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Quantum geometry and topological quantum field theory.

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QUANTUM GEOMETRY AND TOPOLOGICAL QUANTUM FIELD THEORY

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ABSTRACT - We present a geometrization of Relativistic Quantum Mechanics (R.Q.M.) and a quantization of gravitation, in terms of the Riemann-Cartan-Weyl (RCW) geometries with Weyl-torsion and their associated diffusion processes. We extend these diffusions of scalar fields to differential forms, and relate the RCW Laplacian with Witten's deformed Laplacian in the topological quantum field theories. We prove that Bohm's relativistic quantum potential is $\frac{1}{12}$ th- of $R(g)$, the metric scalar curvature. We introduce the two Hilbert spaces of quantum gravitation and relate the usual heat kernel for the conformal invariant wave equation to the kernel of the RCW diffusion for spin 0 fields. We relate our theory with Witten's formulation of supersymmetric quantum mechanics. We discuss the relation between the RCW geometries and the symplectic structure of loop space.

I. Introduction.

Since B. de Witt's first proposal of quantization of the theory of gravitation by the path-integral representation of the heat kernel [9], a considerable progress has been achieved in relating this quantization to diffusion processes [16]. This program has lead to the Feynman path integral representation of the transition density in terms of a classical lagrangian [14,16].

Yet, in this program the geometry associated to the diffusion processes and the gravitational field, is considered to be Riemannian. The purpose of this article is to present a solution to the problem of establishing a link between relativistic quantum mechanics and gravitation, by revealing that this geometry is

non-Riemannian and given by a RCW (Riemann-Cartan-Weyl) geometry with torsion -in addition to a Riemannian metric- given by the logarithmic differential of a scalar field [3]. Yet, the introduction of the torsion is essential to the theory, since it introduces the mean velocity (an observable) of the generalized Brownian processes which have continuous **non-differentiable** trajectories. The object in terms of which we shall present the theory is the Laplacian operator of the RCW structure, and its central role stems from the fact that it contains **both** the 1/2-order approximation proper of Brownian motions (described by the Laplace-Beltrami operator of the Riemannian metric), and the linear approximation introduced by the conjugate vector field to the trace-torsion 1-form given by the Weyl exact 1-form, d ln ψ ; as it is well know, the first two moments of a Markov probability measure described here by grad lnψ and the Riemannian metric g, determine all higher moments of the Markov measure, and thus all the probabilistic features are completely determined by the metric and the Weyl torsion one-form.

II. Diffusion Processes and Cartan Geometries

II.A. Cartan Laplacians and Markovian semigroups

In 1931 Schrödinger introduced in non-relativistic Q.M. the notion of trajectories described by the solutions of stochastic differential equations (s.d.eqts.) associated to non-Markovian stochastic processes [1]. Such a probabilistic approach leads to the consideration of **ensembles** of trajectories and thus to the problem of determining measures on trajectories, which is the genesis of Feynman path integrals, or more properly, its Euclidean version with measures on Brownian trajectories.

In this article, we shall deal instead with Markovian diffusion processes which we shall construct in terms of certain Cartan connections on a smooth space-time manifold. The central notion in the construction of Markovian diffusion processes is the infinitesimal generator of the Markov semigroup, or what is the same, the differential generator which defines the stochastic derivative. This is a second order elliptic operator which we shall introduce in this section.

Let us consider for a start, a smooth n-dimensional manifold M , on which we shall consider a second-order smooth differential operator L. On a local coordinate system, $(x^\alpha), \alpha = 1, \ldots, n$, L is written as

$$L = \frac{1}{2} g_{\alpha\beta}(x)\partial_\alpha\partial_\beta + B^\alpha(x)\partial_\alpha + c(x). \qquad (1).$$

From now on, we shall fix this coordinate system, and all local expressions shall be written in it.

We wish to give an invariant description of L, i.e. a description independent of the local coordinate system. This is an essential prerequisite of covariance.

For this, we shall introduce an arbitrary connection on M, whose covariant derivative we shall denote as ∇. We remark here that ∇ need **not** be the Levi-Civita connection associated to g; we shall precise this below. Let $\sigma(\nabla)$ denote the second-order part of L, and let us denote by $X_0(\nabla)$ the vector field on M given by the first-order part of L. Finally, the zero-th order part of L is given by $L(1)$, where 1 denotes the constant function on M equal to 1.

Then, for $f : M \to R$ of class C^2, we have

$$\sigma(\nabla)(x) = \frac{1}{2} trace(\nabla^2 f)(x) = \frac{1}{2}(\nabla df)(x)), \qquad (5)$$

where the trace is taken in terms of g, and ∇df is thought as a section of $L(T^*M, T^*M)$. Also, $X_0(\nabla) = L - L(1) - \sigma(\nabla)$. If $\Gamma^\alpha_{\beta\gamma}$ is the local representation for the Christoffel symbols of the connection, then the local representation of $\sigma(\nabla)$ is:

$$\sigma(\nabla)(x) = \frac{1}{2} g_{\alpha\beta}(x)(\partial_\alpha\partial_\beta + \Gamma^\gamma_{\alpha\beta}(x)\partial_\gamma), \qquad (5)$$

and,

$$X_0(\nabla)(x) = B^\alpha(x)\partial_\alpha - \frac{1}{2} g_{\alpha\beta}(x)\Gamma^\gamma_{\alpha\beta}\partial_\gamma. \qquad (6)$$

If ∇ is the Levi-Civita connection associated to g, which we shall denote as ∇^g, then for any $f : M \to R$ of class C^2:

$$\sigma(\nabla^g)(df) = \frac{1}{2} tr((\nabla^g)^2 f) = \frac{1}{2} tr(\nabla^g df) = -\frac{1}{2} \text{div grad } f = \frac{1}{2}\Delta_g f.$$

Here, Δ_g is the Levi-Civita laplacian operator on functions; locally,it is written as

$$\Delta_g = (\det g)^{\frac{-1}{2}} \partial_\alpha((\det g)^{\frac{1}{2}} g^{\alpha\beta}\partial_\beta).$$

We now take ∇ to be a Riemann-Cartan connection [5], which we additionally assume to be compatible with g, i.e. $\nabla g = 0$. Then $\sigma(\nabla) = \frac{1}{2}tr(\nabla^2)$. Let us compute this. Denote the Christoffel coefficients of ∇ as $\Gamma^{\alpha}_{\beta\gamma}$; then,

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + 1/2K^{\alpha}_{\beta\gamma}, \tag{7}$$

where the first term in (7) stands for the Christoffel Levi-Civita coefficients of the metric g, and

$$K^{\alpha}_{\beta\gamma} = T^{\alpha}_{\beta\gamma} + S^{\alpha}_{\beta\gamma} + S^{\alpha}_{\gamma\beta},$$

is the cotorsion tensor, with $S^{\alpha}_{\beta\gamma} = g^{\alpha\nu}g_{\beta\kappa}T^{\kappa}_{\nu\gamma}$, and $T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$ the skew-symmetric torsion tensor Let us consider the Laplacian operator associated to this Cartan connection [3], defined -in extending the usual definition- by

$$H(\Gamma) = 1/2g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}, \tag{8}$$

where ∇ stands for the covariant derivative operator with respect to Γ; then, $\sigma(\nabla) = H(\Gamma)$. A straightforward computation shows that that $H(\Gamma)$ only depends in the trace of the torsion tensor and g:

$$H(\Gamma) = 1/2\Delta_g + g^{\alpha\beta}Q_{\beta}\partial_{\alpha}, \tag{9}$$

with $Q = T^{\nu}_{\nu\beta}dx^{\beta}$, the trace-torsion one-form.

Therefore, for the Riemann-Cartan connection ∇ defined in (7), we have that

$$\sigma(\nabla) = \frac{1}{2}tr(\nabla^2) = \frac{1}{2}\Delta_g + \hat{Q}, \tag{10}$$

with $\hat{Q}$ the vector-field conjugate to the 1-form Q: $\hat{Q}(f) =< Q, \text{grad } f >)$, $f : M \to R$. We further have:

$$X_0(\nabla) = B - \frac{1}{2}g^{\alpha\beta}\left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}\partial_{\gamma} - \hat{Q}, \tag{11}$$

Therefore, the invariant decomposition of L is

$$\frac{1}{2}tr(\nabla) + X_0(\nabla) + L(1) = \frac{1}{2}\Delta_g + b + L(1). \tag{12}$$

with

$$b = B - \frac{1}{2}g^{\alpha\beta}\left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}\partial_{\gamma}.$$

Notice that (11) can be thought as arising from a gauge transformation: $\tilde{b} \rightarrow \tilde{b} - Q$, with $\tilde{b}$ the 1-form conjugate to b: $\tilde{b}(Y) = < b, Y >$, for any vector field Y on M.

If we take for a start ∇ with Christoffel symbols of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)} \left\{ \delta^{\alpha}_{\beta} \, Q_{\gamma} - g_{\beta\gamma} \, Q^{\alpha} \right\} \tag{13}$$

with

$$Q = \tilde{b}, \quad \text{i.e.} \hat{Q} = b,$$

we have

$$X_0(\nabla) = 0$$

and

$$\sigma(\nabla) = \frac{1}{2} tr(\nabla^2) = H(\nabla) = \frac{1}{2} tr((\nabla^g)^2) + \hat{Q} = \frac{1}{2}\Delta_g + b.$$

Therefore,

$$L = \sigma(\nabla) + L(1) = \frac{1}{2} tr((\nabla^g)^2) + \hat{Q} + L(1). \tag{12'}$$

The restriction we have placed in ∇ to be as in (13), i.e. only the trace component of the irreducible decomposition of the torsion tensor is taken, is due to the fact that all other components of this tensor do not appear at all in the laplacian of (the otherwise too general) ∇. In the particular case of dimension 2, this is automatically satisfied. In the case we actually have assumed that g is Riemannian, the expression (12′) is the most general invariant laplacian acting on functions defined on a smooth manifold. This restriction, will allow us to establish a one-to-one correspondance between Riemann-Cartan connections of the form (12) with Markovian diffusion processes. These metric compatible connections we shall call RCW geometries (short for Riemann-Cartan-Weyl), since the trace-torsion is a Weyl 1-form [3]. Thus, these geometries do not have the historicity problem which lead to Einstein's rejection of the first gauge theory ever proposed by Weyl.

We shall further assume in the following that Q reduces to the exact form: $Q : Q = \mathrm{d} \ln \psi$, where ψ is a real function on M. In this case, the RCW geometry is determined by the Riemannian metric g and the function ψ. The corresponding laplacian, which we shall write from now on as $H(g, \psi)$ is defined by its formal action on functions $f : M \rightarrow R$ by

$$H(g, \psi) f = \frac{1}{2}\Delta_g f + < \text{grad} \ln\psi, \text{grad} f > . \tag{14}$$

The theory we shall construct is determined by this laplacian, which we shall call the RCW laplacian.

We are interested in Markovian semigroups $\{P_\tau, \tau \geq 0\}$ with infinitesimal generator given by $H(g,\psi)$:

$$H(g,\psi)f = \text{str } \lim_{\tau \to 0} \frac{P_\tau f - f}{\tau}$$

for f in the domain of $H(g,\psi)$; here, the limit is taken in the strong (operator) sense [36]. We further assume that $\{P_\tau, \tau \geq 0\}$ preserves probability, i.e. $P_\tau(1) = 1$, for any $\tau \geq 0$; consequently, $L(1) = 0$, i.e. the zero-order ("potential") term of the operator L is identically zero.

The role of $b = \text{grad } \ln\psi$ the vector field conjugate to the trace-torsion 1-form, is that of the drift (average velocity) of the continuous sample curves of the diffusion processes associated to $H(g,\psi)$. Therefore, the introduction of the torsion is a most essential feature of the diffusion processes associated to $\{P_\tau, \tau \geq 0\}$, since Brownian processes have continuous non-differentiable sample paths. (Actually, they are fractals). We shall see further below that $H(g,\psi)$ is a negative symmetric operator on a Hilbert space which has $C_0^\infty(M)$ as a dense subspace of definition of $\{P_\tau, \tau \geq 0\}$ which can formally be written as $exp(\tau H(g,\psi))$, $\tau \geq 0$.

We must remark that τ is not to be confused with the relativistic time coordinate of M; it is to be thought as an internal time evolution parameter of the diffusion we shall describe below, as originally conceived by B. de Witt [9]. This time parameter is Liouville's time in Prigogine's theory of non-equilibrium statistical mechanics [29].

The transition density $p_\psi(\tau, x, y)$ is determined as the fundamental solution of the "heat" equation on the first variable x:

$$\frac{\partial u}{\partial \tau} = H(g,\psi)(x)u \ . \tag{15}$$

It will be very important for the following, to note that the semigroup $\{P_\tau : \tau \geq 0\}$ has a unique τ- independent **invariant** probability density ρ determined as the fundamental weak solution (in the sense of the theory of generalized functions) of the τ-independent Fokker-Planck-Kolmogorov equation: $H(g,\psi)^\dagger(\rho) = 0$, where $H(g,\psi)^\dagger = 1/2\Delta_g - \text{div}_g$, is the adjoint of $H(g,\psi)$. One readily proves that $\rho = \psi^2 \text{vol}_g$; this relativistic **Born** density can also be proved to be a relaxation density for the Markov process, since one can prove that $p_\psi(\tau, x, y)$ tends exponentially in τ, with x fixed on a compact set, to $\psi^2(y)$.

II.B Riemann-Cartan-Weyl Diffusions

By embedding M on R^d, with $d \leq 2n+1$, we can obtain a section Y (at least locally Lipschitz, or still, satisfying Sobolev regularity conditions) of $L(R^d, TM)$, so that if Y^* denotes the dual section of $L(TM, R^d)$, then for all $x \in M$,

$$\sigma(x) = Y(x)Y^*(x).$$

Given an orthonormal basis $\{e_i, i = 1, \ldots, n\}$ of R^d, we may define vector fields

$$Y_i(x) = Y(x)(e_i).$$

Taking Y to be smooth, we can define the second-order differential operator

$$L_Y^2 = (Y_i)^2.$$

For L as in (1) and Y locally of the form

$$Y_i(x) = Y_\beta^\alpha(x)\partial_\alpha,$$

then, locally

$$L_Y^2(x) = g_{\alpha\beta}\partial_\alpha\partial_\beta + Y_\alpha^\beta(x)\partial_\beta Y_\alpha^\gamma(x)\partial_\gamma.$$

If we take the vector field on M given by

$$X_0^Y(x) = b(x) - \frac{1}{2}Y_\alpha^\beta\partial_\beta Y_\alpha^\gamma(x)\partial_\gamma$$

with $b = \hat{Q}$, then

$$L = \frac{1}{2}L_Y^2 + X_0^Y.$$

This decomposition, while still invariant by diffeomorphisms of M, it depends essentially on the choice of the "square root" Y of σ.

For an arbitrary Riemann-Cartan connection ∇ as in (7), we consider its associated Levi-Civita connection ∇^g; with the choice of Y sufficiently regular, we have the following decomposition of L:

$$X_0^Y = X_0(\nabla) - S(\nabla^g, Y),$$

and

$$\frac{1}{2}L_Y^2 = \sigma(\nabla) + S(\nabla^g, Y)$$

their diffusion processes [23] and RCW laplacian operators [3]. The former correspondances and their relation with the Dirichlet problem are the cornerstone for the study of analytic estimates of heat kernels such as hypercontractivity and ultracontractivity, which are important notions in constructive quantum field theory [38]. We shall present below in a rather schematic way the above mentioned correspondances.

We can associate with the diffusion process a Hamiltonean operator on the Hilbert space $L^2(\psi^2\text{vol}_g)$ [3]. With abuse of notation, let us denote still as $H(g,\psi)$ the Friedrichs self-adjoint extension [38] of the infinitesimal generator (14) with domain given by $C_0^\infty(M)$. We can now define the inner product

$$< f_1, f_2 >^\rho = 1/2 \int g(\text{grad} f_1, \text{grad} f_2)\psi^2\text{vol}_g$$

By integration by parts, we obtain

$$< f_1, f_2 >^\rho = -(f_1, H(g,\psi)f_2)_\rho \tag{17}$$

where $(.,.)_\rho$ denotes the weighted inner product in terms of $\rho = \psi^2\text{vol}_g$ which thus defines a Hilbert space which we denote as $L^2(\psi^2 vol_g)$. Let us consider now the closed quadratic form, (the *Dirichlet form*) q associated to $< .,. >$, i.e. $q(f) =< f,f >$ [3,23]. We see from eqt. (19) that there is a unique Hamiltonean operator which generates q, it is the self-adjoint operator $-H(g,\psi)$. Since the quadratic form is positive, $q(f) \geq 0$, for any $f \in L^2(\psi^2 vol_g)$, then $H(g,\psi)$ is a negative self-adjoint operator on $L^2(\psi^2 vol_g)$ and the semigroup $exp(\tau H(g,\psi))$ is defined.

Let us see how this construction is related to the usual formulation of Quantum Mechanics in terms of quadratic forms in $L^2(\text{vol}_g)$, which in the non- relativistic flat case has been elaborated by several authors [22,38].

Consider the mapping $C_\psi : L^2(\psi^2\text{vol}_g) \to L^2(vol_g)$ defined by multiplication by ψ. This maps takes $C_0^\infty(M)$ into itself. For any f in $C_0^\infty(M)$ we have

$$\begin{aligned}
q(\psi^{-1}f) &=< \psi^{-1}f, \psi^{-1}f > \\
&= 1/2 \int \{g(\text{grad } f, \text{grad } f) - 2g(\text{grad } f, \text{grad } ln\ \psi)f + g(b,b)f^2\}\text{vol}_g \\
&= 1/2 \int \{g(\text{grad } f, \text{grad } f) + (\text{div}_g b)f^2 + g(b,b)f^2\}\text{vol}_g \\
&= \int f\{-\frac{1}{2}\Delta_g + V\}f\text{vol}_g = (f, Hf)_{L^2(vol_g)},
\end{aligned}$$

with $b = \text{grad } \ln\psi$ and

$$H = C_\psi \circ H(g,\psi) \circ C_\psi^{-1} = -1/2\Delta_g + V$$

where in the weak sense,

$$V = 1/2(\text{div}_g\ b + g(b,b)) = \frac{\Delta_g \psi}{2\psi}, \tag{17}$$

is the **relativistic quantum potential**. In the case of $n = 3$ and g the Euclidean metric, we retrieve Bohm's potential in non-relativistic Quantum Mechanics [19].

Then, we have proved that $-H(g,\psi)$ is unitarily equivalent to the Hamiltonean operator $H = -1/2\Delta_g + V$ defined on $L^2(\text{vol}_g)$ and ψ is a generalized groundstate eigenfunction of H with 0 eigenvalue. The non-linear dependence of V on the invariant density introduced by ψ introduces non-local (in the sense of Einstein- Podolsky-Rosen) correlations on the quantum system; we recall that the existance of instantaneous quantum correlations have been verified experimentally by Aspect [39]. We shall see below that this dependence of V on ψ is removed due to conformal invariance.

III. The Mean Curvature Extremal Principle

We shall assume that $n = 4$. We start with a general Riemann-Cartan connection (Γ_α^{ab}), (where Greek letters denote space-time indices as until now, and Latin letters denote anholonomic indices), and we introduce its scalar curvature

$$R(\Gamma) = e_a^\alpha e_b^\beta R_{\alpha\beta}^{\cdot\cdot ab}, \tag{18}$$

where the e_a^α is a field of invertible tetrads with $g_{\alpha\beta} = \eta_{ab} e_\alpha^a e_\beta^b$, with η_{ab} the Euclidean metric, and $R_{\alpha\beta}^{\cdot\cdot ab}$ is the curvature tensor of (Γ_α^{ab}) [5]. Definining the following **generalized** conformal (Einstein) Λ transformations [6,4]: i): (Γ_α^{ab}) is invariant, and ii): e_a^α is transformed into $\psi^{-1} e_a^\alpha$; from them we get the usual Weyl transformation on the metric: $g_{\alpha\beta}$ is transformed into $\psi^2 g_{\alpha\beta}$ while $R(\Gamma)$ is transformed into $\psi^{-2} R(\Gamma)$. Since, by above, the scalar fields ϕ transform as $\psi^{-1}\phi$, we get that the functional

$$A(\Gamma,\phi,g) = \int R(\Gamma)\varphi^2 \text{vol}_g, \tag{19}$$

is conformal invariant. Notice that if ϕ generates a drift vector field $b = \text{grad}\ ln\phi$, so that $\phi^2 \text{vol}_g$ is the unique invariant density of the corresponding diffusion process, then (19) is the **mean Riemann-Cartan scalar curvature**. This theory contains the Einstein-Cartan theory which is obtained by fixing the ϕ to be positive constant [5], and consequently it contains the classical Einstein's theory, retrieved for symmetric Γ. Taking variations with respect to g we obtain that

$$R_{\alpha\beta}(\Gamma) - 1/2 g_{\alpha\beta} R(\Gamma) = 0, \tag{20}$$

i.e. the Einstein-Cartan equations for Γ in the vacuum, while by taking variations with respect to $T^{\gamma}_{\alpha\beta}$, we obtain that

$$T^{\gamma}_{\alpha\beta} = \delta^{\alpha}_{\beta}\ \partial\gamma\ ln\phi - \ \delta^{\alpha}_{\gamma}\partial_{\beta}\ ln\phi, \tag{21}$$

so that, up to normalization, $Q = d\ ln\phi$. Taking variations with respect to ϕ we get the teleparallelism: $R(\Gamma) = 0$; replacing (21) in (20) we get the field equations

$$G_{\alpha\beta}(g) = -\frac{6}{\phi^2} T_{\alpha\beta}, \tag{22}$$

with $G_{\alpha\beta}(g)$ the Einstein **metric** tensor, and

$$T_{\alpha\beta} = \partial_{\alpha}\phi\ \partial_{\beta}\phi - 1/2 g_{\alpha\beta}\partial_{\gamma}\phi\partial^{\gamma}\phi - \frac{1}{6}(\nabla_{\alpha}\nabla_{\beta}\phi^2 - g_{\alpha\beta}\Delta_g\phi^2), \tag{26}$$

minus the improved energy-momentum density of the renormalizable gauge theories [20]. Now, by taking the trace in (21) we finally get

$$(\Delta_g - \frac{1}{6}R(g))\phi = 0, \tag{23}$$

so that ϕ is a generalized groundstate of the conformal invariant wave operator defined on $L^2(\text{vol}_g)$. Note that from (23) we conclude that the quantum potential is $\frac{1}{12}R(g)$ which **does not** depend on the scalar field ϕ at all. Therefore, the correlations on the quantum system are mediated by the metric scalar curvature!

Solving the conformal invariant wave equation with Dirichlet boundary conditions we obtain a conformally conjugate Dirichlet form whose associated Hamiltonean operator is $-H(g,\psi)$, and thus the Markovian semigroup determined by it can be reconstructed. Let us show this in detail.

Suppose that there exists a positive compact supported C^2 function ψ on M such that

$$H\psi := (\Delta_g - \tilde{V})\psi = 0,$$

where $\tilde{V}$ denotes the operator of multiplication by $KR(g)$, $K = \frac{1}{6}$. Put $X = \frac{\Delta_g \psi}{\psi}$; then, X is a continuous function. Consider the quadratic form on $C_0^1(M)$, the space of all compact supported C^1 functions on M, given by

$$Q(f) = \frac{1}{2}\int (|df|^2 + \tilde{V} f^2) vol_g ; f \in C_0^1(M).$$

Put $h = \psi^{-1} f$, then

$$Q(f) = \frac{1}{2}\int (|d(\psi h)|^2 + \tilde{V} h^2 \psi^2) vol_g$$

$$= \frac{1}{2}\int (|dh|^2 \psi^2 + h^2 |d\psi|^2 + 2 < dh, d\psi > h\psi + \tilde{V} h^2 \psi^2) vol_g$$

$$= \frac{1}{2}\int (< d\psi, d(\psi h^2) > + |dh|^2 \psi^2 + \tilde{V} h^2 \psi^2) vol_g$$

which, by integration by parts of the first term, gives

$$= \frac{1}{2}\int (|dh|^2 + (\tilde{V} - X) h^2) \psi^2 vol_g$$

$$= \frac{1}{2}\int |dh|^2 \psi^2 vol_g \geq 0.$$

We have proved that Q is positive. Then, the form $Q_\psi(f) := Q(C_\psi f) = Q(\psi f)$, defined on $f \in L^2(\psi^2 vol_g)$, equals to

$$\frac{1}{2}\int |dh|^2 \psi^2 vol_g$$

is positive, and its Hamiltonean operator, $-H(g, \psi)$, is negative self-adjoint on $L^2(\psi^2 vol_g)$; consequently, $exp(\tau H(g, \psi))$ is a Markovian semigroup on $L^2(\psi^2 vol_g)$, or in its dense domain $C_0^\infty(M)$.

Therefore, starting from a positive compact supported solution of the equation (23), we have constructed the Markovian semigroup generated by the RCW Laplacian $H(g, \psi)$. Furthermore, one can reconstruct the Markovian semigroup in $L^2(vol_g)$ with generator $-\frac{1}{2}H$.

We shall finally establish the relation between the heat kernel $p_{\rm conf}(\tau, x, y)$ of the Markovian semigroup $exp(\frac{\tau}{2}H)$ and the heat kernel $p_{\psi}(\tau, x, y)$ of the RCW semigroup. We have

$$exp(\frac{\tau}{2}H(g,\psi))f(x) = \psi^{-1}(x)exp(\frac{\tau}{2}H)(\psi f)(x)$$
$$= \int \psi^{-1}(x)p_{\rm conf}(\tau, x, y)\psi(y)f(y)vol_g(y)$$

so that we conclude that

$$p_{\psi}(\tau, x, y) = \psi^{-1}(x)\psi(y)p_{\rm conf}(\tau, x, y)$$

Thus, we have linked the quantization in the two Hilbert spaces, the ground-state Hilbert space $L^2(\psi^2 vol_g)$, and $L^2(vol_g)$. The former corresponds to the RCW geometry, while the latter is the usual Hilbert space for the quantization of the scalar field ψ in terms of the Riemannian invariants of the manifold M described in terms of g. We remark that the introduction of both spaces and the unitary transformation between them, has allowed us to identify the quantum potential, while working only in the usual Hilbert space would not have allowed for this identification. Thus, in the $L^2(vol_g)$ space we have found the Hamiltonean operator obtained by B.de Witt in his pioneering work [9], and reencountered by several researchers in quantum field theory in Riemannian geometries through the HamiDew expansion of $p_{\rm conf}(\tau, x, x)$ [37]. Yet, our result is in disagreement with the path integral representation of the classical action in a Riemann-Cartan geometry due to Kleinert, in which he obtains *twice* the quantum potential (see, chap. X, [34]).

IV. Witten's deformed laplacian and the RCW stochastic flows

Let us assume in the following that we have a smooth n-dimensional orientable compact manifold M provided with a Riemannian metric, g. We consider the Hilbert space of square summable ω of differential forms of degree q on M, with respect to vol_g. We shall denote this space as $L^{2,q}$ or still as $L^2\Omega^q(M, \mathrm{vol}_g)$. The inner product is

$$< \omega, \phi >= \int_M < \omega(x), \phi(x) > \mathrm{vol}_g$$

where the integrand is given by the natural pairing between the components of ω and the conjugate tensor: $g^{\alpha_1\beta_1}\dots g^{\alpha_q\beta_q}\phi_{\beta_1\dots\beta_q}$; alternatively, we can write in a coordinate independent way: $<\omega(x),\phi(x)> vol_g = \omega(x)\wedge *\phi(x)$, with $*$ the Hodge star operator, for any $\omega,\phi \in L^{2,q}$ [45].

The de Rham-Kodaira operator on $L^{2,q}$ is defined as

$$\Delta = -(d+\delta)^2 = -(d\delta + \delta d), \tag{24}$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator d defined on $L^{2,q}$:

$$<\delta\phi,\omega> = <\phi, d\omega>,$$

for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$. In the case of $q=0$, this is the Laplace-Beltrami operator on functions encountered before; in the general case we have in addition of $tr(\nabla^g)^2$ the contribution of the Weitzenbock curvature term. Let us assume that the real valued function on M ψ is smooth and everywhere positive. We then have an induced smoooth density $\rho = \psi^2 \mathrm{vol}_g$ on M.

We introduce the Hilbert space $L^{2,q,\rho} = L^2\Omega^q(M,\rho)$, of differential forms on M of degree q, square integrable with respect to ρ, with inner product:

$$<\phi_1,\phi_2>^\rho = \int_M <\phi_1(x),\phi_2(x)> \rho, \tag{25}$$

for $\phi_1,\phi_2 \in L^{2,q,\rho}$. We define the quadratic form $q(\phi) = \frac{1}{2} <\phi,\phi>^\rho$, with ϕ on the Hilbert space given by the completion of the space of all smooth q-forms under the $L^{2,\rho}$ inner product. In the case of exact one-forms, this is quadratic form introduced in correspondance with the Brownian processes determined by $H(g,\psi)$. The Markovian semigroups we are interested are associated to the closed extension of q with Hamiltoneans (or what is the same, with infinitesimal generators [23]) which will be given by the Friedrichs self-adjoint extensions of the laplacian operators we shall compute below.

Consider the formal adjoint of d, which we shall denote as δ^ψ defined on $L^{2,q+1,\rho}$ as follows

$$<\delta^\psi\omega,\phi>^\rho = <\omega, d\phi>^\rho, \tag{26}$$

for any $\omega \in L^{2,q,\rho}$ and $\phi \in L^{2,q,\rho}$. Since $d^2=0$, we have

$$(\delta^\psi)^2 = 0. \tag{27}$$

For any smooth function f defined on M, and ω a q-form:

$$\delta(f\omega) = f\delta\omega - i_{\text{grad } f}\omega,$$

where i_X is the interior product derivation on q-forms.

We introduce the operator on $L^{2,q,\rho}$:

$$\Delta^{\psi,q} = -(d + \delta^{\psi})^2, \tag{28}$$

which still writes as

$$-(d\delta^{\psi} + \delta^{\psi} d).$$

Recalling the definition of the Lie-derivative operator $L_X = di_X + i_X d$, X a smooth vector field on M, we finally have

$$\Delta^{\psi,q} = \Delta^q + 2L_{\text{grad } \ln\psi}, \tag{29}$$

Let us define now the deformed exterior differential operator mapping q-forms in $q+1$-forms, by:

$$d^{\psi} = \psi d\psi^{-1}, \tag{30}$$

so that

$$d^{\psi}\omega = d\omega - d\ln\psi \wedge \omega.$$

We have that

$$(d^{\psi})^2 = 0. \tag{31}$$

This operator is the ($\tau = -1$ version [46]) of Witten's deformed differential [33]. We introduce now the deformed co-differential operator as the formal adjoint of d^{ψ}:

$$(d^{\psi})^* = \psi^{-1}\delta\psi. \tag{32}$$

We introduce the deformed Laplacian operator, as a particular case of Witten's [33], defined as:

$$L^{\psi,q} = -(d^{\psi} + d^{\psi *})^2, \tag{33}$$

which can still be written as

$$-(d^{\psi} d^{\psi *} + d^{\psi *} d).$$

We have the following relation between the two Laplacian operators:

$$\Delta^{\psi,q} = \psi^{-1} L^{\psi,q} \psi, \tag{34}$$

so that these two operators are conformally equivalent under conjugation by ψ. Note that $\Delta^{\psi,0} = 2H(g,\psi)$. Note that (34) is the extension to forms of arbitrary degree of the relation found for scalars in **II**.

Now consider the semigroups in $L^{2,q,\rho}$ with infinitesimal generators given by the self-adjoint extensions in $C^\infty\Lambda^q(M)$ of $\frac{1}{2}\Delta^{\psi,q} = \frac{1}{2}\Delta^q + L_{\textbf{grad} \ln\psi}$; we shall denote these semigroups as $P^q_\tau, q = 0,\ldots,n$. Clearly, P^0_τ is the stochastic process with infinitesimal generator given by $H(g,\psi)$.

Thus, starting from the RCW geometry determined by the field equations, we can construct a family of stochastic processes on forms of any degree. Remarkably, the Laplacian introduced by Witten in topological quantum field theory, appears to be related to a wave function which satisfies the field equations and produces the torsion of the RCW geometry.

We would like to note finally that from the fact that $(d^\psi)^2 = 0$, we can define a deformed de Rham complex: $H^q_\psi(M,R)$ as $Ker(d^\psi : \Lambda^q \to \Lambda^{q+1})/Ran(d^\psi : \Lambda^{q-1} \to \Lambda^q)$. Yet, since $Ker(d^\psi) = \psi Ker(d)$, and $Ran(d^\psi) = \psi Ran(d)$, we obtain that $H^q_\psi(M,R) \cong H^q(M,R)$, for any $q = 0,\ldots,n$. Now, by Hodge'theorem: $dim H^q(M,R) = dim(Ker(\Delta^q))$, which by the above construction is clearly equal to $dim(Ker(L^{\psi,q}))$; by (33) we conclude that

$$dim(H^q(M,R)) = dim(Ker\Delta^{\psi,q}). \tag{35}$$

This identification, which we shall not use in this article, is fundamental to the formulation of the ergodic studies of this theory; indeed, if the first Betti number of M, $b_1(M) = dim(H^1(M,R)) \neq 0$, then the solution flow of (16) is unstable. More precisely, it can be proved that the flow of solutions of eqt. (16) corresponding to RCW geometries with ψ a positive solution of the field equation, with g a Riemannian metric with coefficients in C^2, are (moment) unstable [25,35].

V. RCW Geometries and Supersymmetric Systems

In this section we shall elaborate briefly the relation between the family of Laplacian operators $\Delta^{q,\psi}$ associated to a RCW structure connection determined by a Riemannian metric g and a positive function ψ satisfying the field equations (23)., and supersymmetric systems.

this description, M can be recovered as the constant loops of Ω. This infinite dimensional setting is the one considered in the topological quantum field theories, and different choices of ψ yield the supersymmetric σ models, the supersymmetric ϕ^4 theory, etc.. [33] Yet, it is too be remarked that the RCW geometries are connected to the symplectic structure on Ω, in a way we shall describe in the following.

Let us fix $\phi \in \Omega$. Then, the tangent space to Ω at ϕ, $T_\phi\Omega$ can be identified with the space of sections of the pull-back vector bundle $\phi^*(TM)$ of TM to S^1 by ϕ. The metric g on M defines a metric $\cdot_\phi$ on $\phi^*(TM)$, and hence we have an inner product on $T_\phi\Omega$: $(s_1, s_2) = \frac{1}{2\pi}\int_{S^1} s_1 \cdot_\phi s_2$. Thus, $T_\phi\Omega$ has a pre-Hilbert structure.

Next, we introduce a general Riemann-Cartan connection, ∇ on M. This induces a connection on $\phi^*(TM)$, and hence a covariant derivative operator ∇_ϕ which acts on sections of $\phi^*(TM)$ by evaluation on the vector field $\frac{d}{d\sigma}$ of S^1. Now we can define a skew-symmetric bilinear form on $T_\phi\Omega$:

$$\omega(\phi) = \frac{1}{4\pi}\int_{S^1}(\nabla_\phi s_1 \cdot_\phi s_2 - \nabla_\phi s_2 \cdot_\phi s_1). \tag{5.1}$$

Varying $\phi \in \Omega$, we obtain a differential 2-form on Ω. As first noted by Atiyah [42], $d\omega$ equals ($\frac{1}{2\pi}$) the integral over S^1 of the skew-symmetric component of the torsion tensor

Therefore, for a RCW geometry, ω is a closed 2-form on Ω. Yet, ω is not properly a symplectic form since it vanishes at those ϕ for which ∇_ϕ has a 0 eigenvalue, i.e. on any tangent vector which is covariantly constant along ϕ. In the case of a purely Riemannian geometry, i.e. $\psi = 1$ this setting has been applied to obtain an exact computation of the trace of the heat kernel of $2\pi\Delta_g$ and to the direct obtention of the Atiyah- Singer index theorem [42]. It would be interesting to check if this constructions can be carried out for the trace of the heat kernel of $H(g,\psi)$.

As a final remark, we point out that the 2-form (5.1) is related to the zero-modes of string theory [43]; it was first observed by Scherk and Schwarz that the zero modes of string theory lead to RC geometries [18].

VII. Conclusions

We have presented a geometrization of spin 0 R.Q.M. together with a quantization of gravitation in terms of the R.C.W. geometries with torsion of the "trivial"

Weyl trace form. Yet, we have shown that this allows an extension to higher spin diffusions and that the family of laplacians on forms of arbitrary degree generated by the RCW geometry yields a supersymmetric system.

Our construction of the two Hilbert spaces has further allowed to identify the relativistic quantum potential with $\frac{1}{12}R(g)$, and thus it is seen that the quantum correlations are mediated by the scalar curvature up to a constant which makes the field equations in the Riemannian Hilbert space to be conformal invariant.

This has important effects on the classical realizations of the spin 0 quantum motions. Indeed, one can think on determining on the smooth curves which with highest probability realize the quantum motions. The solution of this problem is related to the theory of large deviations in probability theory [44], and its solution demands the introduction of the Onsager-Machlup lagrangian on the smooth most probable realizations of the quantum motions. This lagrangian gives the path integral representation of the heat kernel [14,16]. Due to the identity of the quantum potential with $\frac{1}{12}R(g)$ one obtains that the classical realizations of the quantum system are given by a deviation of the geodesic flow due precisely to $b = \text{grad} \ln\psi$ [4]. Yet, this does not conflict with the principle of equivalence since the quantum system is an interacting system and not a system of test-particles. This contrasts with the classical motions of spinless test particles submitted to a RCW geometry, or still a geometry with more general torsion [2]: they follow the geodesic flow of the metric in the RCW geometry uninfluenced by the torsion; only spin $\neq 0$ test particles submitted to a metric plus torsion deviate from the geodesic flow due to the coupling of the spin density to the Cartan curvature.

The characterization of the quantum potential as related to the metric scalar curvature should, perhaps, not surprise us. There are other non-local phenomenae in physics, such as the Aharonov-Bohm effect or still, Berry's phase which are related to (Yang-Mills) curvature [40]. Remarkable still is the fundamental role that the metric scalar curvature plays in the existance of solutions of the monopole equations in four manifolds [41].

Our theory allows for the formulation of the ergodicity properties of the quantum flows generated by the RCW geometries, extending the ergodic theory of classical dynamical systems [17]. The fundamental fact that allows for this extension is that the quantum flows of (16) yield **diffeomorphisms** of space-time, in spite that their sample curves are continuous non-differentiable. A direct consequence of this is that there is a well determined invariant measure for the quantum flows given by the product of the Born measure with the Wiener measure [25,35]. This factorization is such that the Born measure accounts for the self-interactions of the quantum system consistently with the quantum potential description on the Riemannian Hilbert space, while the Wiener measure points to the "free" quantum field measure.

Our theory is geometrically determined; this is -in our point of view- the core idea of Stochastic Differential Geometry: **Geometry determines Probability** [15,3]. It is remarkable that both the theories of Brownian motion and Relativity, that have provided in this theory for the synthesis of Quantum Mechanics and the latter, through the RCW geometries of Stochastic Differential Geometry, were the creation of Einstein, and thus his -separate- conceptions of geometrical determinism and statistical physics for which he strongly advocated, have been linked.

Yet there are historical antecedents of blending of geometry and probabilistic structures. The causal theory of Quantum Mechanics due to de Broglie, Bohm and Vigier [19,30] relies in a Hamilton-Jacobi theory for the ψ-field which is taken to be complex, and thus it is intimately related with Symplectic-Geometry. More recently and contemporarily [29], in the formulation of statistical mechanics far from equilibrium due to Prigogine and the Brussels-Austin groups'theory of irreversibility and chaoticity [18b], we have a Hamiltonean function H on phase-space, and the theory stems from the infinitesimal generator of the corresponding Perron-Frobenius semigroup, which is the Liouville operator $\{H,.\}$, where $\{.,.\}$ denotes the Poisson bracket defined by H. Thus, also this approach stems from Symplectic Geometry [27], albeit in a seemingly trivial way. In Prigogine's theory, probability densities on phase-space satisfy an evolution equation as in eq.(6), with the Liouville operator instead of $H(g,\psi)$, i.e. a substitution of infinitesimal generators and of phase space instead of configuration space.

A similar thesis to the one of the present article, that **quantization is geometry** has been presented for symplectic structures augmented by metrics on phase-space [32]. Our theory instead has stemmed from Cartan geometries, yet quite remarkably, these geometries provide for a more general and natural setting for Symplectic Geometry [2,28] that the usual setting introduced from the flat canonical 1-form of classical mechanics [27]: the canonically one-form of Classical Mechanics is given by the soldering one-from on the bundle of linear frames, from which the torsion is derived by taking the covariant derivative of the soldering one-form.

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