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# Representation properties, Racah sum rule and Biedenharn-Elliott identity for $U_q(osp(1|2))$.

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**Abstract**

It is shown that the universal $R$-matrix in the tensor product of two irreducible representation spaces of the quantum superalgebra. $U_q(osp(1|2))$ can be expressed by Clebsch-Gordan coefficients. Racah sum rule satisfied by $U_q(osp(1|2))$ Racah coefficients and 6-$j$ symbols is derived from the properties of the universal $R$-matrix in the tensor product of three representation spaces. Considering the tensor product of four irreducible representations, it is shown that Biedenharn-Elliott identity holds for $U_q(osp(1|2))$ Racah coefficients and 6-$j$ symbols. A recursion relation for $U_q(osp(1|2))$ 6-$j$ symbols is derived from Biedenharn-Elliott identity.



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# I. Introduction

In three previous papers, Ref.[1], [2], [3], we have studied the properties of irreducible representations of the quantum superalgebra $U_q(osp(1|2))$. It was shown that it is possible to construct Racah-Wigner calculus for this quantum superalgebra, in a completely similar way as in the classical algebra $su(2)$ [4] and the quantum algebra $U_q(su(2))$ [5], [6], [7] cases. In this paper, in order to complete our study of the representations of this simple quantum superalgebra, we shall consider the properties of its universal $R$-matrix in the tensor product of irreducible representations.

The universal $R$-matrix for $U_q(osp(1|2))$ has been derived and its basic properties have been exhibited in Ref.[8]. It was also considered in the construction of vertex model solutions of the graded Yang-Baxter equation in Ref.[9]. In the present paper we study the properties of the universal $R$-matrix in the tensor product of two irreducible $U_q(osp(1|2))$ representations. We give explicit formulas for the matrix representation of the universal $R$-matrix in different bases of the tensor product of representations. In particular we show that matrix elements of the universal $R$-matrix in the reduced basis of the tensor product of two representations can be expressed with $U_q(osp(1|2))$ Clebsch-Gordan coefficients (denoted $sq$-CGc). Analytic expressions for $sq$-CGc of any tensor product of irreducible representations are known [1], [10], [11], therefore the matrix representation of the universal $R$-matrix can be calculated for any tensor product.

In Ref.[2], we have defined and studied properties of Racah coefficients (denoted $sqRc$) and 6-$j$ symbols (denoted $sq6$-$j$) for the quantum superalgebra $U_q(osp(1|2))$. In particular, it was shown that, as in the classical case, $sq6$-$j$ symbols satisfy not only the usual tetrahedral symmetry but present also an additional symmetry of Regge type. Another property satisfied by $sqRc$ and $sq6$-$j$ symbols is the pseudo-orthogonality relation. This relation can be considered as an algebraic relation satisfied by $sqRc$ and $sq6$-$j$ symbols. It is known that in the classical Racah-Wigner calculus for $su(2)$ or $U_q(su(2))$, Racah coefficients and 6-$j$ symbols satisfy, besides orthogonality relations, other algebraic relations namely Racah sum rule and Biedenharn-Elliott identity. These algebraic identities have been extented to the corresponding features of the superalgebra $osp(1|2)$, Ref.[12]. In this paper we extend Racah sum rule and Biedenharn-Elliott identity for $sqRc$ and $sq6$-$j$ symbols. In both cases, the structure of these relations is completely similar to the correponding classical ones, the only difference concerns the phases which are more complicated in the case of $U_q(osp(1|2))$. As in the classical cases, we also derive from the Biedenharn-Elliott identity, a three-term recurrence relation between $sq6$-$j$ symbols.

This paper is organized in the following way: in section II we study the properties of the universal $R$-matrix in the tensor product of irreducible representations of $U_q(osp(1|2))$. In section III, considering the tensor product of three $U_q(osp(1|2))$ representations, we derive Racah sum rule for $sqRc$ and $sq6$-$j$ symbols and we derive a simple algebraic identity following from the sum rule. Finally, in section IV, we prove Biedenharn-Elliott identity for $sq6$-$j$ symbols and the three-term recurrence relation.

# II. Properties of the universal $R$-matrix in the irreducible representations of the quantum superalgebra $U_q(osp(1|2))$

## A. The $U_q(osp(1|2))$ universal $R$-matrix

The quantum superalgebra $U_q(osp(1|2))$ is generated by 4 elements: $\mathbf{1}$, $H$ (even) and $v_\pm$ (odd) with the following (anti)commutation relations

$$[H, v_\pm] = \pm\frac{1}{2}v_\pm, \quad [v_+, v_-]_+ = -\frac{sh\,(\eta H)}{sh\,(2\eta)}\,, \tag{2.1}$$

where the deformation parameter $\eta$ is real and $q = e^{-\frac{\eta}{2}}$ (we choose $\eta > 0$ so that $q < 1$). The quantum superalgebra $U_q(osp(1|2))$ is a Hopf algebra with the following coproduct $\Delta^q$ and antipode $S$

$$\Delta^q(v_\pm) \;=\; v_\pm \otimes q^H + q^{-H} \otimes v_\pm, \tag{2.2}$$

$$\Delta^q(H) \;=\; H \otimes \mathbf{1} + \mathbf{1} \otimes H, \quad \Delta^q(\mathbf{1}) \;=\; \mathbf{1} \otimes \mathbf{1}, \tag{2.3}$$

$$S(H) = -H \qquad S(v_\pm) = -q^{\pm\frac{1}{2}}v_\pm. \tag{2.4}$$

One can consider another coalgebra structure and antipode defined by

$$\Delta'^q(v_\pm) \;=\; v_\pm \otimes q^{-H} + q^H \otimes v_\pm, \tag{2.5}$$

$$S'(H) \;=\; -H \qquad S'(v_\pm) = -q^{\mp\frac{1}{2}}v_\pm. \tag{2.6}$$

which, together with relations (2.1), define another Hopf algebra structure that we denote $U'_q(osp(1|2))$. Both Hopf algebra structures are related by

$$U'_q(osp(1|2)) = U_{q^{-1}}(osp(1|2)). \tag{2.7}$$

It is known [8] that it exists a canonical element $R^q \in U_q(osp(1|2)) \otimes U_q(osp(1|2))$, called the universal $R$-matrix, that defines a similarity relation between $\Delta'^q$ and $\Delta^q$ :

$$R^q\Delta^q = \Delta'^q R^q. \tag{2.8}$$

This universal $R$-matrix is :

$$R^q = q^{4H\otimes H}\sum(-1)^k q^{\frac{k(k-1)}{4}}\frac{(1+q^{-1})^k}{[k]!\gamma^k}(q^H v_+)^k \otimes (q^H v_-)^k, \tag{2.9}$$

and it satisfies the following relations :

$$R^{q13}R^{q23} \;=\; (\Delta \otimes id)R^q \equiv R^{q12,3}, \tag{2.10}$$

$$R^{q13}R^{q12} \;=\; (id \otimes \Delta)R^q \equiv R^{q1,23}, \tag{2.11}$$

where the indices $i,j=1,2,3, \quad i\neq j$, in $R^{qij}$ show the embeding of $R^q=\sum r_n\otimes r'_n$ into the tensor product $U_q(osp(1|2))\otimes U_q(osp(1|2))\otimes U_q(osp(1|2))$. The above relations imply Yang-Baxter equation for the universal $R$-matrix

$$R^{q12}R^{q13}R^{q23}=R^{q23}R^{q13}R^{q12}. \tag{2.12}$$

In the following we will consider the properties of the universal $R$-matrix in the tensor product of irreducible representations of the quantum superalgebra $U_q(osp(1|2))$.

## B. The $U_q(osp(1|2))$ finite dimensional representations

A representation of a quantum superalgebra $U_q(osp(1|2))$ in a finite dimensional graded space $V$ is a homomorphism $T: U_q(osp(1|2)) \rightarrow L(V,V)$ of the associative graded algebra $U_q(osp(1|2))$ into the associative graded algebra $L(V,V)$ of linear operators in $V$, such that

$$[T(H),T(v_\pm)]=\pm\frac{1}{2}T(v_\pm), \quad [T(v_+),T(v_-)]_+=-\frac{sh\,(\eta T(H))}{sh\,(2\eta)}\,. \tag{2.13}$$

Let us recall the main results concerning these representations, Ref.[1]. Any finite dimensional grade star representation of $U_q(osp(1|2))$ is characterized by four parameters : the highest weight $l$ (a non negative integer), the parity $\lambda$ of the highest weight vector in the representation space and the signature parameters $\varphi,\psi=0,1$ of the Hermitean form in the representation space $V$. The parity $\lambda$ and the signature $\varphi$ define the class $\varepsilon=0,1$ of the grade star representation by the relation $\varepsilon=\lambda+\varphi+1 \pmod 2$. The irreducible representation space of highest weight $l$, $V=V^l(\lambda)$ is a graded vector space of dimension $2l+1$ with basis $e^{lq}_m(\lambda)$, where $-l<m<l$, the parameter $\lambda=0,1$ is the parity of the highest weight vector $e^l_l(\lambda)$ and $deg(e^l_m(\lambda))=l-m+\lambda \pmod 2$. The vectors $e^{lq}_m(\lambda)$ are constructed from the vector $e^l_l(\lambda)$ in a standard way so they depend on $q$ via the normalisation factor. The vectors $e^{lq}_m(\lambda)$ are pseudo-orthogonal with respect to an Hermitian form in the representation space, denoted ( , ), and their normalization is determined by the signature parameters $\varphi,\psi$ :

$$(e^{lq}_m(\lambda),e^{lq}_{m'}(\lambda))=(-1)^{\varphi(l-m)+\psi}\delta_{mm'}\,. \tag{2.14}$$

The operators $T(v_\pm)$ and $T(H)$ act on the basis $e^{lq}_m(\lambda)$ in the following way :

$$\begin{aligned} T(H)e^{lq}_m(\lambda) &= \frac{m}{2}\,e^{lq}_m(\lambda)\,, \\ T(v_+)e^{lq}_m(\lambda) &= (-1)^{l-m}\sqrt{[l-m][l+m+1]\gamma}\;e^{lq}_{m+1}(\lambda)\,, \\ T(v_-)e^{lq}_m(\lambda) &= \sqrt{[l+m][l-m+1]\gamma}\;e^{lq}_{m-1}(\lambda)\,, \end{aligned} \tag{2.15}$$

where the symbol $[n]$ is the graded quantum symbol defined by

$$[n]=\frac{q^{-\frac{n}{2}}-(-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}}+q^{\frac{1}{2}}}, \tag{2.16}$$

and $\gamma = \dfrac{\cosh(\frac{\eta}{4})}{\sinh(2\eta)}$. Note that although the symbol $[n]$ is not invariant under the exchange $q \leftrightarrow q^{-1}$, the relations (2.15) are invariant with respect to this substitution. Furthermore, it follows from the explicit formulas for $e_m^{lq}(\lambda)$, Ref. [11] that

$$e_m^{lq}(\lambda) = e_m^{lq^{-1}}(\lambda), \tag{2.17}$$

thus both quantum algebras $U_q(osp(1|2))$ and $U_{q^{-1}}(osp(1|2))$ define the same basis in the representation space $V^l(\lambda)$. In the next subsection, we shall see that this is no more true when we consider tensor product of representation spaces.

## C. Tensor product of representations and the $\mathcal{T}$-matrix

In the following, the representation $T$ of class $\varepsilon$ which acts in the representation space $V^l(\lambda)$ is denoted by $T^l$. The generators $v_\pm$ and $H$ are represented in the tensor product of two irreducible representations of the same class, $T^{l_1} \otimes T^{l_2}$ by the operators

$$v_\pm^{\otimes}(1,2) \;=\; (T^{l_1} \otimes T^{l_2})(\Delta(v_\pm)) \;=\; T^{l_1}(v_\pm) \otimes q^{T^{l_2}(H)} + q^{-T^{l_1}(H)} \otimes T^{l_2}(v_\pm), \tag{2.18}$$

$$H^{\otimes}(1,2) = (T^{l_1} \otimes T^{l_2})(\Delta(H)) \;=\; T^{l_1}(H) \otimes T^{l_2}(\mathbf{1}) + T^{l_1}(\mathbf{1}) \otimes T^{l_2}(H)\,. \tag{2.19}$$

The tensor product $T^{l_1} \otimes T^{l_2}$is simply reducible. The standard basis $e_{m_1}^{l_1 q}(\lambda_1) \otimes e_{m_2}^{l_2 q}(\lambda_2)$ and reduced basis $e_m^{lq}(l_1, l_2, \lambda)$ of the representation space $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ are related by Clebsch-Gordan coefficients in the following way :

$$e_m^{lq}(l_1, l_2, \lambda) = \sum_{m_1, m_2} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q \; e_{m_1}^{l_1 q}(\lambda_1) \otimes e_{m_2}^{l_2 q}(\lambda_2)\,, \tag{2.20}$$

or equivalently,

$$\sum_{l,m} (-1)^{(l-m)L} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q \; e_m^{lq}(l_1, l_2, \lambda) = (-1)^{(l_1 - m_1)(l_2 - m_2)} e_{m_1}^{l_1 q}(\lambda_1) \otimes e_{m_2}^{l_2 q}(\lambda_2), \tag{2.21}$$

where $m_1 + m_2 = m$ , $L = l_1 + l_2 + l$ and $l$ is an integer satisfying the conditions

$$|l_1 - l_2| \leq l \leq l_1 + l_2\,, \qquad \lambda = L + \lambda_1 + \lambda_2 \;(\mathrm{mod}\, 2). \tag{2.22}$$

Both bases $e_{m_1}^{l_1 q}(\lambda_1) \otimes e_{m_2}^{l_2 q}(\lambda_2)$ and $e_m^{lq}(l_1, l_2, \lambda)$ are orthogonal but not positive definite, i.e., we have

$$(e_{m_1}^{l_1 q}(\lambda_1) \otimes e_{m_2}^{l_2 q}(\lambda_2), e_{m_1'}^{l_1 q}(\lambda_1) \otimes e_{m_2'}^{l_2 q}(\lambda_2)) = (-1)^{(l_1 - m_1 + \lambda_1)(l_2 - m_2 + \lambda_2)} \delta_{m_1 m_1'} \delta_{m_2 m_2'}, \tag{2.23}$$

$$(e_m^{lq}(l_1, l_2, \lambda), e_{m'}^{l'q}(l_1, l_2, \lambda)) = \delta_{ll'}\; \delta_{mm'} (-1)^{\varphi(l-m)+\psi}, \tag{2.24}$$

where

$$\varphi = L + \lambda_1 + \varphi_2 \;\; (\mathrm{mod}\; 2), \qquad \psi = (L + \lambda_2)\lambda_1 + \varphi_2 L + \psi_1 + \psi_2 \;(\mathrm{mod}\, 2)\,, \tag{2.25}$$

We may consider also another reduced basis built with the basis $e_m^{lq^{-1}}(\lambda)$ :

$$e_m^{lq^{-1}}(l_1,l_2,\lambda) = \sum_{m_1,m_2} (l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_{q^{-1}}\ e_{m_1}^{l_1q^{-1}}(\lambda_1)\otimes e_{m_2}^{l_2q^{-1}}(\lambda_2), \quad (2.26)$$

$$= \sum_{m_1,m_2} (l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_{q^{-1}}\ e_{m_1}^{l_1q}(\lambda_1)\otimes e_{m_2}^{l_2q}(\lambda_2), \quad (2.27)$$

where the second line is deduced from Eq.(2.17). This basis is normalized in the same way (2.24) as the basis $e_m^{lq}(l_1,l_2,\lambda)$, but the two reduced bases are different because we have in general

$$(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_{q^{-1}} \neq (l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_q. \quad (2.28)$$

Thus, there exist an automorphism $\mathcal{T}^q(l_1,l_2)$ of the space $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)$ that relates both bases :

$$\mathcal{T}^q(l_1,l_2)(e_m^{lq}(l_1,l_2,\lambda)) = e_m^{lq^{-1}}(l_1,l_2,\lambda).$$

Using the orthogonality relations (2.20), (2.21), one can show that this operator has the following matrix form in the basis $e_m^{lq}(l_1,l_2,\lambda)$ :

$$[\mathcal{T}^q(l_1,l_2)]_{l'm',lm} = (-1)^{(l_1+l_2+l')(l'-m')}\sum_{m_1m_2}(-1)^{(l_1-m_1)(l_2-m_2)}$$
$$\times(l_1m_1\lambda_1, l_2m_2\lambda_2|l'm'\lambda')_q(l_1m_1\lambda_1, l_2m_2\lambda_2|lm\lambda)_{q^{-1}}. \quad (2.29)$$

From this expression it follows that $[\mathcal{T}^q(l_1,l_2)]^{-1} = \mathcal{T}^{q^{-1}}(l_1,l_2)$. Note also that the matrix $\mathcal{T}^q(l_1,l_2)$ is diagonal in the indices $m',m$ but it is not diagonal in the indices $l',l$. The symmetries of Clebsch-Gordan coefficients imply the following properties for the matrix elements $[\mathcal{T}^q(l_1,l_2)]_{l'm',lm}$ :

$$[\mathcal{T}^q(l_1,l_2)]_{lm,l'm'} = (-1)^{(l_1+l_2+l'+l+m)(l'+l)}\left[\mathcal{T}^{q^{-1}}(l_1,l_2)\right]_{l'm',lm}, \quad (2.30)$$

$$\left[\mathcal{T}^{q^{-1}}(l_1,l_2)\right]_{l'm',lm} = (-1)^{m(l'+l)}(-1)^{\frac{1}{2}l(l-1)+\frac{1}{2}l'(l'-1)}[\mathcal{T}^q(l_1,l_2)]_{l'-m',l-m}, \quad (2.31)$$

$$[\mathcal{T}^q(l_2,l_1)]_{l'm',lm} = (-1)^{(l_1+l_2+\lambda_1+\lambda_2)(l'+l)}(-1)^{\frac{1}{2}l(l-1)+\frac{1}{2}l'(l'-1)}\left[\mathcal{T}^{q^{-1}}(l_1,l_2)\right]_{l'm',lm}. \quad (2.32)$$

## D. The permutation operator $\tau$ and the universal $R$-matrix

We shall now consider the tensor product of the previous irreducible representations in reverse order, namely $T^{l_2}\otimes T^{l_1}$. For this we introduce the permutation operator $\tau$ acting in any graded tensor product space as follows

$$\tau(a\otimes b) = (-1)^{\deg(a)\deg(b)}(b\otimes a). \quad (2.33)$$

Therefore, for the quantum superalgebra $U_q(osp(1|2))$ we have

$$\tau\Delta^q(v_\pm) = \Delta^{q^{-1}}(v_\pm) = R^q\Delta^q(v_\pm)(R^q)^{-1}, \quad (2.34)$$

$$\tau\Delta^q(H) = \Delta^{q^{-1}}(H) = R^q\Delta^q(H)(R^q)^{-1}, \quad (2.35)$$

which means that the action of the operator $\tau$ for the coproduct of generators of $U_q(osp(1|2))$ coïncides with the action of the universal $R$-matrix. The action of the operator $\tau$ in the tensor product of representations $T^{l_1} \otimes T^{l_2}$ reverse the order of the tensor product and we have

$$\tau(e^{l_1 q}_{m_1}(\lambda_1) \otimes e^{l_2 q}_{m_2}(\lambda_2)) = (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)} e^{l_2 q}_{m_2}(\lambda_2) \otimes e^{l_1 q}_{m_1}(\lambda_1), \tag{2.36}$$

and

$$\tau\, v^{\otimes q}_{\pm}(l_1, l_2)\, \tau^{-1} = v^{\otimes q^{-1}}_{\pm}(l_2, l_1), \tag{2.37}$$

$$\tau\, H^{\otimes q}(l_1, l_2)\, \tau^{-1} = H^{\otimes q^{-1}}(l_2, l_1). \tag{2.38}$$

Using the following symmetry property of Clebsch-Gordan coefficients [1]

$$\begin{aligned}(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_q &= (-1)^{(l_1-m_1+\lambda_1)(l_2-m_2+\lambda_2)} (-1)^{(l_1+l_2+l)(\lambda_1+\lambda_2)+\lambda_1\lambda_2} \\ &\quad \times (-1)^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l+1)} (l_2 m_2 \lambda_2, l_1 m_1 \lambda_1 | l m \lambda)_{q^{-1}},\end{aligned} \tag{2.39}$$

one can check that the action of $\tau$ on the reduced basis reads

$$\tau(e^{lq}_m(l_1, l_2, \lambda)) = (-1)^{(l_1+l_2+l+\lambda_1)(l_1+l_2+l+\lambda_2)} (-1)^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l-1)} e^{lq^{-1}}_m(l_2, l_1, \lambda). \tag{2.40}$$

Thus the operator $\tau$ that relates the bases $e^{lq}_m(l_1, l_2, \lambda)$ and $e^{lq^{-1}}_m(l_2, l_1, \lambda)$ is a representation isomorphism :

$$\tau : (T^{l_1} \otimes T^{l_2})\,(U_q(osp(1|2))) \to (T^{l_2} \otimes T^{l_1})\,(U_{q^{-1}}(osp(1|2)))\,. \tag{2.41}$$

The operator $R^q$ in the representation $T^{l_1} \otimes T^{l_2}$ will be denoted $R^q(l_1, l_2) \equiv R^q(1, 2)$. In such a representation its action is

$$R^q(1,2)\, v^{\otimes q}_{\pm}(l_1, l_2)\, R^q(1,2)^{-1} = v^{\otimes q^{-1}}_{\pm}(l_1, l_2), \tag{2.42}$$

$$R^q(1,2)\, H^{\otimes q}(l_1, l_2)\, R^q(1,2)^{-1} = H^{\otimes q^{-1}}(l_1, l_2). \tag{2.43}$$

Since $R^q(1,2)$ does not exchange the two spaces, in this case the action of the operator $\tau$ does not coïncide with the action of $R^q(1,2)$.

### D.1. Matrix elements of $R^q(1,2)$ in the tensor product basis

Using relations (2.15) one can show that, in the standard basis $e^{l_1 q}_{m_1}(\lambda_1) \otimes e^{l_2 q}_{m_2}(\lambda_2)$, the operator $R^q(1,2)$ has the following matrix form

$$\begin{aligned}[\mathbb{R}^q(1,2)]_{m_1 m_2, n_1 n_2} &= (-1)^{k(\lambda_1+1)+\frac{1}{2}k(k-1)} q^{\frac{1}{4}k(3k+1)} q^{\frac{1}{2}k(n_1-n_2)+(n_1+k)(n_2-k)} \frac{(1+q^{-1})^k}{[k]!} \\ &\quad \times \left( \frac{[l_1-n_1]!\,[l_1+n_1+k]!\,[l_2+n_2]!\,[l_2-n_2+k]!}{[l_1+n_1]!\,[l_1-n_1-k]!\,[l_2-n_2]!\,[l_2+n_2-k]!} \right)^{\frac{1}{2}} \delta_{m_1, n_1+k} \delta_{m_2, n_2-k}.\end{aligned} \tag{2.44}$$

For the simplest case when $l_1 = l_2 = 1$, $R^q(1,2)_{m_1m_2,n_1n_2}$ has the following form

$$[\mathcal{R}^q(1,1)]_{m_1m_2,n_1n_2} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & b & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}, \tag{2.45}$$

where

$$a = d = (-1)^{\lambda_1}(q - q^{-1}), \quad c = b = q^{-\frac{1}{2}}a, \quad e = (-1)^{\lambda_1}(1 + q^{-1})a. \tag{2.46}$$

When $\lambda_1 = 0$, this matrix coïncides with the corresponding matrix given in Ref.[8].

### D.2. Matrix elements of $R^q(1,2)$ and $R^{q^{-1}}(1,2)$ in the reduced bases

Relations (2.42), (2.43) suggest that the vector $e^{lq^{-1}}_m(l_1,l_2,\lambda)$ is proportional to the vector $R^q(1,2)e^{lq}_m(l_1,l_2,\lambda)$. In the appendix we prove that indeed we have

$$R^q(1,2)e^{lq}_m(l_1,l_2,\lambda) = q^{\frac{1}{2}(l(l+1)-l_1(l_1+1)-l_2(l_2+1))}e^{lq^{-1}}_m(l_1,l_2,\lambda). \tag{2.47}$$

From this relation it follows that

$$[R^q(1,2)]^{-1} = R^{q^{-1}}(1,2). \tag{2.48}$$

Furthermore, formula (2.47) allows one to calculate the matrix elements of the operators $R^q(1,2)$ and $R^{q^{-1}}(1,2)$ in the reduced bases $e^{lq}_m(l_1,l_2,\lambda)$ and $e^{lq^{-1}}_m(l_1,l_2,\lambda)$. Let us introduce the following notation:
$\mathbb{R}^q_e$ ($\mathbb{R}^{q^{-1}}_e$) is the matrix of the operator $R^q(1,2)$ ( $R^{q^{-1}}(1,2)$) in the basis $e^{lq}_m(l_1,l_2,\lambda)$,
$\mathbb{R}^q_{\tilde{e}}$ ($\mathbb{R}^{q^{-1}}_{\tilde{e}}$) is the matrix of the operator $R^q(1,2)$ ( $R^{q^{-1}}(1,2)$) in the basis $e^{lq^{-1}}_m(l_1,l_2,\lambda)$,
and let $\mathbb{D}^q$ be a diagonal matrix of the form

$$[\mathbb{D}^q]_{lm,l'm'} = q^{\frac{1}{2}((l(l+1)-l_1(l_1+1)-l_2(l_2+1))}\delta_{l,l'}\delta_{m,m'}. \tag{2.49}$$

Using relation (2.47) and the orthogonality relations (2.20), (2.21) for $sq$-CGc, one can derive the following matrix relations :

$$\mathbb{R}^q_e = \mathcal{T}^q\mathbb{D}^q, \qquad \mathbb{R}^{q^{-1}}_e = \mathbb{D}^{q^{-1}}\mathcal{T}^{q^{-1}}, \tag{2.50}$$

$$\mathbb{R}^q_{\tilde{e}} = \mathbb{D}^q\mathcal{T}^q, \qquad \mathbb{R}^{q^{-1}}_{\tilde{e}} = \mathcal{T}^{q^{-1}}\mathbb{D}^{q^{-1}}, \tag{2.51}$$

where $\mathcal{T}^q$ is the matrix given by the formula (2.29). In particular, for the first relation we have

$$[\mathbb{R}^q_e]_{l'm',lm} = q^{\frac{1}{2}((l(l+1)-l_1(l_1+1)-l_2(l_2+1))}(-1)^{(l_1+l_2+l')(l'-m')} \times \sum_{m_1 m_2} (-1)^{(l_1-m_1)(l_2-m_2)}(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l' m' \lambda')_q (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 | l m \lambda)_{q^{-1}}, \tag{2.52}$$

i.e., the matrix elements of the operator $R^q(1,2)$ can be expressed in terms of $sq$-CGc. The values of the $sq$-CGc for arbitrary tensor product of irreducible representations can be calculated from the analytical expression given in Ref.[2], [10], [11], therefore the matrix elements of $R^q(1,2)$ in any tensor product of irreducible representations can be deduced from formula (2.52). For example, calculating the values of corresponding $sq$-CGc for $l_1 = l_2 = 1$, we get from formula (2.52)

$$[\mathbb{R}^q_e]_{l'm',lm} = \begin{pmatrix} q^{-2}a & 0 & 0 & 0 & 0 & 0 & qc & 0 & 0 \\ 0 & q^{-1}b & 0 & 0 & 0 & qd & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1}b & 0 & 0 & 0 & -qd & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & -q^{-1}d & 0 & 0 & 0 & qb & 0 & 0 & 0 \\ -q^{-2}c & 0 & 0 & 0 & 0 & 0 & qa & 0 & 0 \\ 0 & 0 & 0 & q^{-1}d & 0 & 0 & 0 & qb & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}, \tag{2.53}$$

where

$$a = \frac{q}{q^2-q+1}, \quad b = \frac{2q}{q^2+1}, \quad c = \frac{q^3-q^2+q-1}{\sqrt{q^2+1}(q^2-q+1)}, \quad d = (-1)^{\lambda_1}\frac{q^2-1}{q^2+1}. \tag{2.54}$$

From expressions (2.42), (2.43), (2.47), it follows that the operator $R^q(1,2)$ yields an isomorphism of two represntations

$$R^q(1,2) : (T^{l_1} \otimes T^{l_2})\,(U_q(osp(1|2))) \rightarrow (T^{l_1} \otimes T^{l_2})\,(U_{q^{-1}}(osp(1|2)))\,. \tag{2.55}$$

## E. The operator $\mathcal{R}^q$

Finally, in the following we will use the operator $\mathcal{R}^q$ that is a composition of the operators $\tau$ and $R^q$

$$\mathcal{R}^q = \tau \circ R^q. \tag{2.56}$$

Properties of the operators $\tau$ and $R^q$, in particular Yang-Baxter equation (2.12) satisfied by $R^q$, implies that $\mathcal{R}^q$ satisfies the following relation

$$(\mathcal{R}^q \otimes id)\,(id \otimes \mathcal{R}^q)\,(\mathcal{R}^q \otimes id) = (id \otimes \mathcal{R}^q)\,(\mathcal{R}^q \otimes id)\,(id \otimes \mathcal{R}^q). \tag{2.57}$$

Further, from relations (2.10), we get the relations

$$(\mathcal{R}^q \otimes id)\ (id \otimes \mathcal{R}^q) = \tau(12,3)R^{q12,3} \equiv \mathcal{R}^{q12,3}, \tag{2.58}$$

$$(id \otimes \mathcal{R}^q)\ (\mathcal{R}^q \otimes id) = \tau(1,23)R^{q1,23} \equiv \mathcal{R}^{q1,23}, \tag{2.59}$$

where

$$\tau(12,3) = (\tau \otimes id)(id(\otimes\tau), \qquad \tau(1,23) = (id \otimes \tau)(\tau \otimes id). \tag{2.60}$$

In the tensor product $T^{l_1} \otimes T^{l_2}$ the operator $\mathcal{R}^q$ is represented by the operator

$$\mathcal{R}^q(1,2) = \tau \circ R^q(1,2)$$

and formulas (2.58), (2.59) take the form

$$(\mathcal{R}^q(1,3) \otimes id(2))\ (id(1) \otimes \mathcal{R}^q(2,3)) = \mathcal{R}^{q12,3}(12,3), \tag{2.61}$$

$$(id(2) \otimes \mathcal{R}^q(1,3))\ (\mathcal{R}^q(1,2) \otimes id(3)) = \mathcal{R}^{q1,23}(1,23). \tag{2.62}$$

Using relations (2.40), (2.47) we derive immediately the action of $\mathcal{R}^q(1,2)$ on the reduced basis

$$\begin{aligned}\mathcal{R}^q(1,2)e^{lq}_m(l_1,l_2,\lambda) \;&=\; (-1)^{(l_1+l_2+l+\lambda_1)(l_1+l_2+l+\lambda_2)}(-1)^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l-1)} \\ &\quad \times q^{\frac{1}{2}[l(l+1)-l_1(l_1+1)-l_2(l_2+1)]}e^{lq}_m(l_2,l_1,\lambda),\end{aligned} \tag{2.63}$$

and its action on the operators $v^{\otimes q}_{\pm}(l_1,l_2)$, $H^{\otimes q}(l_1,l_2)$ reads

$$\mathcal{R}^q(1,2)\ v^{\otimes q}_{\pm}(l_1,l_2)\ \mathcal{R}^q(1,2)^{-1} = v^{\otimes q}_{\pm}(l_2,l_1), \tag{2.64}$$

$$\mathcal{R}^q(1,2)\ H^{\otimes q}(l_1,l_2)\ \mathcal{R}^q(1,2)^{-1} = H^{\otimes q}(l_2,l_1), \tag{2.65}$$

where

$$\mathcal{R}^q(1,2)^{-1} = R^{q^{-1}}(1,2) \circ \tau. \tag{2.66}$$

The operator $\mathcal{R}^q(1,2)$ defines then an isomorphism of representations

$$\mathcal{R}^q(1,2) : (T^{l_1} \otimes T^{l_2})\,(U_q(osp(1|2))) \rightarrow (T^{l_2} \otimes T^{l_1})\,(U_q(osp(1|2)))\,. \tag{2.67}$$

The universal $R$-matrix is the appropriate tool for obtaining numerous relations between Racah coefficients and $sq6$-$j$ symbols. In the following section we shall first use it to derive Racah sum rule for $sq6$-$j$ symbols.

# III. Racah sum rule for the $sq-6j$ symbols.

In order to derive Racah sum rule, we will consider the action of the operator $\mathcal{R}^q$ on the reduced bases of the tensor product of three irreducible representations of the same class $T^{l_1}\otimes T^{l_2}\otimes T^{l_3}$.The operators $H, v_\pm$ are represented in this tensor product by

$$v_\pm^{\otimes}(1,2,3) = (T^{l_1}\otimes T^{l_2}\otimes T^{l_3})((\Delta\otimes id)\Delta(v_\pm)), \tag{3.1}$$
$$H^{\otimes}(1,2,3) = (T^{l_1}\otimes T^{l_2}\otimes T^{l_3})((\Delta\otimes id)\Delta(H)). \tag{3.2}$$

The reduction of the tensor product $V^{l_1}(\lambda_1)\otimes V^{l_2}(\lambda_2)\otimes V^{l_3}(\lambda_3)$ of representation spaces can be done, as in the classical case, in two different schemes :

$$T^l\subset((T^{l_1}\otimes T^{l_2})\otimes T^{l_3}),\ \ T^l\subset(T^{l_1}\otimes(T^{l_2}\otimes T^{l_3})). \tag{3.3}$$

The bases $e^{lq}_m(l_1,l_{23},\lambda)$ and $e^{lq}_m(l_{12},l_3,\lambda)$ corresponding to these reduction schemes are orthogonal and normalized in the following way

$$(e^{lq}_m(l_{12},l_3,\lambda)\ ,\ e^{l'q}_{m'}(l'_{12},l_3,\lambda)) = (-1)^{\varphi_{12,3}(l-m)+\psi_{12,3}}\ \delta_{ll'}\ \delta_{mm'}\ \delta_{l_{12}l'_{12}} = g_{ll_{12}m,l'l'_{12}m'}, \tag{3.4}$$

$$(e^{lq}_m(l_1,l_{23},\lambda)\ ,\ e^{l'q}_{m'}(l_1,l'_{23},\lambda)) = (-1)^{\varphi_{1.23}(l-m)+\psi_{1,23}}\ \delta_{ll'}\ \delta_{mm'}\ \delta_{l_{23}l'_{23}} = g_{ll_{23}m,l'l'_{23}m'},, \tag{3.5}$$

where

$$\varphi_{1,23} = \varphi_{12,3} = \mathcal{L}+\lambda_1+\lambda_2+\varphi_3 \pmod 2, \tag{3.6}$$
$$\psi_{12,3} = (l_1+l_2+l_{12})(\mathcal{L}+1)+(\lambda_1+\lambda_3+\varphi_2)\mathcal{L}+\sum_{i<j}\lambda_i\lambda_j+\sum_{i=1}^{3}\psi_i \pmod 2, \tag{3.7}$$
$$\psi_{1,23} = (l_2+l_3+l_{23})(\mathcal{L}+1)+(\lambda_1+\lambda_3+\varphi_2)\mathcal{L}+\sum_{i<j}\lambda_i\lambda_j+\sum_{i=1}^{3}\psi_i \pmod 2, \tag{3.8}$$

with

$$\mathcal{L}=l_1+l_2+l_3+l,\quad \lambda=\sum_{i=1}^{3}(\lambda_i+l_i)+l. \tag{3.9}$$

The $sq$Racah Coefficients $U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)$ of the quantum superalgebra $U_q(osp(1|2))$ are defined, in the standard way, as the coefficients that relate reduced bases in two different reduction schemes [2]

$$e^{lq}_m(l_{12},l_3,\lambda)=\sum_{l_{23}}(-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)}\ U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)\ e^{lq}_m(l_1,l_{23},\lambda), \tag{3.10}$$

or equivalently

$$e^{lq}_m(l_1,l_{23},\lambda) = \sum_{l_{12}}(-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)}U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)e^{lq}_m(l_{12},l_3,\lambda). \tag{3.11}$$

We have the following relation between $sqRc$ and $sq6$-$j$ symbols [2]

$$U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) = (-1)^{(\lambda_2+1)(l_2+l+l_{23}+l_{12})}(-1)^{(l_1+l_2+l_3)(l+1)}(-1)^{l_{12}(l_1+l_{23}+l)} \times(-1)^{l_{23}(l_{12}+l_3+l)}(-1)^{\frac{\mathcal{L}(\mathcal{L}-1)}{2}}\sqrt{[2l_{12}+1][2l_{23}+1]}\left\{\begin{array}{ccc} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23}\end{array}\right\}_q^s. \quad (3.12)$$

Let us consider the matrix $W_{l'm',lm}$ of the operator $id(1)\otimes\mathcal{R}^q(3,2)$ in the bases $e^{lq}_m(l_{12},l_3,\lambda)$ and $e^{lq}_m(l_{13},l_2,\lambda)$

$$W_{l'm',lm} = g_{l'l''_{12}m',l''l''_{12}m''}(e^{l''q}_{m''}(l''_{12},l_3,\lambda), id(1)\otimes\mathcal{R}^q(3,2)e^{lq}_m(l_{13},l_2,\lambda)\ ) \quad (3.13)$$

Using relations (2.20),(3.10),(2.63), one obtains

$$\begin{aligned} id(1)\otimes\mathcal{R}^q(3,2)&e^{lq}_m(l_{13},l_2,\lambda) \\ &= \sum_{l_{12},l_{23}}(-1)^{(l_1+l_3+l_{12}+l_{23})(\mathcal{L}+1)}(-1)^{(l_2+l_3+l_{23}+\lambda_2)(l_2+l_3+l_{23}+\lambda_3)} \\ &\times(-1)^{\frac{1}{2}(l_2+l_3-l_{23})(l_2+l_3-l_{23}-1)}q^{\frac{1}{2}[l_{23}(l_{23}+1)-l_2(l_2+1)-l_3(l_3+1)]} \\ &\times U^s(l_1,l_3,l_2,l,l_{13},l_{23},q)\ U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)\ e^{lq}_m(l_{12},l_3,\lambda) \end{aligned} \quad (3.14)$$

Therefore, the matrix $W_{l'm',lm}$ has the form

$$\begin{aligned} W_{l'm',lm} &= \sum_{l_{23}}(-1)^{(l_1+l_3+l_{12}+l_{23})(\mathcal{L}+1)}(-1)^{(l_2+l_3+l_{23}+\lambda_2)(l_2+l_3+l_{23}+\lambda_3)} \\ &\times(-1)^{\frac{1}{2}(l_2+l_3-l_{23})(l_2+l_3-l_{23}-1)}\ q^{\frac{1}{2}[l_{23}(l_{23}+1)-l_2(l_2+1)-l_3(l_3+1)]} \\ &\times U^s(l_1,l_3,l_2,l,l_{13},l_{23},q)\ U^s(l_1,l_2,l_3,l,l_{12},l_{23},q)\ \delta_{l',l}\delta_{m',m}. \end{aligned} \quad (3.15)$$

It is diagonal and it does not depend on $m$.

On the other hand we have

$$\begin{aligned} id(1)\ \otimes\ &\mathcal{R}^q(3,2)\ e^{lq}_m(l_{13},l_2,\lambda) \\ &= ([\mathcal{R}^q(1,2)]^{-1}\otimes id(3))\ (\mathcal{R}^q(1,2)\otimes id(3))\ (id(1)\otimes\mathcal{R}^q(3,2))\ e^{lq}_m(l_{13},l_2,\lambda), \quad (3.16) \\ &= (-1)^{(l_2+l_{13}+l+\lambda_2)(l_2+l_{13}+l+\lambda_{13})}(-1)^{\frac{1}{2}(l_2+l_{13}-l)(l_2+l_{13}-l-1)} \\ &\qquad\times q^{\frac{1}{2}[l(l+1)-l_2(l_2+1)-l_{13}(l_{13}+1)]}([\mathcal{R}^q(1,2)]^{-1}\otimes id(3))\ e^{lq}_m(l_2,l_{13},\lambda), \quad (3.17) \end{aligned}$$

where we have used relations (2.61), (2.63). Applying further relations (2.20), (3.10), (2.63), we find that

$$\begin{aligned} ([\mathcal{R}^q(1,2)]^{-1}\otimes id(3))e^{lq}_m(l_2,l_{13},\lambda) &= \sum_{l_{12}}(-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)}(-1)^{(l_1+l_2+l_{12}+\lambda_1)(l_1+l_2+l_{12}+\lambda_2)} \\ &\times(-1)^{\frac{1}{2}(l_1+l_2-l_{12})(l_1+l_2-l_{12}-1)}q^{-\frac{1}{2}[l_{12}(l_{12}+1)-l_1(l_1+1)-l_2(l_2+1)]} \\ &\times U^s(l_2,l_1,l_3,l,l_{12},l_{13},q)\ e^{lq}_m(l_{12},l_3,\lambda). \end{aligned} \quad (3.18)$$

Substitution of this relation into (3.16) and of the resulting formula into relation (3.13) yields another expression for the matrix $W_{l'm',lm}$

$$\begin{aligned} W_{l'm',lm} &= (-1)^{(l_2+l_{23}+l+\lambda_2)(l_2+l_{13}+l+\lambda_{13})}\ (-1)^{(l_1+l_2+l_{12}+\lambda_1)(l_1+l_2+l_{12}+\lambda_2)}\ (-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)} \\ &\times(-1)^{\frac{1}{2}(l_2+l_{13}-l)(l_2+l_{13}-l-1)}\ (-1)^{\frac{1}{2}(l_1+l_2-l_{12})(l_1+l_2-l_{12}-1)}\ q^{\frac{1}{2}[l(l+1)-l_2(l_2+1)-l_{13}(l_{13}+1)]} \\ &\times q^{-\frac{1}{2}[l_{12}(l_{12}+1)-l_1(l_1+1)-l_2(l_2+1)]}\ U^s(l_2,l_1,l_3,l,l_{12},l_{13},q)\ \delta_{l',l}\delta_{m',m}. \end{aligned} \tag{3.19}$$

Comparison of this expression with relation (3.15) furnishes Racah sum rule for $U_q(osp(1|2))$ Racah coefficients :

$$\begin{aligned} &\sum_{l_{23}}(-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)}\ (-1)^{(l_2+l_3+l_{23}+\lambda_2)(l_2+l_3+l_{23}+\lambda_3)}\ (-1)^{\frac{1}{2}(l_2+l_3-l_{23})(l_2+l_3-l_{23}-1)} \\ &\qquad\times q^{\frac{1}{2}[l_{23}(l_{23}+1)-l_2(l_2+1)-l_3(l_3+1)]}\ U^s(l_1,l_3,l_2,l,l_{13},l_{23},q)\ U^s(l_1,l_2,l_3,l,l_{12},l_{23},q) \\ &= (-1)^{(l_2+l_{23}+l+\lambda_2)(\mathcal{L}+\lambda_1+\lambda_3)}\ (-1)^{(l_1+l_2+l_{12}+\lambda_1)(l_1+l_2+l_{12}+\lambda_2)}\ (-1)^{\frac{1}{2}(l_2+l_{13}-l)(l_2+l_{13}-l-1)} \\ &\qquad\times(-1)^{\frac{1}{2}(l_1+l_2-l_{12})(l_1+l_2-l_{12}-1)}\ q^{\frac{1}{2}[l(l+1)+l_1(l_1+1)-l_{12}(l_{12}+1)-l_{13}(l_{13}+1)]}\ U^s(l_2,l_1,l_3,l,l_{12},l_{13},q) \end{aligned} \tag{3.20}$$

Substituting in this expression relation (3.12) that relates $sq$Racah coefficients and $sq6$-$j$ symbols, we derive Racah sum rule satisfied by the latter :

$$\begin{aligned} &\sum_{l_{23}}(-1)^{(l_1+l_2+l_{13}+l_{23})(\mathcal{L}+1)+\mathcal{L}l}(-1)^{(l_2+l_3+l_{23})l_{23}+(l_1+l_2+l_{12})l_{12}+(l_1+l_3+l_{13})l_{13}}(-1)^{\frac{\mathcal{L}(\mathcal{L}+1)}{2}} \\ &\quad\times(-1)^{\frac{1}{2}(l_2+l_3-l_{23})(l_2+l_3-l_{23}-1)}\ (-1)^{\frac{1}{2}(l_1+l_2-l_{12})(l_1+l_2-l_{12}-1)}\ (-1)^{\frac{1}{2}(l_2+l_{13}-l)(l_2+l_{13}-l-1)} \\ &\quad\times q^{\frac{1}{2}[l_{23}(l_{23}+1)+l_{12}(l_{12}+1)+l_{13}(l_{13}+1)]}\ [2l_{23}+1] \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix}^s_q \begin{Bmatrix} l_1 & l_3 & l_{13} \\ l_2 & l & l_{23} \end{Bmatrix}^s_q \\ &\qquad= q^{\frac{1}{2}[l(l+1)+l_1(l_1+1)+l_2(l_2+1)+l_3(l_3+1)]} \begin{Bmatrix} l_2 & l_1 & l_{12} \\ l_3 & l & l_{13} \end{Bmatrix}^s_q \end{aligned} \tag{3.21}$$

Using the orthogonality relation for the $sq6$-$j$ symbols

$$\begin{aligned} &\sum_{l_{12}}(-1)^{(l_1+l_2+l_{12})(\mathcal{L}+1)}\ [2l_{12}+1]\,[2l_{23}+1] \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix}^s_q \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l'_{23} \end{Bmatrix}^s_q \\ &\qquad= (-1)^{(l_2+l_3+l_{23})(\mathcal{L}+1)}\ \delta_{l_{23},l'_{23}}, \end{aligned} \tag{3.22}$$

one can rewrite the Racah sum rule (3.21) in the following form

$$\sum_{l_{13}}(-1)^{\Theta}\, q^{-\frac{1}{2}[l_{23}(l_{23}+1)+l_{12}(l_{12}+1)+l_{13}(l_{13}+1)]}\,[2l_{13}+1]\begin{Bmatrix} l_2 & l_1 & l_{12} \\ l_3 & l & l_{13} \end{Bmatrix}^{s}_{q}\begin{Bmatrix} l_1 & l_3 & l_{13} \\ l_2 & l & l_{23} \end{Bmatrix}^{s}_{q}$$

$$= q^{-\frac{1}{2}[l(l+1)+l_1(l_1+1)+l_2(l_2+1)+l_3(l_3+1)]}\begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix}^{s}_{q}, \tag{3.23}$$

where the phase argument $\Theta$ is

$$\begin{aligned}\Theta &= \frac{1}{2}(l_2+l_3-l_{23})(l_2+l_3-l_{23}-1)+\frac{1}{2}(l_1+l_2-l_{12})(l_1+l_2-l_{12}-1)\\ &+\frac{1}{2}(l_2+l_{13}-l)(l_2+l_{13}-l-1)+\frac{1}{2}\mathcal{L}(\mathcal{L}+1)+\mathcal{L}l+(l_2+l_3+l_{23})l_{23}\\ &+(l_1+l_2+l_{12})l_{12}+(l_1+l_3+l_{13})l_{13}.\end{aligned} \tag{3.24}$$

In the limit $q \longrightarrow 1$ this formula becomes Racah sum rule for the 6-$j$ symbols for the superalgebra $osp(1|2)$ and it takes the form given in Ref.[12].
If $l_{12}=0$ and $l_1=l_2$, $l_3=l$, then Racah sum rule (3.23) leads to the summation formula

$$\sum_{l_{13}}(-1)^{(l_1+l_3)(l_{13}+l'_{13})+l'_{13}}q^{-\frac{1}{2}\left[l_{13}(l_{13}+1)+l'_{13}(l'_{13}+1)\right]}\,[2l_{13}+1]\begin{Bmatrix} l_1 & l_3 & l_{13} \\ l_1 & l_3 & l'_{13} \end{Bmatrix}^{s}_{q} = q^{-\frac{1}{2}[l_1(l_1+1)+l_3(l_3+1)]}. \tag{3.25}$$

# IV. Biedenharn-Elliott identity for the quantum superalgebra $U_q(osp(1|2))$

The tensor product $T^{l_1}\otimes T^{l_2}\otimes T^{l_3}\otimes T^{l_4}$ of four irreducible representations of the same class can be reduced into irreducible representations according to one of the following five reduction schemes :

$$[(T^{l_1}\otimes T^{l_2})\otimes T^{l_3}]\otimes T^{l_4}, \tag{4.1}$$

$$[T^{l_1}\otimes (T^{l_2}\otimes T^{l_3})]\otimes T^{l_4}, \tag{4.2}$$

$$T^{l_1}\otimes [(T^{l_2}\otimes T^{l_3})\otimes T^{l_4}], \tag{4.3}$$

$$T^{l_1}\otimes [T^{l_2}\otimes (T^{l_3}\otimes T^{l_4})], \tag{4.4}$$

$$(T^{l_1}\otimes T^{l_2})\otimes (T^{l_3}\otimes T^{l_4}) \tag{4.5}$$

Biedenharn-Elliott identity is concerned with the relation between the schemes (4.1) and (4.4). One can go from one to the other either by the chain

$$(4.1)\longrightarrow(4.2)\longrightarrow(4.3)\longrightarrow(4.4), \tag{4.6}$$

or by the chain

$$(4.1)\longrightarrow(4.5)\longrightarrow(4.4). \tag{4.7}$$

each step involving $sq$Racah coefficients. Since the overall matrices that connect the two reduction schemes, (4.1) and (4.4), must be independent of the chain, there must exist a relation between combinations of products of three $sq$Racah coefficients for the firrst chain and combinations of products of two $sq$Racah coefficients for the second chain. This is precisely the Biedenharn-Elliott identity. Writing down the equality of corresponding matrix elements, Biedenharn-Elliott identity for $sq$Racah coefficients reads

$$\begin{aligned}
&\sum_{l_{23}} (-1)^{(l_2+l_3+l_{23})(l_2+l_3+l_4+l_{234}+1)} (-1)^{(l_1+l_2+l_{12})(l_1+l_2+l_3+l_{123}+1)} (-1)^{(l_1+l_{23}+l_{123})(l_1+l_{23}+l_4+l+1)} \\
&\quad U^s(l_1,l_2,l_3,l_{123},l_{12},l_{23},q)\; U^s(l_1,l_{23},l_4,l,l_{123},l_{234},q)\; U^s(l_2,l_3,l_4,l_{234},l_{23},l_{34},q) \\
&= (-1)^{(l_1+l_2+l_{12})(l_1+l_2+l_{34}+l+1)} (-1)^{(l_{12}+l_3+l_{123})(l_{12}+l_3+l_4+l+1)} \\
&\quad U^s(l_{12},l_3,l_4,l,l_{123},l_{34},q)\; U^s(l_1,l_2,l_{34},l,l_{12},l_{234},q). \qquad (4.8)
\end{aligned}$$

For $sq6$-$j$ symbols it is rewritten as

$$\begin{aligned}
&\sum_{l_{13}} (-1)^{\Theta}\,[2l_{23}+1] \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l_{123} & l_{23} \end{Bmatrix}^s_q \begin{Bmatrix} l_1 & l_{23} & l_{123} \\ l_4 & l & l_{234} \end{Bmatrix}^s_q \begin{Bmatrix} l_2 & l_3 & l_{23} \\ l_4 & l_{234} & l_{34} \end{Bmatrix}^s_q \\
&\qquad = \begin{Bmatrix} l_{12} & l_3 & l_{123} \\ l_4 & l & l_{34} \end{Bmatrix}^s_q \begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_{34} & l & l_{234} \end{Bmatrix}^s_q, \qquad (4.9)
\end{aligned}$$

where the phase argument $\Theta$ is given by

$$\begin{aligned}
\Theta &= \frac{1}{2}\left(\sum_{all} l_i\right)\left(\sum_{all} l_i - 1\right) + l_{23}(l+l_{12}+l_{34}) + (l_4+l_{234})(l_3+l+l_{123}+l_{34}) \\
&\quad +(l_3+l_{12}+l_{123})(l_3+l_2+l+l_{12}+l_{34}) + (l_1+l_2+l_{12})(l_1+l_3+l_4+l_{34}+l+l_{234}) \\
&\quad +l_2+l_4+l+l_{34}+l_{123}+l_{234}. \qquad (4.10)
\end{aligned}$$

Similarly as in the cases of $su(2)$ and $u_q(su(2))$ three-term recurrence relation for $sq6$-$j$)symbols follows from the Biedenharn-Elliott identity (4.9). Putting $l=1$ into this identity and using the known analytical expressions of the $sq6$-$j$ symbols with one argument equal to 1 (see the table in Ref.[3]) we derive a recurrence relation that can be written, after a change in the notation,

$$A \begin{Bmatrix} a & b & e \\ c & d & f-1 \end{Bmatrix}^s_q + B \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}^s_q + C \begin{Bmatrix} a & b & e \\ c & d & f+1 \end{Bmatrix}^s_q = 0, \qquad (4.11)$$

where

$$
\begin{aligned}
A &= (-1)^{a+d+f+1}[2][2f+2]([a+d+f+1][a-d+f][a+d-f+1][-a+d+f] \\
&\quad \times[b+c+f+1][b-c+f][b+c-f+1][-b+c+f])^{\frac{1}{2}}, \qquad (4.12) \\
B &= (-1)^{a+c+f+1}[2f+1]([a-d+f][a+d-f]+(-1)^{a+b+f}[-a+d+f][a+d+f+2]) \\
&\quad \times([b+c-f][b-c+f]+(-1)^{b+c+f}[-b+c+f][b+c+f+2]) \\
&\quad +(-1)^{\Psi}[2f][2f+1][2f+2]([e+c-d][e-c+d] \\
&\quad +(-1)^{e+c+d}[-e+c+d][e+c+d+2]), \qquad (4.13) \\
C &= (-1)^{b+c+f}[2][2f]([a+d+f+2][a-d+f+1][a+d-f][-a+d+f+1] \\
&\quad \times[b+c+f+2][b-c+f+1][b+c-f][-b+c+f+1])^{\frac{1}{2}}, \qquad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi &= (a+b+e)(a+b+c+d)+(d+e+c)(b+d+e+f) \\
&\quad +(a+d+f)(a+c+e+f)+c+d. \qquad (4.15)
\end{aligned}
$$

Appendix

# A Action of the universal $R$-matrix on the reduced basis $e^{lq}_m(l_1, l_2, \lambda)$

In this appendix we prove the relation (2.47). For this, we recall that the vector $e^{lq}_m(l_1, l_2, \lambda)$ can be expressed in the form (ref.[1])

$$e^{lq}_m(l_1, l_2, \lambda) = \frac{1}{(N(l, m, q))^{\frac{1}{2}}} P^{lq\otimes}_{ml}(1, 2)\ e^{l_1 q}_{l_1}(\lambda_1) \otimes e^{l_2 q}_{l-l_1}(\lambda_2), \quad . \tag{A.1}$$

where

$$P^{lq\otimes}_{ml}(1, 2) = \left( \frac{[l + m]!}{[2l]!\,[l - m]!} \gamma^{-(l-m)} \right)^{\frac{1}{2}} v^{\otimes q}_-(l_1, l_2)^{l-m} P^{lq\otimes}(1, 2), \tag{A.2}$$

$$P^{lq\otimes}(1, 2) = (T^{l_1} \otimes T^{l_2})(\Delta(P^{lq})) = \sum_{r=0}^{\infty} c_r(l)(v^{\otimes}_-(1, 2))^r (v^{\otimes}_+(1, 2))^r, \tag{A.3}$$

and $P^{lq}$ is the projection operator on the highest weight vector of weight $l$. The coefficients $c_r(l)$ are of the form

$$c_r(l) = \frac{[2l + 1]!}{[2l + r + 1]![r]!\gamma^r}. \tag{A.4}$$

The value of the normalisation factor $N(l, m, q)$ is given by the formula

$$N(l, m, q) = q^{\frac{1}{2}(l_1+l_2-l)(l_1+l_2-l-1)} \frac{[2l + 1]![2l_1]!}{[l_1 + l_2 + l]!\,[l_1 + l_2 + l + 1]!}. \tag{A.5}$$

From relations (2.42) and (A.2, A.3) it follows that

$$R^q(1, 2)\ P^{lq\otimes}_{ml}(1, 2)\ R^{q^{-1}}(1, 2) = P^{lq^{-1}\otimes}_{ml}(1, 2),$$

and one deduces successively the relations

$$\begin{aligned} R^q(1, 2)\ e^{lq}_m(l_1, l_2, \lambda) &= (N(l, m, q))^{-\frac{1}{2}}\ R^q(1, 2)\ P^{lq\otimes}_{lm}(1, 2)\ R^{q^{-1}}(1, 2) \\ &\quad \times R^q(1, 2)\ e^{l_1 q}_{l_1}(\lambda_1) \otimes e^{l_2 q}_{l-l_1}(\lambda_2), && \text{(A.6)} \\ &= (N(l, m, q))^{-\frac{1}{2}}\ q^{l_1(l-l_1)}\ P^{lq^{-1}\otimes}_{ml}(1, 2)\ e^{l_1 q}_{l_1}(\lambda_1) \otimes e^{l_2 q}_{l-l_1}(\lambda_2), && \text{(A.7)} \\ &= \left( \frac{N(l, m, q^{-1})}{N(l, m, q)} \right)^{\frac{1}{2}} q^{l_1(l-l_1)}\ e^{lq^{-1}}_m(l_1, l_2, \lambda). && \text{(A.8)} \end{aligned}$$

The norm ratio $\frac{N(l,m,q^{-1})}{N(l,m,q)}$ is calculated from the definition (A.5), and one gets finally the action of the universal $R$-matrix on the reduced basis $e^{lq}_m(l_1, l_2, \lambda)$ :

$$R^q(l_1, l_2)\ e^{lq}_m(l_1, l_2, \lambda) = q^{\frac{1}{2}[l(l+1)-l_1(l_1+1)-l_2(l_2+1)]}\ e^{lq^{-1}}_m(l_1, l_2, \lambda). \tag{A.9}$$

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# References


[1] P. Minnaert and M. Mozrzymas, J. Phys. A: Math. Gen. **28,** 669-683 (1995).

[2] P. Minnaert and M. Mozrzymas, J. Math. Phys. **36**, 907 (1995).

[3] P. Minnaert and M. Mozrzymas, J. Math. Phys. **38**, 2676 (1997).

[4] L. C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics*, (Addison-Wesley, London, 1981).

[5] A. N. Kirillov and N. Yu. Reshetikhin, Preprint E-9-88, Leningrad Branch, V. A. Steklov Mathematics Institute, Leningrad (1988).

[6] M. Nomura, J. Math. Phys. **30**, 2397 (1989).

[7] Yu. F. Smirnov, V. N. Tolstoy and Yu. I. Kharitonov, Sov. J. Nucl. Phys. **53** (6), 1068 (1991).

[8] P. P. Kulish and N. Yu. Reshetikhin, Lett. Math. Phys. **18,** 143 (1989).

[9] H. Saleur, Nucl. Phys. **B336,** 363 (1990).

[10] P. P. Kulish, LOMI Preprint published in Zapiski Nautchnovo Seminaria, LOMI, 1990.

[11] P. Minnaert, and M. Mozrzymas, J. Math. Phys. **35**, 3132 (1994).

[12] P. Minnaert and S. Toshev, J. Math. Phys. **35**, 5057 (1994).