\mu \ln \psi - \delta_\mu^\alpha \partial_\beta \ln \psi) \quad (1.23)$$

This shows that only the trace of the torsion tensor is conformally transformed, i.e. the 1-form $Q = Q_\mu dx^\mu = T_{\alpha\mu}^\alpha dx^\mu$ of the original connection is transformed as $W(Q) = Q + 3/2 d \ln \psi$.

There are various interesting instances of the transformation (1.22). Firstly, if the original connection space-time connection coefficients are torsionless, and given by the Levi-Civita coefficients determined by a space-time metric g , we have introduced torsion associated to the conformal field ψ on a purely Riemannian geometry. More general is the case in which the original torsion reduces to the trace-component given by a non-exact one-form, Q , which we may interpretate

as an electromagnetic potential with non-trivial field. Then, the above conformal transformations induce a gauge transformation so that Q is transformed to $Q + 3/2 d \ln \psi$. The final case we want to consider is the one in which the original space-time connection has non-zero torsion yet such that its trace vanishes completely. In this case, the original space-time connection is associated to the rotational degrees of freedom of the Dirac-Hestenes spinor field Ψ which produce a completely skew-symmetric torsion, while ψ is the scalar field that enters in the canonical decomposition of Ψ ,

$$\Psi = \psi \exp(\gamma_5 S) R,$$

where $R(x) \in \text{Spin}_+(1,3) \cong \text{Sl}(2, \mathbb{C})$ with the property that $R(x)R(x)^{-1} = \text{Id}$. For the details of this we refer to article of (Rapoport et al, 1994). The identification of ψ as a Schrodinger field was obtained from the field equations for the geometries introduced by the transformation (1.22); (Rapoport, 1994 b).

Now, the connection defined by (1.22) is not metric compatible, yet the modified connection given by (we normalize the 3/2 factor)

$$\Gamma_{\beta\mu}^\alpha = \{\alpha_{\beta\mu}\} + 2/3 (\delta_\beta^\alpha \partial_\mu \ln \psi - g_{\beta\mu} g^{\gamma\alpha} \partial_\gamma \ln \psi) \quad (1.24)$$

where $\{\alpha_{\beta\mu}\}$ are the coefficients of the Levi-Civita connection associated to g . is a metric compatible connection. Then, $Q = d \ln \psi$, the logarithmic differential of the scale field ψ , is a Weyl one-form of a RC metric compatible structure. For any metric, in account of the fact we have proved in (Rapoport, 1991) that ψ^2 is an invariant probability density, then $2K_B Q$, where K_B is Boltzmann's constant, can be thought as an entropy one-form.

The metric compatibility of these RC structures produced by the general action of the conformal group, distinguishes them from the usual Weyl geometry produced by the transformations on the space-time metric (1.21). In the latter, it is the Weyl one-form which precisely expresses the lack of preservation of lengths under parallel Weyl-transport. So the introduction of these structures solves a long pending problem of compatibilisation of the RC structures with the local action of the Weyl group.

Therefore, this geometry, which we shall call of Riemann-Cartan-Weyl (RCW, for short) has no **historicity problem** which moved Einstein to reject Weyl's

attempt to construct the first gauge theory in which he associated the Weyl form to the electromagnetic field , this in spite of Q not being a complex field (yet, a non-obvious association).

It is of great importance that we should remark that our above constructions can be carried out for the case of a general configuration space M of dimension m , on taking instead of the Poincaré group, the group given by the semi-direct sum $O(m) + R^m$. This is of relevance for the formulation of quantum mechanics for a system of n particles for which M is their configuration manifold (so that $m = 4n$ in the relativistic case) and ψ denotes their joint wave function, and is valid also for the Dirac-Hestenes spinor fields; see (Rapoport et al, 1994).

II. THE D'ALEMBERT AND WAVE OPERATORS OF THE RCW STRUCTURES

In this section we shall construct the Laplacian operators associated to a RCW geometries. As explained in the Introduction, the relevance of this follows from the fact that these operators plays a central role in the formulation of quantum mechanics as a theory of Brownian motion, as well as a Hilbert space operator theory.

We shall study then, the D'Alembert -in the case g is Lorentzian- or Laplacian -in the Riemannian case- operator associated to the RC structures. (Of course, for diffusion processes, we only consider the Riemannian case). The construction we shall present is valid for both cases undistinctly, and from now on we shall simply speak of Laplacian operators. Our treatment differs from the original treatment given by (Rapoport,1991) and follows (Rapoport 1994,1995). A completely different derivation of this operator can be found as well in (Kleinert, 1991).

Henceforth, in this paragraph, the dimension of M will be arbitrary n . We start with a RC connection described by an arbitrary metric g and an arbitrary torsion tensor. Let ∇ denote its covariant derivative operator which we additionally assume to be compatible with g , i.e. $\nabla g = 0$. Denote the Christoffel coefficients of ∇ as $\Gamma_{\beta\gamma}^{\alpha}$; then,

$$\Gamma_{\beta\gamma}^{\alpha} = \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} + 1/2K_{\beta\gamma}^{\alpha}. \quad (2.1)$$

where the first term in (2.1) stands for the Christoffel Levi-Civita coefficients of the metric g , and

$$K_{\beta\gamma}^{\alpha} = T_{\beta\gamma}^{\alpha} + S_{\beta\gamma}^{\alpha} + S_{\gamma\beta}^{\alpha},$$

is the cotorsion tensor, with

$$S_{\beta\gamma}^{\alpha} = g^{\alpha\nu} g_{\beta\kappa} T_{\nu\gamma}^{\kappa}.$$

Let us consider the Laplacian operator on functions associated to this Cartan connection, defined -in extending the usual definition- by

$$H(\nabla)f := 1/2\text{tr}(\nabla^2)f = 1/2g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}f, \quad (2.2)$$

for any continuously differentiable function f on M , where ∇ stands for the covariant derivative operator with respect to Γ . A straightforward computation shows that that $H(\nabla)$ only depends in the trace of the torsion tensor and g :

$$H(\nabla) = 1/2\Delta_g + g^{\alpha\beta}Q_{\beta}\partial_{\alpha}, \quad (2.3)$$

with $Q = T_{\nu\beta}^{\nu}dx^{\beta}$, the trace-torsion one-form, and

$$\Delta_g = \text{tr}((\nabla^g)^2) = (\det g)^{-1/2}g^{\alpha\beta}\partial_{\alpha}\left((\det g)^{1/2}g^{\alpha\beta}\partial_{\beta}\right),$$

the Laplace-Beltrami operator associated to the Levi-Civita connection ∇^g . Therefore, for the Riemann-Cartan connection ∇ we have that, on smooth functions defined on M

$$H(\nabla) = \frac{1}{2}\text{tr}((\nabla)^2) = \frac{1}{2}\Delta_g + \hat{Q}, \quad (2.3')$$

with \hat{Q} the vector-field dual to the 1-form Q : $\hat{Q}(f) = \langle Q, \text{grad } f \rangle$, $f: M \rightarrow \mathbb{R}$.

Notice that $H(\nabla)$ only depends on g and the trace-torsion of the connection ∇ ; the other terms of the invariant decomposition of the torsion tensor of ∇ do not intervene in the Laplacian. Thus, to obtain a one-to-one correspondance (in general dimension other than 2 when this is satisfied trivially) between Cartan connections and their Laplacian operators, we restrict ourselves to ∇ with Christoffel symbols of the form

$$\Gamma_{\beta\gamma}^{\alpha} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)} \{ \delta_{\beta}^{\alpha} Q_{\gamma} - g_{\beta\gamma} Q^{\alpha} \}. \quad (2.4)$$

We have

$$H(\nabla) = \frac{1}{2} \text{tr}((\nabla^g)^2) + \hat{Q} \quad (2.5)$$

In the case that g is Riemannian, then the expression (2.3) is the most general invariant laplacian acting on functions defined on a smooth manifold associated to a Markovian semigroup that preserves probability; (Rapoport, 1995 c). This restriction, will allow us to establish a one-to-one correspondance between Riemann-Cartan-Weyl connections (2.4) with Markovian diffusion processes.

We shall further assume in the following that Q reduces to the exact form: $Q = d \ln \psi$, where ψ is a real function on M . In this case, the RCW geometry is determined by the Riemannian metric g and the function ψ . The corresponding Laplacian, which we shall write from now on as $H(g, \psi)$ is defined by its action on function $f : M \rightarrow \mathbb{R}_{>0}$ by

$$H(g, \psi)f = \frac{1}{2} \Delta_g f + \langle \text{grad} \ln \psi, \text{grad} f \rangle. \quad (2.6)$$

The theory we shall construct is determined by this laplacian, which we shall call the RCW laplacian.

III. RCW LAPLACIANS AND WITTEN'S DEFORMED LAPLACIAN OF THE TOPOLOGICAL QUANTUM FIELD THEORIES

Let us assume in the following that we have a smooth n -dimensional orientable compact manifold M provided with a Riemannian metric, g . We consider the Hilbert space of square summable ω of differential forms of degree q on M , with respect to vol_g . We shall denote this space as $L^{2,q}$ or still as $L^2\Omega^q(M, \text{vol}_g)$. The inner product is

$$\langle \omega, \phi \rangle = \int_M \langle \omega_x, \phi_x \rangle \text{vol}_g$$

where the integrand is given by the natural pairing between the components of ω and the conjugate tensor: $g^{\alpha_1 \beta_1} \dots g^{\alpha_q \beta_q} \phi_{\beta_1 \dots \beta_q}$; alternatively, we can write in a coordinate independent way: $\langle \omega(x), \phi(x) \rangle \text{vol}_g = \omega(x) \wedge * \phi(x)$, with $*$ the Hodge star operator, for any $\omega, \phi \in L^{2,q}$ [45].

The de Rham-Kodaira operator on $L^{2,q}$ is defined as

$$\Delta = -(d + \delta)^2 = -(d\delta + \delta d), \quad (24)$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator d defined on $L^{2,q}$:

$$\langle \delta\phi, \omega \rangle = \langle \phi, d\omega \rangle,$$

for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$. In the case of $q = 0$, this is the Laplace-Beltrami operator on functions encountered before; in the general case we have in addition of $\text{tr}(\nabla^g)^2$ the contribution of the Weitzenbock curvature term. Let us we are given a C^2 positive function ψ on M . We then have an induced smooth density $\rho = \psi^2 \text{vol}_g$ on M .

We introduce the Hilbert space $L^{2,q,\rho} = L^2\Omega^q(M, \rho)$, of differential forms on M of degree q , square integrable with respect to ρ , with inner product:

$$\langle \phi_1, \phi_2 \rangle^\rho = \int_M \langle \phi_1(x), \phi_2(x) \rangle \rho, \quad (3.1)$$

for $\phi_1, \phi_2 \in L^{2,q,\rho}$. We define the quadratic form $q(\phi) = \frac{1}{2} \langle \phi, \phi \rangle^\rho$, with ϕ on the Hilbert space given by the completion of the space of all smooth q -forms under the $L^{2,\rho}$ inner product. In the case of exact one-forms, this is (twice) the quadratic form introduced in correspondance with the Brownian processes determined by $H(g, \psi)$; (Rapoport 1991, 1994, 1995).

Consider the formal adjoint of d , which we shall denote as δ^ψ defined on $L^{2,q+1,\rho}$ as follows

$$\langle \delta^\psi \omega, \phi \rangle^\rho = \langle \omega, d\phi \rangle^\rho, \quad (3.2)$$

for any $\omega \in L^{2,q,\rho}$ and $\phi \in L^{2,q+1,\rho}$. Since $d^2 = 0$, we have

$$(\delta^\psi)^2 = 0. \quad (3.3)$$

For any smooth function f defined on M , and ω a q -form:

$$\delta(f\omega) = f\delta\omega - i_{\text{grad } f}\omega,$$

where i_X is the interior product derivation on q -forms.

We introduce the operator on $L^{2,q,\rho}$:

$$\Delta^{\psi,q} = -(d + \delta^\psi)^2, \quad (3.4)$$

which still writes as

$$-(d\delta^\psi + \delta^\psi d).$$

Recalling the definition of the Lie-derivative operator $L_X = di_X + i_X d$, X a smooth vector field on M , we finally have

$$\Delta^{\psi,q} = \Delta^q + 2L_{\text{grad } \ln \psi}, \quad (3.5)$$

Let us define now the deformed exterior differential operator mapping q -forms in $q+1$ -forms, by:

$$d^\psi = \psi d \psi^{-1}, \quad (3.6)$$

so that

$$d^\psi \omega = d\omega - d \ln \psi \wedge \omega.$$

We have that

$$(d^\psi)^2 = 0. \quad (3.7)$$

This operator is the ($\tau = -1$ version) of Witten's deformed differential (Witten, 1982). We introduce now the deformed co-differential operator as the formal adjoint of d^ψ :

$$(d^\psi)^\ast = \psi^{-1} \delta \psi. \quad (3.8)$$

We introduce the Witten deformed Laplacian operator, defined as:

$$L^{\psi,q} = -(d^\psi + d^{\psi^\ast})^2, \quad (3.9)$$

which can still be written as

$$-(d^\psi d^{\psi^\ast} + d^{\psi^\ast} d).$$

We have the following relation between the two Laplacian operators:

$$\Delta^{\psi,q} = \psi^{-1} L^{\psi,q} \psi, \quad (3.10)$$

so that these two operators are conformally equivalent under conjugation by ψ . Note that $\Delta^{\psi,0} = 2H(g, \psi)$.

The key to the construction of Quantum Mechanics as a theory of diffusion processes and a Hilbert space operator theory, rests on choosing the operators $\frac{1}{2}\Delta^{\psi,q}$ as infinitesimal generators of Markovian semigroups [1], $P_\tau^q, q = 0, \dots, n$. P_τ^0 is the stochastic process with infinitesimal generator given by $H(g, \psi)$, which was constructed in (Rapoport, 1991, 1995). Yet, the construction of this family of diffusion processes on differential forms of arbitrary degree, rests on the knowledge of the data g and ψ which determine the RCW structure; these data are determined from a stochastic extension of the Einstein-Hilbert variational principle to RC geometries; one proves that ψ is a solution of the conformal invariant wave equation; (Rapoport, 1995 a).

Thus, starting from the RCW geometry determined by the field equations, we can construct a family of stochastic processes on forms of any degree. Remarkably, the Laplacian introduced by Witten in topological quantum field theory, appears to be related to a wave function which satisfies the field equations and produces the torsion of the RCW geometry.

We would like to note finally that from the fact that $(d^\psi)^2 = 0$, we can define a deformed de Rham complex: $H_\psi^q(M, R)$ as $\text{Ker}(d^\psi : \Lambda^q \rightarrow \Lambda^{q+1}) / \text{Ran}(d^\psi : \Lambda^{q-1} \rightarrow \Lambda^q)$. Yet, since $\text{Ker}(d^\psi) = \psi \text{Ker}(d)$, and $\text{Ran}(d^\psi) = \psi \text{Ran}(d)$, we obtain that $H_\psi^q(M, R) \cong H^q(M, R)$, for any $q = 0, \dots, n$. Now, by Hodge's theorem: $\dim H^q(M, R) = \dim(\text{Ker}(\Delta^q))$, which by the above construction is clearly equal to $\dim(\text{Ker}(L^{\psi,q}))$: by (3.10) we conclude that

$$\dim(H^q(M, R)) = \dim(\text{Ker}\Delta^{\psi,q}). \quad (35)$$

This identification, which we shall not use in this article, is fundamental to the formulation of the ergodic studies of the flows; indeed, if the first Betti number of M , $b_1(M) = \dim(H^1(M, R)) \neq 0$, then it can be proved that the flows corresponding to RCW geometries with infinitesimal generators given by $\frac{1}{2}H(g, \psi)$ is (moment) unstable; (Rapoport, 1995, b, c).

IV. RCW GEOMETRIES AND SUPERSYMMETRIC SYSTEMS

That Laplacian operators on smooth compact manifolds are examples of supersymmetric systems was a profound observation due to Witten; (Witten, 1982).

One starts with a Hamiltonian H on a Hilbert space \mathbf{H} , together with a self-adjoint operator Q and a bounded self-adjoint operator P both defined on \mathbf{H} , such that

$$H = Q^2 \geq 0, P^2 = 1, \text{ and } \{Q, P\} = QP + PQ = 0.$$

Then, the triple $\{H, P, Q\}$ is said to be a supersymmetric system, or still, to have supersymmetry. Since P is self-adjoint and $P^2 = 1$, then P has for eigenvalues 1 and -1 . Denote

$$\mathbf{H}_{\text{ferm}} = \{\phi \in \mathbf{H}, P\phi = -\phi\}$$

and

$$\mathbf{H}_{\text{bos}} = \{\phi \in \mathbf{H}, P\phi = \phi\},$$

which are called the fermionic and bosonic states, respectively. Then, $Q : \mathbf{H}_{\text{ferm}} \rightarrow \mathbf{H}_{\text{bos}}$ and $Q : \mathbf{H}_{\text{bos}} \rightarrow \mathbf{H}_{\text{ferm}}$, or in other words, Q maps fermionic states into bosonic states and viceversa.

In the present theory, we take for Hilbert space $\mathbf{H} = \bigoplus_{q=0}^n L^{2,q,\rho}$, and the Hamiltonian operator H is $\Delta^\psi = \frac{1}{2}\Delta + L_{\text{grad } \ln \psi}$ as an operator on forms of arbitrary degree, where $\Delta_g = -(d\delta + \delta d)^2$. Now we take $Q = i(d + \delta^\psi)$ and P is defined on \mathbf{H} by its restriction to q -forms: $P|L^{2,q,\rho} = (-1)^q$, $q = 0, \dots, n$. i.e., the operator of multiplication by $(-1)^q$. Then, it is easily seen that $\{H, P, Q\}$ is a supersymmetric system. Thus, in this setting, fermionic (bosonic) states are given by odd (even) forms.

Yet we would like to remark that the quantization of gravitation suggested in the above sections by taking $\frac{1}{2}\Delta^{g,\psi}$, with $g = 0, \dots, n$ for infinitesimal generators of diffusion processes -corresponding to fermions and bosons- determined by solving the heat kernel of each Markovian family P^g , $0 \leq g \leq n$, depends on the knowledge of g and ψ . Thus, the knowledge of the RCW geometry determines the quantum theory for bosons and fermions alike.

V. RCW GEOMETRIES AND THE SYMPLECTIC STRUCTURE OF LOOP SPACE

It may seem rather strange that in the present theory, in contrast with TQFT, ψ is defined on space-time M instead of being a functional on loop space, i.e. the infinite dimensional manifold $\Omega = \{\phi : S^1 \rightarrow M, \phi \in C^\infty\}$. In this description, M can be recovered as the constant loops of Ω . This infinite dimensional setting is the one considered in the topological quantum field theories, and different choices of ψ yield the supersymmetric σ models, the supersymmetric ϕ^4 theory, etc.; Witten 1982. Yet, it is to be remarked that the RCW geometries are connected to the symplectic structure on Ω , in a way we shall describe in the following.

Let us fix $\phi \in \Omega$. Then, the tangent space to Ω at ϕ , $T_\phi\Omega$ can be identified with the space of sections of the pull-back vector bundle $\phi^*(TM)$ of TM to S^1 by ϕ . The metric g on M defines a metric \cdot_ϕ on $\phi^*(TM)$, and hence we have an inner product on $T_\phi\Omega$: $(s_1, s_2) = \frac{1}{2\pi} \int_{S^1} s_1 \cdot_\phi s_2$. Thus, $T_\phi\Omega$ has a pre-Hilbert structure.

Next, we introduce a general Riemann-Cartan connection, ∇ on M . This induces a connection on $\phi^*(TM)$, and hence a covariant derivative operator ∇_ϕ which acts on sections of $\phi^*(TM)$ by evaluation on the vector field $\frac{d}{d\sigma}$ of S^1 . Now we can define a skew-symmetric bilinear form on $T_\phi\Omega$:

$$\omega(\phi) = \frac{1}{4\pi} \int_{S^1} (\nabla_\phi s_1 \cdot_\phi s_2 - \nabla_\phi s_2 \cdot_\phi s_1). \quad (6.1)$$

Varying $\phi \in \Omega$, we obtain a differential 2-form on Ω . As first noted by Atiyah (Atiyah, 1985), $d\omega$ equals $(\frac{1}{2\pi})$ the integral over S^1 of the skew-symmetric component of the torsion tensor; for the details, see (Bowick & Rajeev, 1987).

Therefore, for a RCW geometry, ω is a closed 2-form on Ω . Yet, ω is not properly a symplectic form since it vanishes at those ϕ for which ∇_ϕ has a 0 eigenvalue, i.e. on any tangent vector which is covariantly constant along ϕ . In the case of a purely Riemannian geometry, i.e. $v = 1$ this "symplectic" setting has been applied to obtain an exact computation of the trace of the heat kernel of $2\pi\Delta_g$ and to the direct obtention of the Atiyah-Singer index theorem: (Atiyah, 1985). It would be interesting to check if this constructions can be carried out for the trace of the heat kernel of $H(g, \psi)$.

We close this section with the observation that the 2-form of (6.1), is related to the zero modes of string theory, (Bowick & Rajeev, 1987); the relevance of RC geometries to string theory was assessed in (Scherk and Schwarz; 1974).

VI. THE CARTAN CLASSICAL COPYING METHOD

As a final section of this first part of the article, we wish to describe the Cartan classical copying method as a preparation for its stochastic extension; (Rapoport, 1995 d).

For this, we need to study in further detail the structure of horizontal vector fields on the bundle of orthogonal frames, P_H , which we shall do next.

The total space of P_H is described as the space of all pairs $r = (x, e)$ where $x \in M$ and $e = [e_1, \dots, e_n]$ is an orthonormal frame at x , i.e. the vectors e_1, \dots, e_n are a basis for $T_x M$ satisfying the condition

$$g_{\alpha\beta} e_a^\alpha e_b^\beta = \eta_{ab} , \quad (7.1)$$

where $\eta = (\eta_{ab})$ is the Minkowski metric in the case g is Lorentzian (or the Euclidean metric in the Riemannian case); as we already saw,

$$e_a^\alpha e_b^\beta = g^{\alpha\beta} . \quad (7.2)$$

Every vector field L on M induces a vector field, \tilde{L} , on P_H , defined as follows. If f is a smooth function on P_H , then $\tilde{L}f$ is given by

$$(\tilde{L}f)(r) = \frac{d}{dt} f((\exp tL)x, (\exp tL)_* e) \Big|_t = 0$$

where $r = (x, e)$ and $(\exp tL)_* : T_x M \rightarrow T_{(\exp tL)x} M$ is the tangent mapping to $\exp tL$, so that $(\exp tL)_*(e) = [(\exp tL)_* e_1, \dots, (\exp tL)_* e_n]$. Here, we recall, $\exp tL$ is the local diffeomorphism $x \rightarrow x(t, x)$ of M defined by the flow of the differential equation

$$\begin{aligned} \frac{dx^\alpha}{dt} &= a^\alpha(x, t), \quad \text{where } L = a^\alpha(x) \partial_\alpha \\ x(0, x) &= x. \end{aligned}$$

If in P_H we have a Cartan connection $\tilde{\nabla}$ with coefficients $(\Gamma_{\beta\gamma}^\alpha)$ which is compatible with g , we can describe the horizontal sub-bundle of P_H : at each $r \in P_H$ as,

$$H_r = \{X \text{ is a vector field on } P_H : X = A^\alpha(x)\partial_\alpha - \Gamma_{\beta\gamma}^\alpha(x)e_\alpha^\gamma A^\beta(x)\partial/\partial e_\alpha^\beta\} .$$

H_r is clearly a subspace of $T_r(P_H)$ which is clearly independent of the choice of local coordinates $(x^\alpha, e_\alpha^\alpha)$ on P_H . We recall that the horizontal lift, $\tilde{\xi}$, of a vector $\xi \in T_x M$, is uniquely determined by $\pi_*(\tilde{\xi}) = \xi$ and $\pi(e) = x$.

Given a vector field X on M , the horizontal lift \tilde{X} , is the unique vector field of such that \tilde{X}_r is the horizontal lift of $X(\pi(r))$, for all $r \in P_H$. In a local coordinate system $(x^\alpha, e_\alpha^\alpha)$, if $X = X^\alpha(x)\partial_\alpha$, then

$$\tilde{X} = X^\alpha\partial_\alpha - \Gamma_{\alpha\beta}^\gamma X^\alpha e_\alpha^\beta \frac{\partial}{\partial e_\alpha^\gamma} . \quad (7.3)$$

Let $\Theta : [0, \infty) \rightarrow M$ be a smooth curve on M ; then the horizontal lift, $\tilde{\Theta}$, of Θ , is the unique curve on P_H such that $d\tilde{\Theta}/dt(t)$ is horizontal and for any $t \geq 0$, we have that $\pi(\tilde{\Theta}(t)) = \Theta(t)$. Clearly, if $r = (x, e)$ is given where $x = \Theta(0)$, then a horizontal lift $\tilde{\Theta}$ of Θ starting at r exists and is unique. Indeed $\tilde{\Theta}(t) = (\Theta(t), [e_1(t), \dots, e_n(t)])$, where $e_a(t) \in T_{\Theta(t)}M$ is obtained from e_a by parallel transport along Θ through $\tilde{\nabla}$, for each $a = 1, \dots, n$; there exists a unique vector field \tilde{L}_a on P_H such that $(\tilde{L}_a)_r$ is the horizontal lift of $e_a \in T_x M$, for every $r = (x, e)$ and $a = 1, \dots, n$. In local coordinates as above,

$$\tilde{L}_a = e_a^\alpha\partial_\alpha - \Gamma_{\beta\gamma}^\epsilon e_\alpha^\beta e_b^\gamma \frac{\partial}{\partial e_b^\epsilon} . \quad (7.4)$$

The set $\{\tilde{L}_1, \dots, \tilde{L}_n\}$ is called the system of horizontal vector fields, or still basic vector fields.

We are finally ready to introduce to the Cartan classical copying method. Let M be a manifold provided with a Cartan connection $\tilde{\nabla}$ compatible with a metric g on M . The Cartan connection enables us to roll M along a curve $\gamma(t)$ on R^n (where n is the dimension of M) to obtain a curve $\Theta(t)$ on M as the trace of the curve γ . An important comment is here in order: γ is arbitrary, i.e. it can be the trajectory of an arbitrary dynamical system. The second and most important

is the fact that the method we shall present works because the tangent space at every point is identical to the quotient of the affine group of H (H direct sum with translations) with H , so that we can place a Cartan connection $\tilde{\nabla}$ on P_H . To precise this, let $r = (x, e) \in P_H$ and $\gamma : [0, \infty) \rightarrow R^n$ be a smooth curve. Define $\tilde{\Theta} : [0, \infty) \rightarrow P_H$, where $\tilde{\Theta}(t) = (\Theta(t), e(t))$ by

$$\text{(Copying equation)} \quad \frac{d\Theta}{dt}(t) = e_a(t) \frac{d\gamma^a}{dt}(t) \quad (7.5)$$

$$\text{(Parallel transport)} \quad \tilde{\nabla}_{d\Theta/dt} e_a(t) = 0 \quad (7.6)$$

for every $a = 1, \dots, n$, with the initial conditions

$$\Theta(0) = x, \quad e(0) = e \quad (7.7)$$

Equations (7.6) and (7.7) have the componentwise form

$$\frac{d\Theta^\alpha}{dt}(t) = e_a^\alpha(t) \frac{d\gamma^a}{dt}(t) \quad (7.6')$$

$$\frac{de_a^\alpha}{dt}(t) = -\Gamma_{\beta\gamma}^\alpha(\Theta(t)) c_a^\gamma(t) \frac{d\Theta^\beta}{dt}(t) \quad (7.7')$$

which themselves are the componentwise expression of the single equation

$$\frac{d\tilde{\Theta}}{dt}(t) = \tilde{L}_a(\tilde{\Theta}(t)) \frac{d\gamma}{dt}(t) \quad (7.8)$$

and the initial condition for $\tilde{\Theta}$ is

$$\tilde{\Theta}(0) = r \quad (7.9)$$

where \tilde{L}_a is the system of horizontal vector fields on P_H . The solution curve $\Theta(t) = \pi(\tilde{\Theta}(t))$ depends on the initial frame at x : we shall denote it by $\Theta(t) = \Theta(t, r, \gamma)$, with $r = (x, e)$. It follows easily that $\Theta(t, Ar, \gamma) = \Theta(t, r, A\gamma)$ for any $t \in [0, \infty)$ and $A \in H$, where the curve $A\gamma$ is defined by $(A\gamma)(t) = A \cdot \gamma(t)$.

Thus we have completed the formulation of the Cartan copying method for classical curves, i.e. for smooth curves. We shall call this instance of the Cartan method as the **classical Cartan copying method**.

The striking point is that one can generalize this method to the copying of Brownian motions in R^n . Yet, it is more striking still, that to construct Quantum Mechanics as a theory of diffusions, one does not start with an arbitrary continuous stochastic process on R^n , but the most simple elementary case of an homogeneous isotropic process, the Wiener process whose transition probability is the standard Gaussian density on R^n . The diffusion process on M becomes completely determined by the RCW connection given by only the trace part of the torsion of $\tilde{\nabla}$. Yet to carry over this construction to the stochastic case, we are lacking one point: the substitution of the usual rules of calculus (more specifically a chain rule) for smooth functions taken along smooth curves, for rules that are applicable in the case that the curves are sample paths of Wiener processes, which are continuous non-differentiable (furthermore, they are fractals). This shall be presented in a forthcoming article, together with the formulation of quantum mechanics as Dirichlet forms associated to the RCW Laplacians.

CONCLUSIONS

We have constructed the RCW geometries and their associated laplacian operators in view of the construction of quantum mechanics as diffusion processes as Markovian semigroups having these laplacians as infinitesimal generators. Yet, in this article we have not presented the field equations for the RCW geometries, which are essentially related to the solution of a Dirichlet problem for the conformal invariant wave equation in the canonical Hilbert space determined by the volume form. Thus, the quantization of gravitation envisaged in this program appears to be related to the usual quantization scheme through the heat kernel expansion through *Riemannian invariants*; (Fulling, Birell and Davies).

Yet, what matters centrally in this quantization, and has been unnoticed by all authors, is that from the field equations for the RCW geometry, the quantum potential is found to be non other than $\frac{1}{12}$ the metric scalar curvature!, which has assimilated the dependence of the quantum system on the square root of the invariant density ψ^2 . In fact, one proves that the explicit dependence of the quantum system on the RCW geometry shows up while working in the Hilbert space $L^2(\psi^2 vol_g)$, where vol_g is the canonical volume density associated to the

metric g , and $\psi^2 vol_g$ is the invariant density of the quantum diffusion. Yet, by conformal transformation to the canonical Hilbert space $L^2(vol_g)$, the heat kernel representation of the diffusion in the RCW Hilbert space, goes to the heat kernel representation for the conformal invariant wave operator on the canonical Hilbert space; (Rapoport, 1995 c). Thus, in the $L^2(\psi^2 vol_g)$ Hilbert space the role of torsion is essential, while in the Hilbert space $L^2(vol_g)$, the role of torsion is lost due to the identity of the quantum potential with $\frac{1}{12}R(g)$, $R(g)$ the metric scalar curvature. This settles the question as to the Riemannian or Cartanian character of the geometry of quantum mechanics and gravitation. It seems that Anandan (Anandan, 1988) was the first author to point out that London description of quantum mechanics is related to the Weyl geometry, yet that this does not rule out a possible need of incorporating torsion into quantum mechanics.

As closing remarks, we would like to point out that the present theory leads to the formulation of the ergodicity studies of the diffusion processes generated by the RCW geometries. The striking fact that allows for such a formulation, is that the flows of the diffusion processes generated by the RCW geometries, are *diffeomorphisms* of space-time, in spite that they arise from a non-differentiable dynamics. As a resultant of this, the evolution of densities governed by a quantum Perron-Frobenius semigroup, leads to the fact that the tensor product of the Wiener measure with the Born invariant measure of the diffusions, $\psi^2 vol_g$, yields an equilibrium measure for the RCW diffusions; (Rapoport, 1995 c).

REFERENCES

- [1] A Markovian semigroup in a Hilbert space H is a family of bounded positive linear operators $\{P_\tau, \tau \geq 0\}$ with dense domain contained in H , such that $P_0 = Id$ verifying the following properties: i)(semigroup property) $P_\tau \circ P_{\tau'} = P_{\tau+\tau'}$, $\tau, \tau' \geq 0$, ii)(contraction property) $\|P_\tau\| \leq 1$, $\tau \geq 0$, iii) $\tau \rightarrow P_\tau$ is strongly continuous.
- Anandan, J. (1988), Non-local aspects of quantum phases, *Annales of Inst. Henri Poincaré, Phys. Theor.*, **49**, No.3, 271.
- Atiyah, M..(1985) Circular Symmetry and Stationary Phase Approximation, *Asterisque* **2**,43-60.
- Bohm, D.. (1952) *Phys. Rev.* **85**,166; Bohm,D. and Vigier, J.P. (1953) *Phys. Rev.* **96**, 208.

- Bowick, M. and Rajeev, S. (1987). The Complex Geometry of String Theory and Loop Space, in *Proceedings, Current Problems in Particle Theory*, G. Domokos et al (eds.), World Scientific, Singapore.
- Cartan, E. and Einstein, A. (1979). *Lettres sur le parallélisme absolu. The Einstein-Cartan correspondance*, Debever, R. (ed.), Princeton University Press, Princeton, N.J., and Royal Academy of Sciences of Belgium, Brussels.
- Einstein, A. and Kauffman. (1955) *Annals Maths*, **56**, (1955); Obukhov, Yu (1982) *Phys. Letts.* **90 A**, 13.
- G. de Rham (1984). *Differentiable Manifolds*, Springer Verlag, Berlin.
- Fulling, S.A. (1989), *Aspects of Quantum Field Theory in Curved Space-Time*, Cambridge Univ. Press; Birell, N.D. and Davies, P.C. (1982). *Quantum Field Theory in Curved Space*, Cambridge Univ. Press.
- Hehl, F. et al. (1976). *Review in Modern Physics*, **48**, 3.
- Graham, R. (1977) *Z. Physik B*, **26**, 281, and references therein.
- Holland, P. (1993) *The Quantum Theory of Motion*, Cambridge Univ. Press, Cambridge, U.K.
- Ikeda, N. and Watanabe, S. (1981) *Stochastic Differential Equations and Diffusion Processes*, North Holland/Kodansha. Amsterdam and Tokyo.
- Kleinert, H. (1989), *The Gauge Theory of Defects, I and II*, World Scientific Publishers, Singapore.
- Kleinert, H. (1991) *Path integrals in Quantum Mechanics, Statistics and Polymer Physics*, World Scientific, Singapore.
- Rapoport, D. and Sternberg, S. (1984), *Lett. N. Cimento* **80 A**, 371.
- Rapoport, D. and Sternberg, S. (1984). On the interactions of Spin with Torsion, *Annals of Phys.* **158**, 447.
- Rapoport, D. (1991) Stochastic processes in conformal Riemann-Cartan- Weyl gravitation, *Int. J. Theor. Physics* **30**, 11, 1497.
- Rapoport, D., Rodrigues, W., de Souza, Q. and Vaz, J. (1994) The Einstein-Cartan effective geometry associated to a Dirac-Hestenes spinor field, *Algebras, Groups and Geometries*, vol **14**, 27.
- Rapoport, D. (1995 a) The Cartan structure of Quantum and Classical Gravitation, *Gravitation. The space-time Structure, Proceedings*, P. Letelier and W. Rodrigues (eds.), Word Scientific, Singapore.
- Rapoport, D. (1995 b). The Geometry of Quantum Fluctuations, the Quantum Lyapunov exponents and the Perron-Frobenius Stochastic Semigroups; in *Proceedings, International Conference on Dynamical Systems and Chaos, Tokyo, May*

1994, K.Shiraiwa et al (edts.),vol. 2, World Scientific, Singapore.

Rapoport,D.. (1995 c). *Cartan Geometries of Gravitation, Witten's Laplacian and Ergodic Structures*, SMR 847/13, ICTP; Lecture Notes, Conference on Topological and Geometrical Problems Related to Quantum Field Theory, ICTP, Trieste, March 1995.

Rapoport,D.. (1995 d). To be submitted.

Sciama, D.W. (1962). *Review of Modern Physics*, 36, 463.

Scherk, J. and Schwarz, J., *Phys. Letts.* 528, (1974),347.

E. Witten.(1982) Supersymmetry and Morse theory, *J. Diff. Geom.* 17, 661-692.

E.Witten. (1994), Monopoles and four-manifolds, *Math. Res. Lett.* 1 (1994), 769.

Received in August, 1995.

SERIE I - TRABAJOS DE MATEMATICA

PUBLICACIONES PREVIAS.

Director de Publicaciones: **Alberto P. Calderón**

- N°244 *"Infinite dimensional homogeneous reductive spaces and finite index conditional expectations"*, E.Andruchow, A.Larotonda, L.Recht, D.Stojanoff (1995).
- N°245 *"The convolution products of $K^j \{(m^2 + P \pm i0)^{\lambda+j}\} * \delta^{(k-1)}(m^2 + P)$ and others"*, M. Aguirre Téllez (1995).
- N°246 *"On Heinz inequality"*, E. Andruchow, G. Corach, D. Stojanoff (1995).
- N°247 *"Pseudoquaternions and unitary null curves"*, G. S. Birman (1995).
- N°248 *"On the inversion of causal Riesz potentials"*, R. A. Cerutti (1995).
- N°249 *"Some properties of the causal Riesz potentials"*, R. A. Cerutti (1995).
- N°250 *"Some properties of the generalized causal and anticausal Riesz potentials"*, R. A. Cerutti, S. E. Trione (1995).
- N°251 *"Hyperbolic geometry and multifractal spectra. Part I."*, M. Piacquadio Losada (1995).
- N°252 *"Hyperbolic geometry and multifractal spectra. Part II."*, S. Grynberg, M. Piacquadio Losada (1995).
- N°253 *"The expansion of $\delta^{(k-1)}(m^2 + P)$ "*, M. Aguirre Téllez (1995).
- N°254 *"On the inversion of Marcel Riesz ultrahyperbolic causal operator"*, R. A. Cerutti, S. E. Trione (1995).
- N°255 *"Quantum geometry and topological quantum field theory"*, D. L. Rapoport (1995).
- N°256 *"The Riemann-Cartan-Weyl quantum geometry and supersymmetric systems"*, D. L. Rapoport (1995).

Toda correspondencia relativa a publicaciones debe dirigirse a:

Instituto Argentino de Matemática
 Viamonte 1636 - 1er. Cuerpo - 1er. Piso
 1055 - Buenos Aires
 Argentina.

